LIMIT THEOREMS FOR RANDOM WALK IN THE HYPERBOLIC SPACE

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LSA – Winter Conference, Voronovo, 18/11/2024 – 22/11/2024/

Let $\mathbb{H}^n (n \geq 2)$ denote the real hyperbolic space of dimension $n \geq 2$ This is the complete and simply connected Riemannian manifold with constant negative sectional curvature equal to -1. We will consider the Poincare ball model for the hyperbolic space $\mathbb{H}^n (n \geq 2)$.

$$\mathbb{B}^n := \{x \in \mathbb{R} : ||x|| < 1\}$$

where $|| \bullet ||$ stands for the Euclidean norm. Recall that the Riemannian metric, the Riemannian volume in Euclidean coordinates and the hyperbolic distance to the origin are equal, respectively

$$ds^2 = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1 - ||x||^2)^2}$$

$$d\mu_{\mathbb{B}^n} = 2^n(1 - ||x||^2)^{-n}dx$$

$$\eta = \ln\left(\frac{1 + ||z||}{1 - ||z||}\right), z \in \mathbb{B}^n$$

In the geodesic polar coordinates $(\eta, \Theta) \in [0, +\infty) \times S^{n-1}$, we have

$$d\mu_{\mathbb{B}^n} = \sinh(\eta)^{n-1}d\eta d\Theta,$$

where $d\Theta$ stands for the Riemannian measure on S^{n-1}

To define a random walk on $\mathbb{B}^n$ we will introduce the Möbius addition $(x, y) \in \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{B}^n$ which writes in Euclidean coordinates

$$x \oplus y = \frac{(1 + 2 < x, y > +||y||^2)x + (1 - ||x||^2)y}{1 + 2 < x, y > +||x||^2||y||^2},$$

where$\langle \ , \ \rangle$ stands for the usual Euclidean scalar product on $\mathbb{R}^n$. We also introduce the Möbius multiplication, i.e. for all $\gamma \in \mathbb{R}_+, z \in \mathbb{B}^n$

$$\gamma \otimes z := \tanh(\gamma \text{atanh}(||z||))\frac{z}{||z||}$$

Remarks

1) In the two-dimensional case, the Möbius transformation first appeared in the seminal paper by Karpelevich, Tutubalin and Shur [1] in the study of the problem of radio waveguides with random inhomogeneities, the problem first considered by Gerzenshtein and Vasilyev [2]. However, they wrote the summation differently considered the disk as a subset of the complex plane

$$z_1 \oplus z_2 = \frac{z_1 + z_2}{1 + \bar{z}_1 z_2}$$

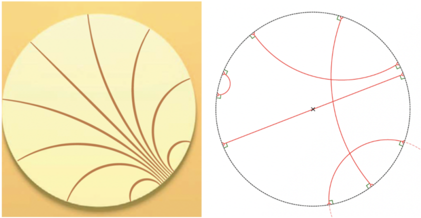
For fixed z_1 as a function of z_2 this transformation can be written as

$$\frac{z_1 + z_2}{1 + \bar{z}_1 z_2} = \frac{E z_2 + F}{\bar{F} z_2 + \bar{E}}$$

where

$$E = \rho, F = \rho z_1, \rho = \frac{1}{\sqrt{1 - z_1 \bar{z}_2}}, E\bar{E} - F\bar{F} = 1.$$

This is the isometry of the Lobachevsky plane, which preserves orientation and turns geodesics into geodesics.



Geodesics of the Poincare disk

2) Mobius addition on $\mathbb{B}^n$ is neither commutative nor associative but it is left gyroassociative

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$$

and gyrocommutative

$$a \oplus b = gyt[a, b](b \oplus a)$$

under gyrations defined by

$$gyr[a, b]c = \ominus(a \oplus b) \oplus \{a \oplus (b \oplus c)\}, a, b, c \in \mathbb{B}^n$$

The mapping $gyr[a,b]c$ is an isometry with the origin as a fixed point, $gyr[a,b]0 = 0$. Gyration looks especially simple in the two-dimensional case.

$$gyr[a,b]c = (1 + a\bar{b})c(1 + a\bar{b})^{-1}, a, b, c \in \mathbb{B}^2,$$

which represent rotation of the disc in turn of the origin

$$(1 + a\bar{b}) = re^{i\phi} \Rightarrow gyr[a,b]c = e^{2i\phi}c,$$

Gyrocommutative property clarifies the condition $\mathbb{R}$ below on the density to be rotationally invariant. As we see, gyroautomorphisms are rotations and for radial densities

$$f_{Z_1 \oplus Z_2} = f_{Z_2 \oplus Z_1}.$$

3) Later we will explain why such multiplication is naturally associated with the addition introduced. Note also that $1 \otimes z = z$

Random walk

Let Z be a $\mathbb{B}^n$ – valued random variable defined on some probability space (Ω, F, P). We will later assume that

[R] Z has radial density $f_z \in C_0^\infty(\mathbb{B}^n)$ w.r.t. the Riemannian volume.

Let now $(Z^j)_{j\leq 1}$ be a sequence of i.i.d random variables which have the same law as Z. Define then

$$\bar{S}_N := \oplus_{j=1}^N \frac{1}{N} \otimes Z^j$$

$$S_N := \oplus_{j=1}^N \frac{1}{\sqrt{N}} \otimes Z^j$$

Then the following results hold:

Theorem 1. (Law of large numbers) Under **[R]**

$$\bar{S}_N \xrightarrow{P} 0, \quad N \to \infty$$

It is shown that S_N has a density f_{S_N}, which can be expressed as the non-Euclidean convolution of the densities of the $(Z^j)_{j\in[1,N]}$. The central limit theorem then quantifies the asymptotic behavior of that density.

Various forms of the law of large numbers for metric spaces of non-positive curvature are known, we refer to [3] and references therein. The peculiarity of our formulation is that we use addition and multiplication by Mobius, and the law takes on a familiar classical form.

Theorem 2.

Theorem 2. (Central limit theorem) Under **[R]** it holds that for measurable sets A in $\mathbb{B}^n$

$$\int_{\mathbb{B}^n} I_A(x) f_{S_N} \mu_{\mathbb{B}^n}(dx) \to \int_{\mathbb{B}^n} I_A(x)\Psi(t,x)\mu_{\mathbb{B}^n}(dx), N \to \infty$$

with $t = \int_0^\infty \eta^2 \mu_Z(d\eta)$, where μ_Z stands for the measure induced by the law of Z in geodesic polar coordinates.

Theorem 2.

Also, $\Psi(t,\bullet)$ stands for the hyperbolic heat kernel in the model B^n for H^n. Namely, it denotes the fundamental solution of the equation

$$\frac{1}{2}\Delta_{\mathbb{B}^n}\Psi(t,x) = \partial_t\Psi(t,x), \quad \Psi(0,\bullet) = \delta(\bullet)$$

The specific expression of $\Psi(t,\bullet)$ will be given below. It plays the same role in the current setting as the normal density in the classical Euclidean central limit theorem.

It should be said that this result was known, we can refer to [3] and references therein for related issues. It could be directly proved modifying the arguments developed for our main result (Theorem 3) which specifies the CLT giving a convergence rate.

Theorem 3.(Local limit theorem) Under **[R]** there exists $C := C(n, \mu_Z)$ s.t. for all $x \in \mathbb{B}^n$ and N large enough

$$|f_{S_N}(x) - \Psi(t,x)| \leq \frac{C}{N}$$

The hyperbolic heat kernel $\Psi(t,x)$ in Theorems 2 and 3 is equal to $\Psi(t,x) = \Psi(\frac{t}{2}, x)$ where

$$\Psi(t,x) = \Psi\left(t, \frac{\eta}{2}\right) =$$

$$\begin{cases} \frac{\exp\left(-\frac{m^2 t}{2}\right)}{(2\pi)^m\sqrt{2\pi t}} \left(-\frac{1}{\sinh \eta}\partial_\eta\right)^m \exp\left(-\frac{\eta^2}{2t}\right), n = 2m+1, \\ \frac{\exp\left(-\frac{\left(m-\frac{1}{2}\right)^2 t}{2}\right)}{(2\pi)^m\sqrt{\pi t}} \int_\eta^{+\infty} \frac{ds}{\sqrt{\cosh(s) - \cosh(\eta)}} (-\partial_s) \left(-\frac{1}{\sinh s}\partial_s\right)^{m-1} \exp\left(-\frac{s^2}{2t}\right), n = 2m. \end{cases} \quad (1)$$

The main result of the work is the local limit theorem, which, as far as we know, is new. The method of proof is harmonic analysis in a Poincare ball. Let's outline the main points of this approach.

The Fourier-Helgason transform and its inverse.

For a radial function $f \in C_0^\infty(\mathbb{B}^n, \mathbb{R}) and \lambda \in \mathbb{R}_+$, its Fourier – Helgason transform is given by the expression

$$\hat{f}(\lambda) = \Omega_{n-1} \int_0^\infty (\sinh \eta)^{n-1} f(\tanh(\frac{\eta}{2}))\phi_\lambda(\tanh(\frac{\eta}{2}))d\eta,$$

where $\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ denotes the area of the unit sphere S^{n-1} and the radial functions ϕ_λ are eigenfunctions of the Laplace – Beltrami operator in B^n expressed in radial coordinates. The function ϕ_λ solves the differential equation

$$\begin{cases} \Delta_{\mathbb{B}^n}\phi_\lambda(r) + (\lambda^2 + \rho^2)\phi_\lambda(r) = 0, \rho = \dfrac{n-1}{2}, \\ \phi_\lambda(0) = 1. \end{cases}$$

This equation is solved using the functions

$$e_{\lambda,\omega}(x) = \frac{(1 - ||x||^2)^{\frac{1}{2}(n-1+i\lambda)}}{||x - \omega||^{n-1+i\lambda}},$$

$\omega \in S^{n-1}$, which are eigenfunctions of $\Delta_{\mathbb{B}^n}$ associated with the eigenvalue $-(\lambda^2 + \rho^2)$ and therefore these functions actually play in the current context a similar role to the complex exponential $\exp(ix \cdot y)$ in the usual Fourier analysis on the Euclidean space $\mathbb{R}^n$

Averaging over the surface of a sphere we then define the corresponding elementary spherical function ϕ_λ setting for all $x \in \mathbb{B}^n$

$$\phi_\lambda(x) = \frac{1}{\Omega_{n-1}} \int_{\mathbb{S}^{n-1}} e_{\lambda,\omega}(x)\Lambda(d\omega)$$

It is clear that ϕ_λ is also an eigenfunction of $\Delta_{\mathbb{B}^n}$ with the eigenvalue $-(\lambda^2 + \rho^2)$ and $\phi_\lambda(0) = 1$.

The inversion formula for a radial function $f \in C_0^\infty(\mathbb{B}^n, \mathbb{R})$ reads for all $||z|| = r \in [0, 1)$

$$f(z) = f(r) = C_n \int_0^{+\infty} d\lambda |c(\lambda)|^{-2} \hat{f}(\lambda) \phi_\lambda(r),$$

where $c(\lambda)$ is the generalized Harish-Chandra function

$$c(\lambda) = \frac{2^{3-n-i\lambda}\Gamma(\frac{n}{2})\Gamma(i\lambda)}{\Gamma(\frac{n-1+i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})},$$

and $C_n = \frac{1}{2^{n-3}\pi\Omega_{n-1}}$.

The heat kernel on $\mathbb{B}^n$

The heat kernel will provide the limit law which is somehow the analogue in the current non-Euclidean setting of the normal law. The normal density of parameter $t > 0$ in $\mathbb{B}^n$ in defined as the solution to

$$\frac{1}{2}\Delta_{\mathbb{B}^n}\Psi(t,x) = \partial_t \Psi(t,x), \Psi(0,\cdot) = \delta(\cdot).$$

In the literature the usual heat equation considered is

$$\Delta_{\mathbb{B}^n}\psi(t,x) = \partial_t \psi(t,x), \psi(0,\cdot) = \delta(\cdot).$$

The heat equation can be solved through the Fourier - Helgason transform (assuming the solution is radial), one derives that for all $\lambda \in \mathbb{R}$

$$\hat{\psi}(t,\lambda) = \exp(-(\lambda^2+\rho^2)t), \rho = \frac{n-1}{2}$$

With the terminology of [1] the last expression corresponds to the so-called characteristic function of the first kind. To match the Euclidean probabilistic set-up, we will consider the characteristic functions of the second kind

$$\hat{\psi}_2(t,\lambda) := \frac{\hat{\psi}(t,\lambda)}{\hat{\psi}(t,0)} = \exp(-\lambda^2 t),$$

so that in particular $\hat{\psi}(t,0) = 1$.

By inversion we obtain the heat kernel on $\mathbb{B}^n$ given by the formula (1). The heat kernel (1) is naturally to call the n-dimensional normal density in $\mathbb{B}^n$. This heat kernel is expressed in term of the hyperbolic distance to the origin. We can refer to [5] for the derivation of (1) through the Abel transform and its inverse or to [6] in which the authors derive the heat kernel through the fundamental solution of the wave equation.

Some additional tools from harmonic analysis on $\mathbb{B}^n$.

Following Ahlfors [7], we define for fixed $a \in \mathbb{B}^n$ the translation operator.

$$T_a : x \in \mathbb{B}^n \to T_a(x) = -a \oplus x \in \mathbb{B}^n.$$

This mapping is bijective, it is easily checked that $T_a^{-1} = T_{-a}$ and has the next important properties

1)

$$1 - ||T_a(x)||^2 = \frac{(1-||a||^2)(1-||x||^2)}{1-2<x,a>+||x||^2||a||^2}.$$

2)

$$\det(\mathrm{D}_x T_a(x)) = \left(\frac{1-||a||}{1-2<x,a>+||x||^2||a||^2}\right)^n, |||D_x T_a(x)||| = \frac{1-||a||^2}{1-2<x,a>+||x||^2||a||^2},$$

where $|||\cdot|||$ denotes the spectral norm.

3) T_a preserves the Riemannian measure. Namely,

$$\frac{|||D_x T_a(x)|||}{1-||T_a(x)||^2} = \frac{1}{1-||x||^2}$$

4)

$$T_a(x) = -\frac{D_x T_a(x)}{|||D_x T_a(x)|||} T_x(a).$$

In particular, for a radial function $f : \mathbb{B}^n \to \mathbb{R}$, it holds that

$$f(T_a(x)) = f(T_x(a)).$$

The translation operator allows to define in a quite natural way the convolution. Namely, for $f, g \in C_0^\infty(\mathbb{B}^n, \mathbb{R})$, we set

$$\forall x \in \mathbb{B}^n, f * g(x) = \int_{\mathbb{B}^n} f(-y \oplus x) g(y) \mu_{\mathbb{B}^n(dy)} = \int_{\mathbb{B}^n} f(T_y(x)) g(y) \mu_{\mathbb{B}^n}(dy).$$

This definition enlarges the one in [1] to the current multi-dimensional setting. Let's list the useful properties of convolution that are used to prove limit theorems.

- Commutativity: it holds that

$$f * g(x) = g * f(x).$$

-Stability of the radial property trough convolution: $f * g$ is a radial function.

- Fourier – Helgason transform of the convolution: it holds that for all $\lambda \in \mathbb{R}$

$$\widehat{f * g}(\lambda) = \hat{f}(\lambda)\hat{g}(\lambda).$$

Non-Euclidean mean, variance and scaling.

We define, coherently with the Euclidean case, the mean and variance, associated with a $\mathbb{B}^n$ - valued random variable Z defined on some probability space (Ω, F, P) satisfying assumption **[R]**. The analog of the characteristic function (of the second kind) writes:

$$\Phi_Z(\lambda) = \frac{\hat{f}_Z(\lambda)}{\hat{f}_Z(0)}, \Phi_Z(0) = 1.$$

We emphasize that, since we assumed the density to be radial, it follows that for all $m = 2j + 1, j \in \mathbb{N}$

$$\partial_\lambda^m \Phi_Z(\lambda)|_{\lambda=0}$$

In other terms, the odd moments of the random variable are 0. From the above definition we define the analogue of the variance as

$$V_Z = -\partial_\lambda^2 \Phi_Z(\lambda)|_{\lambda=0}$$

Scaling property.

The authors in [1] considered scaled variables Z_ε (associated with Z) for which they assume that the corresponding measure $\mu_{Z_\varepsilon,R}$ in radial geodesic coordinates satisfies for all $\tau > 0$

$$\mu_{Z_\varepsilon,R}(\{\eta \in \mathbb{R}_+ : \eta \leq \tau\varepsilon\}) = \mu_{Z,R}(\{\eta \in \mathbb{R}_+ : \eta \leq \tau\})$$

Importantly, from the definition of the Möbius multiplication it holds that the scaling property holds if and only if

$$Z_\varepsilon \overset{\text{(law)}}{=} \varepsilon \otimes Z.$$

The main properties of the variances.

1) Let Z_1 and Z_2 be two $\mathbb{B}^n$ - valued independent random variables with radial densities $f_{Z_1}, f_{Z_2} \in C_0^\infty(\mathbb{B}^n, \mathbb{R})$ w.r.t. the Riemannian volume of $\mathbb{B}^n$. It then holds that

$$V_{Z_1 \oplus Z_2} = V_{Z_1} + V_{Z_2}.$$

2) Let Z satisfy [R]. Set for $\varepsilon > 0, Z_\varepsilon = \varepsilon \otimes Z$. It then holds that there exists $C \geq 1$ s.t.

$$V_{Z_\varepsilon} \leq C\varepsilon^2.$$

Choosing $\varepsilon = \frac{1}{\sqrt{N}}$, the above control can be specified to derive that

$$V_{S_N} = V_{\oplus_{j=1}^N \frac{1}{\sqrt{N}} \otimes Z^j} \underset{N}{\rightarrow} t := \frac{1}{n} \int_0^{+\infty} \tilde{\eta}^2 \mu_{Z,R}(d\tilde{\eta}),$$

which is precisely the asymptotic variance appearing in theorems 2 and 3.

Furthermore, there exists $C \geq 1$ s.t.

$$\|V_{S_N} - t\| \leq \frac{C}{N},$$

and

$$V_{\bar{S}_N} = V_{\oplus_{j=1}^N \frac{1}{N} \otimes Z^j} \underset{N}{\rightarrow} 0,$$

which is also coherent with the statement of theorem 1.

Proof of the main results.

The proofs are based on the analysis of the Fourier-Helgason transform. We will briefly focus on the proof of the local limit theorem, since this proof is the most difficult part of the paper. The idea of the proof is traditional: use the inversion formula to analyze the density difference and estimate the difference in the characteristic functions on the right side of the inversion formula. The density difference writes:

$$f_{S_N}(\tanh(\frac{\eta}{2})) - \Psi(t, \tanh(\frac{\eta}{2}))$$

$$= C_n \int_0^\infty [\hat{f}_{S_N} - \hat{\psi}(t,\lambda)]\phi_\lambda(\tanh(\tfrac{\eta}{2}))|c(\lambda)|^{-2} d\lambda$$

$$= C_n \int_0^\infty [\prod_{j=1}^N \hat{f}_{\frac{1}{\sqrt{N}}\otimes Z^j}(\lambda) - \exp(-\tfrac{(\rho^2+\lambda^2)t}{2})]\phi_\lambda \tanh(\tfrac{\eta}{2})|c(\lambda)|^{-2} d\lambda$$

$$= C_n \int_0^\infty (\mathbb{I}_{\lambda \le D_N} + \mathbb{I}_{\lambda > D_N})[(\hat{f}_{\frac{1}{\sqrt{N}}\otimes Z})^N -$$

$$\exp(-\tfrac{(\rho^2+\lambda^2)t}{2})]\phi_\lambda(\tanh(\tfrac{\eta}{2}))|c(\lambda)|^{-2} d\lambda =: (B_N + \tau_N)(\eta)$$

where $C_n = \frac{1}{2^{n-3}\pi\Omega_{n-1}}, \Omega = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ denotes the area of the unit sphere $S^{n-1}, D_N = N^{\frac{1}{4}}$ is a cutting level to be specified which will allow to balance the contribution for the terms $B_N(\eta)$ and $\tau_N(\eta)$ corresponding respectively to the bulk and tails of the Fourier – Helgason integral.

Contribution of the bulk.

Here it is possible to adapt some techniques for evaluating characteristic functions in the classical Euclidean case. Additionally, an estimate of the Harish-Chandra function and some estimates of spherical functions are used. As a result, we get an estimate

$$|B_N(\eta)| \leq \frac{c}{t^2 N}\left(\frac{1}{t^{\frac{1}{2}}} \bigwedge \frac{1}{t^{\frac{n}{2}}}\right) \tag{2}$$

Contribution of the tails.

When estimating the "tails", we use the expression of spherical functions in terms of the Gauss hypergeometric function

$$\varphi_\lambda(\tanh(\frac{\eta}{2})) = {}_2F_1(\rho + i\frac{\lambda}{2}, \rho - i\frac{\lambda}{2}, \frac{n}{2}, -\sinh^2(\frac{\eta}{2})).$$

Then we use the decomposition of the hypergeometric function into a series and analyze the terms of this series for large values of λ. We get an estimate

$$|\tau_N(\eta)| \le \frac{C}{N}(\frac{1}{t^2}(\frac{1}{t^{\frac{1}{2}}} \bigwedge \frac{1}{t^{\frac{n}{2}}}) + 1) \tag{3}$$

Theorem 3 now follows from (2) and (3).

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