Some Problems on Pursuit and Approximation of Random Processes

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This mini-course is based on joint work with

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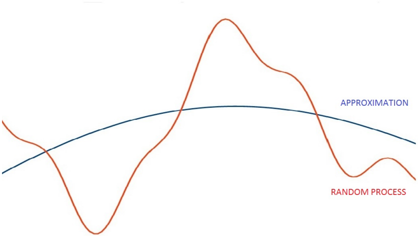
Approximation of a random process



Some aspects to specify:

- What is a good approximation? Having low kinetic energy or some other kind of energy?
- How to measure the closeness of a process and its approximation? Uniformly? In average (using L_2 distance or some other potential)?
- Which data of the process are available for constructing an approximation? Non-adaptive or adaptive approximation?
- Which types of processes to handle? Stationary or with stationary increments.

Example: running after a Brownian dog

How to keep the Brownian dog on a leash in the energy saving mode?

Let the dog walk in $\mathbb{R}$ according to a Brownian motion $W(t)$.

You must follow it by moving with a finite speed and always stay not more than 1 away from the dog.

If $x(t)$ is your trajectory, then the goal is to follow the dog, i.e. keep $|x(t) - W(t)| \leq 1$ and expend minimal kinetic energy per unit of time

$$\frac{1}{T} \int_0^T x'(t)^2 \, dt$$

in a long run, $T \to \infty$.

Diffusion strategy for the pursuit

Let $X(t) := x(t) - W(t)$ be the signed distance to the dog.
A reasonable strategy is to determine the speed $x'(t)$ as a function of $X(t)$ by accelerating when $X(t)$ approaches the boundary ± 1. So let $x'(t) := b(X(t))$. Then X becomes a diffusion satisfying $dX = b(X)dt - dW$. One-dimensional diffusions are well understood.
Consider a probability density

$$p(x) = C\, e^{B(x)}, \qquad \text{where } B(x) := 2 \int^x b(y)dy.$$

If non-exit conditions $\int_{-1} \frac{dx}{p(x)} = \int^{+1} \frac{dx}{p(x)} = \infty$ are satisfied, then the diffusion is ergodic and $p(x)dx$ is its invariant measure.
By ergodic theorem, in the stationary regime

$$\frac{1}{T} \int_0^T x'(t)^2\, dt \to \int_{-1}^1 b(x)^2\, p(x)\, dx = \frac{1}{4} \int_{-1}^1 \frac{p'(x)^2}{p(x)^2}\, p(x)\, dx := \frac{1}{4}\, I(p).$$

We have to minimize Fisher information I(p) !

Solution: optimal strategy

Minimizing Fisher information on the interval is a classical problem arising in Statistics, Data Analysis, etc (Zipkin, Huber, Levit, Shevlyakov, etc).
By simple variational calculus we obtain the optimal density

$$p(x) = \cos^2(\pi x/2), \qquad x \in [-1, 1],$$

and the optimal speed strategy

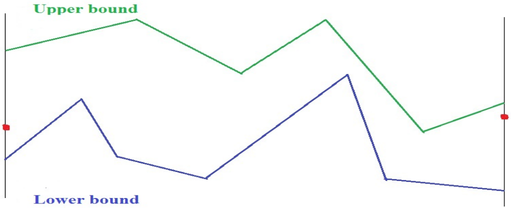
$$b(x) = -\pi \tan(\pi x/2)$$

exploding at the boundary.
This leads to the asymptotic minimal reduced energy

$$\frac{1}{T} \int_0^T x'(t)^2 \, dt \to \frac{1}{4} I(p) = \frac{\pi^2}{4}.$$

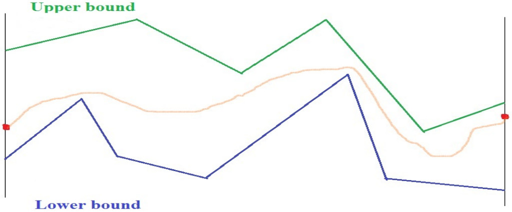
Non-adaptive setting: taut string

Let on some time interval two functions be given – an upper boundary and a lower boundary, as well as some initial value and some final value located between the boundaries.



Non-adaptive setting: taut string

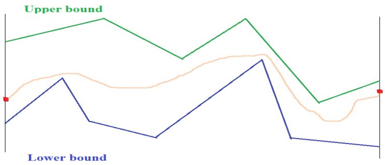
Consider all functions running between the boundaries and having given initial and final values.



Consider a class of optimization problems

$$\begin{cases} \int_{T_0}^{T_1} \varphi(h'(t))dt \searrow \min \\ h(T_0) = h_0, \quad h(T_1) = h_1, \\ F(t) \le h(t) \le G(t), \end{cases}$$

Non-adaptive setting: taut string



$$\begin{cases} \int_{T_0}^{T_1} \varphi(h'(t))dt \searrow \min \\ h(T_0) = h_0, \quad h(T_1) = h_1, \\ F(t) \le h(t) \le G(t), \end{cases}$$

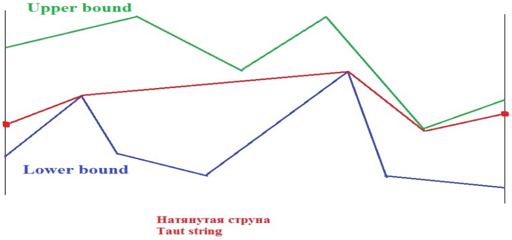
Examples:

- $\int_{T_0}^{T_1} \sqrt{1 + h'(t)^2}\, dt$ is the graph length;
- $\int_{T_0}^{T_1} h'(t)^2\, dt$ is the kinetic energy;
- $\int_{T_0}^{T_1} |h'(t)|\, dt$ is the total variation;

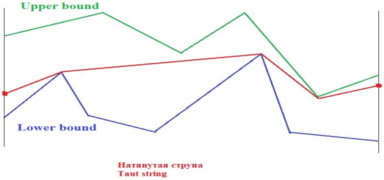
Non-adaptive setting: taut string

$$\begin{cases} \int_{T_0}^{T_1} \varphi(h'(t))dt \searrow \min \\ h(T_0) = h_0, \quad h(T_1) = h_1, \\ F(t) \le h(t) \le G(t), \end{cases}$$

Magic: for all convex functions φ there is a common solution called *taut string*.

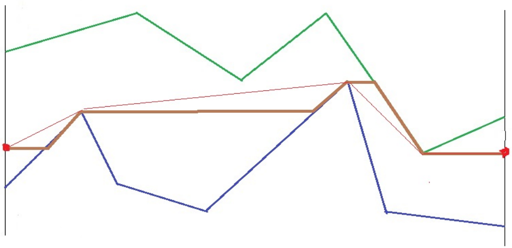


Taut string: properties



- Taut string is piecewise linear when running strictly between the boundaries; It can be non-linear when running along one of the boundaries; It can have infinitely many linear pieces if a boundary is non-smooth;
- When eliminating initial and the final values, the solution may depend on φ;
- If φ is strictly convex, taut string is a unique solution; If φ is non-strictly convex, other solutions may exist. Example: $\varphi(t) = |t|$ (the total variation functional).

Alternative minimizers for total variation: lazy function vs taut string



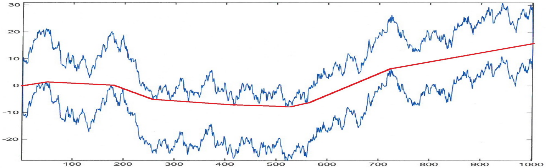
Lazy function ant taut string

Lazy function does not change its value as long as it can.
Both a lazy function and a taut string minimize the total variation of the functions running between the boundaries.

A taut string for Brownian motion

$$\begin{cases} \int_0^T h'(t)^2 dt \searrow \min \\ h(0) = W(0), \quad h(T) = W(T), \\ W(t) - r/2 \le h(t) \le W(t) + r/2, \qquad 0 \le t \le T. \end{cases}$$

Taut string for Brownian motion



Non-adaptive approximation: formal setting

We consider uniform norm

$$||h||_T := \sup_{0\le t\le T} |h(t)|, \qquad h \in \mathbb{C}[0,T],$$

and Sobolev-type norm (spent kinetic energy)

$$|h|_T^2 := \int_0^T h'(t)^2 dt, \qquad h \in AC[0,T].$$

Let W be a Brownian motion. We are mostly interested in its approximation characteristics
$I(T,r)$:= minimal kinetic energy over approximations h such that $||h-W||_T \le r/2$ and $h(0)=0$,
and

$$I^0(T,r) := \inf\{|h|_T^2; h \in AC[0,T], ||h-W||_T \le r/2, h(0)=0, h(T)=W(T)$$

First results on non-adaptive approximation, ML+E.Setterqvist, 2015

Theorem (weak LLN)

There exists $\mathcal{C} \approx 0,63$ *such that for any* $q > 0$ *if* $\frac{r}{\sqrt{T}} \to 0$*, then*

$$\frac{r^2}{T}\, I(T,r) \xrightarrow{L_q} 4\mathcal{C}^2 \qquad \textit{and} \qquad \frac{r^2}{T}\, I^0(T,r) \xrightarrow{L_q} 4\mathcal{C}^2.$$

We may complete the mean convergence with a.s. convergence to $4\mathcal{C}^2$.

Theorem (strong LLN)

For any fixed $r > 0$*, when* $T \to \infty$*, we have*

$$\frac{r^2}{T}\, I(T,r) \xrightarrow{a.s.} 4\mathcal{C}^2 \quad \textit{and} \quad \frac{r^2}{T}\, I^0(T,r) \xrightarrow{a.s.} 4\mathcal{C}^2.$$

Theorem (strong LLN (2015))

There exists $\mathcal{C} \approx 0,63$ such that for any fixed $r > 0$, when $T \to \infty$, we have

$$\frac{r^2}{T} I(T,r) \xrightarrow{a.s.} 4\mathcal{C}^2 \quad \textit{and} \quad \frac{r^2}{T} I^0(T,r) \xrightarrow{a.s.} 4\mathcal{C}^2.$$

In this result, the value of $\mathcal{C}$ was found by computer simulation. Now, as a special case of a fairly general result we know that this limit equals $\frac{\pi^2}{6}$, so that $\mathcal{C}^2 = \frac{\pi^2}{24}$, i.e. $\mathcal{C} \approx 0,64$.
Recall that

$$I^0(T,r) = \int_0^T \eta'_{T,r}(t)^2 dt, \qquad \text{where } \eta_{T,r} \text{ is a taut string.}$$

We show that, as $T \to \infty$,

$$\int_0^T \varphi(\eta'_{T,r}(t))dt \xrightarrow{\text{a.s.}} \int_{\mathbb{R}} \varphi(u)\nu(du).$$

for fairly general φ and some explicitly given measure ν on $\mathbb{R}$.

Convergence of occupation measures

We want to find

$$R(\varphi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(\eta'_{T,r}(t))dt.$$

Rewrite

$$\frac{1}{T}\int_0^T \varphi(\eta'_{T,r}(t))dt = \int_{\mathbb{R}} \varphi(u)\nu_T(du)$$

via taut string derivative's occupation measure

$$\nu_T(du)(A) := \frac{1}{T}\int_0^T \mathbf{1}_{\eta'_{T,r}(t))\in A}dt.$$

If we prove that $\nu_T \Rightarrow \nu$ a.s. with some limiting non-random measure ν (as $T \to \infty$), then it is likely that

$$R(\varphi) = \lim_{T\to\infty} \int_{\mathbb{R}} \varphi(u)\nu_T(du) = \int_{\mathbb{R}} \varphi(u)\nu(du).$$

Convergence of occupation measures (continued)

We will prove that the limiting occupation measure of taut string's derivative ν exists and has a density

$$p_\nu(u) = r\,\frac{ru\coth(ru) - 1}{\sinh(ru)^2}$$

(the limiting value at $u = 0$ exists and is equal to $\frac{r}{3}$). This is a good symmetric density exponentially decreasing at $\pm\infty$. It follows that $R(\varphi) = \int_{\mathbb{R}} \varphi(u) p_\nu(u) du$. In particular, for kinetic energy $\varphi(u) = u^2$ we have

$$R(\varphi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \eta'_{T,r}(t)^2 dt = \int_{\mathbb{R}} u^2 p_\nu(u) du = \frac{\pi^2}{6r^2},$$

as claimed before.
We will find the limiting occupation measure ν via truncated variation (a notion to be explained now).

Truncated variation (R. Lochowski)

Let f be a continuous function on an interval $[0, T]$.
Consider partitions of $[0, T]$ by points

$$0 = t_0 < t_1 < \cdots < t_n = T.$$

The usual total variation is defined as

$$TV(f) := \sup_{n \geq 1, t_0 \ldots t_n} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|.$$

If f is absolutely continuous, then $TV(f) = \int_0^T |f'(t)| dt$. Let $r > 0$. Then r-truncated variation is defined by

$$TV^r(f) := \sup_{n \geq 1, t_0 \ldots t_n} \sum_{i=1}^{n} \max\{|f(t_i) - f(t_{i-1})| - r, 0\}.$$

There is no contribution from increments smaller than r !
We have $TV^0(f) = TV(f)$.

Truncated variation (continued)

The advantage of truncated variation

$$TV^r(f) := \sup_{n\geq 1, t_0 \ldots t_n} \sum_{i=1}^{n} \max\{|f(t_i) - f(t_{i-1})| - r, 0\}.$$

is that it can be finite for functions having infinite total variation such as Brownian motion.

There is a variational representation of truncated variation

$$\begin{aligned} TV^r(f) &= \inf\{TV(g), g : ||f-g||_T \leq r/2\} \\ &= \inf\left\{\int_0^T |g'(t)|dt, g : g \in AC[0,T], ||f-g||_T \leq r/2\right\}. \end{aligned}$$

Here we come closer to taut strings!

Truncated variation of Brownian motion

Theorem (Lochowski–Milos)

Let $u \in \mathbb{R}$, let W be a Brownian motion and let $X_u(t) := W(t) - ut$ be a Brownian motion with drift u. Then

$$\lim_{T\to\infty} \frac{1}{T} TV^r(X_u;[0,T]) = m_r(u) = \begin{cases} u\coth(ru), & u \neq 0, \\ r^{-1}, & u = 0. \end{cases}$$

By variational representation $TV^r(X_u;[0,T])$ equals to

$$\begin{aligned} & \inf\left\{ \int_0^T |g'(t)|dt, g : g \in AC[0,T], \|W(t) - ut - g\|_T \le r/2 \right\} \\ = \quad & \inf\left\{ \int_0^T |h'(t) - u|dt, h : g \in AC[0,T], \|W(t) - h(t)\|_T \le r/2 \right\} \\ \approx \quad & \int_0^T \varphi_u(\eta'_{T,r}(t))dt, \qquad \varphi_u(v) := |v - u|. \end{aligned}$$

First conclusions on taut string

We infer from Lochowski–Milos theorem that for the function φ_u defined by $\varphi_u(v) := |v - u|$

$$R(\varphi_u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi_u(\eta'_{T,r}(t))dt = m_r(u).$$

We will slightly modify this result. Let $\chi(v) := v$ be identity function. Then

$$\begin{aligned} R(\chi) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \chi(\eta'_{T,r}(t))dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \eta'_{T,r}(t)dt \\ &= \lim_{T\to\infty} \frac{1}{T}\eta_{T,r}(T) = \lim_{T\to\infty} \frac{1}{T} W(T) = 0. \end{aligned}$$

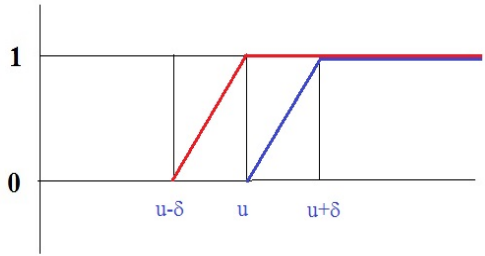
We introduce the functions $\psi_u = \frac{\varphi_u + \chi - u}{2}$, i.e. $\psi_u(v) = (v-u)_+$ and infer

$$R(\psi_u) = \frac{R(\varphi_u) + R(\chi) - u}{2} = \frac{m_r(u) - u}{2}.$$

Computation of the occupation measure

By using the functions $\psi_u(v) := (v-u)_+$ we have

$$\frac{\psi_u - \psi_{u+\delta}}{\delta} \le \mathbf{1}_{[u,\infty)} \le \frac{\psi_{u-\delta} - \psi_u}{\delta}$$



Computation of the occupation measure (continued)

By integrating the inequality

$$\frac{\psi_u - \psi_{u+\delta}}{\delta} \leq \mathbf{1}_{[u,\infty)} \leq \frac{\psi_{u-\delta} - \psi_u}{\delta}$$

w.r.t. the occupation measure ν_T we have

$$\int_{\mathbb{R}} \frac{\psi_u - \psi_{u+\delta}}{\delta} \, \nu_T(dv) \leq \nu_T[u,\infty) \leq \int_{\mathbb{R}} \frac{\psi_{u-\delta} - \psi_u}{\delta} \, \nu_T(dv),$$

and taking the limit in $T \to \infty$, and using $R(\psi_u) = \frac{m_r(u)-u}{2}$ we obtain

$$\begin{aligned} & R\left(\frac{\psi_u - \psi_{u+\delta}}{\delta}\right) = \frac{m_r(u) - m_r(u+\delta) + \delta}{2\delta} \\ \leq \quad & \liminf_{T\to\infty} \nu_T[u,\infty) \leq \limsup_{T\to\infty} \nu_T[u,\infty) \\ \leq \quad & R\left(\frac{\psi_{u-\delta} - \psi_u}{\delta}\right) = \frac{m_r(u-\delta) - m_r(u) + \delta}{2\delta}. \end{aligned}$$

Computation of the occupation measure (end)

Having

$$
\begin{aligned}
& \frac{m_r(u) - m_r(u+\delta) + \delta}{2\delta} \\
\leq \quad & \liminf_{T\to\infty} \nu_T[u,\infty) \leq \limsup_{T\to\infty} \nu_T[u,\infty) \\
\leq \quad & \frac{m_r(u-\delta) - m_r(u) + \delta}{2\delta}
\end{aligned}
$$

we take the limit in $\delta \searrow 0$ and get

$$
\lim_{T\to\infty} \nu_T[u,\infty) = \frac{-m_r'(u) + 1}{2} := \nu[u,\infty).
$$

The density of the limiting occupation measure ν is

$$
p_\nu(u) = -\frac{d}{du}\,\nu[u,\infty) = \frac{m_r''(u)}{2} = r\,\frac{ru\coth(ru) - 1}{\sinh(ru)^2},
$$

as claimed.

Final result on the best non-adaptive approximation of BM

We consider convergence $\lim_{T\to\infty} \frac{1}{T} \int_1^T \varphi(\eta'_{T,r}(t))dt = \int_{\mathbb{R}} \varphi(u)\nu(du)$.

There are some subtle differences between a.s.-convergence and convergence in probability.

Theorem

We have

- $\frac{1}{T} \int_1^T \varphi(\eta'_{T,r}(t))dt \xrightarrow{\mathbb{P}} \int_{\mathbb{R}} \varphi(u)\nu(du)$; *(even if the limit is infinite);*
- $\liminf_{T\to\infty} \frac{1}{T} \int_1^T \varphi(\eta'_{T,r}(t))dt \geq \int_{\mathbb{R}} \varphi(u)\nu(du)$ *a.s.;*
- *if* $\varphi(u) \leq c(1+|u|)^\alpha$ *for come* $c, \alpha > 0$, *then* $\frac{1}{T} \int_1^T \varphi(\eta'_{T,r}(t))dt \xrightarrow{a.s.} \int_{\mathbb{R}} \varphi(u)\nu(du)$;
- *There exists* $\lambda > 0$ *such that for* $\varphi(u) = \exp(\lambda u)$ *we have* $\limsup_{T\to\infty} \frac{1}{T} \int_1^T \varphi(\eta'_{T,r}(t))dt = +\infty$ *a.s. but* $\int_{\mathbb{R}} \varphi(u)\nu(du) < \infty$.

Open problem: other processes

One can get similar results for the Brownian motion with drift.
Open problem: get similar results

$$\frac{1}{T}\int_1^T \varphi(\eta'_{T,r}(t))dt \xrightarrow{\text{a.s.}} \int_{\mathbb{R}} \varphi(u)\nu(du);$$

for taut strings in a tube around sample path of a process other than Brownian motion, e.g., a rather general Gaussian process with stationary increments.

A missing piece: Lochowski–Milos result on truncated variation is exclusively Brownian.

Diffusion strategy for multivariate pursuit

Now the object of pursuit is a multivariate Brownian motion $W(t)$ in $\mathbb{R}^d$, $d > 1$. (Imagine a Brownian dog running across 2-dimensional field). The pursuit trajectory $x(t)$ must stay within unit distance from $W(t)$ and spend the minimal amount of kinetic energy per unit of time. Again, we explore the diffusion strategies: always running towards the target with a speed depending on the distance from the target, i.e., let $X(t) := x(t) - W(t)$ determine the direction to the dog and the pursuit speed is

$$\frac{dx}{dt}(t) = b(||X||)\,\frac{X}{||X||}.$$

Then X is a d-dimensional diffusion and one has to minimize the pursuit energy over $b(\cdot)$.

Diffusion strategy for multivariate pursuit (solution)

The next result (I. Lialinov, 2025) describes the optimal strategy in this class. Let $\nu = \frac{d-2}{2}$. Let J_ν be the first kind Bessel function of order ν, and j_ν its smallest positive root. Then the optimal pursuit speed is

$$b(r) = \frac{j_\nu J_{\nu+1}(j_\nu r)}{J_\nu(j_\nu r)}$$

which gives the minimal asymptotic pursuit energy per unit of time

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left\| \frac{dx}{dt}(t) \right\|^2 dt = j_\nu^2.$$

The simplest expression appears dimension for $d = 3$. Then

$$b(r) = \frac{1}{r} - \pi \cot(\pi r)$$

and the minimal limiting energy equals π^2 .

Open problems

- Prove that the optimal diffusion strategy for Brownian dog pursuit is also optimal among all adaptive strategies.
- Find an optimal diffusion strategy for Brownian dog pursuit on a finite time interval. It will be not time-homogeneous.

An extension: variable width

Now we want to minimize the kinetic energy over the functions running through a band of variable width. Let $t \mapsto r(t)$ be a positive width function. We solve

$$\begin{cases} \int_0^T h'(t)^2 dt \searrow \min \\ h(0) = W(0), \quad h(T) = W(T), \\ W(t) - r(t)/2 \le h(t) \le W(t) + r(t)/2, \qquad 0 \le t \le T. \end{cases}$$

Again, the solution is given by the taut string $\eta_{T,r(.)}$. We have

Theorem (ML+A.Siuniaev, 21)

Let $r(t) \nearrow$ but $\frac{r(t) \log\log t}{t^{1/2}} \searrow 0$. Then, with the same $\mathcal{C}^2 = \frac{\pi^2}{24}$,

$$\int_0^T \eta'_{T,r(.)}(t)^2 dt \sim 4\mathcal{C}^2 \int_0^T \frac{dt}{r(t)^2}, \qquad a.s., \qquad as\ T \to \infty.$$

Further extension: approximation of random walk

Let $X_1, X_2, ...$ be an i.i.d. sequence. Define the partial sums (a random walk) as $S_0 := 0$, $S_k := \sum_{j=1}^{k} X_j$ ($k \geq 1$). Define a random broken line $S(t), t \geq 0$, by $S(k) := S_k, k \geq 0$, and by linear interpolation between integer times. We solve

$$\begin{cases} \int_0^T h'(t)^2 dt \searrow \min \\ h(0) = S(0), \quad h(T) = S(T), \\ S(t) - r(t)/2 \leq h(t) \leq S(t) + r(t)/2, \qquad 0 \leq t \leq T. \end{cases}$$

The approximation results for W extend to those of S if either

- for some $p > 2$ we have $\mathbb{E}|X_j|^p < \infty$ and $r(t) \gg t^{1/p}$,

or

- for some $\lambda > 0$ we have $\mathbb{E}\exp\{\lambda |X_j|\} < \infty$ and $r(t) \gg \log t$.

The proofs go through KMT approximation of S by W.

A famous related problem: Strassen's FLIL

Strassen's functional law of the iterated logarithm:

$$\limsup_{T\to\infty} \inf_{|h|_1\le 1} \left\| \frac{W(\cdot T)}{\sqrt{2T\ln\ln T}} - h \right\|_1 = 0 \qquad \text{a.s.}$$

Convergence rate: Grill, Talagrand $\exists c_1, c_2$ such that

$$c_1 < \limsup_{T\to\infty} (\ln\ln T)^{2/3} \inf_{|h|_T\le 1} \left\| \frac{W(\cdot T)}{\sqrt{2T\ln\ln T}} - h \right\|_1 < c_2 \qquad \text{a.s.}$$

Liminf result (Grill, Griffin and Kuelbs) $\exists c_3, c_4$ such that

$$c_3 < \liminf_{T\to\infty} (\ln\ln T) \inf_{|h|_1\le 1} \left\| \frac{W(\cdot T)}{\sqrt{2T\ln\ln T}} - h \right\|_1 < c_4 \qquad \text{a.s.}$$

In terms of the taut string energy $I(T,r)$ we have

$$\limsup_{T\to\infty} \frac{I(T, c_1(2T)^{1/2}(\ln\ln T)^{-1/6})}{(2\ln\ln T)^{1/2}} > 1, \qquad \text{a.s.}$$

etc. Here the tube is much wider and the string energy is much lower than in our case.

Unilateral approximation

Let W still be a Brownian motion and let $r > 0$. We consider absolutely continuous functions h satisfying unilateral constraints

$$h(t) \geq W(t) - r, \qquad 0 \leq t \leq T,$$

and initial condition $h(0) = 0$ and try to minimize the energy

$$\int_0^T \varphi(h'(t))dt,$$

with some convex non-negative energy function φ, e.g. kinetic energy

$$\int_0^T h'(t)^2 dt,$$

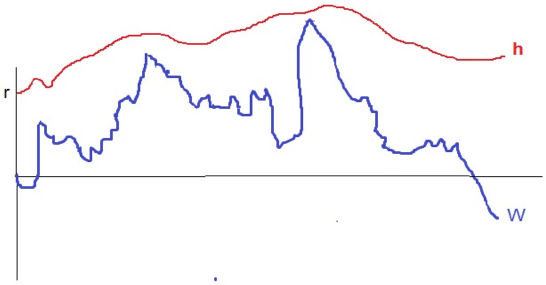
This means that we need not follow the target process when it goes deeply downwards.

Unilateral approximation (continued)

Let us shift the problem upwards in order to unify the constraints. We will minimize the energy on functions h satisfying

$$h(t) \geq W(t), \qquad 0 \leq t \leq T,$$

and initial condition $h(0) = r$.

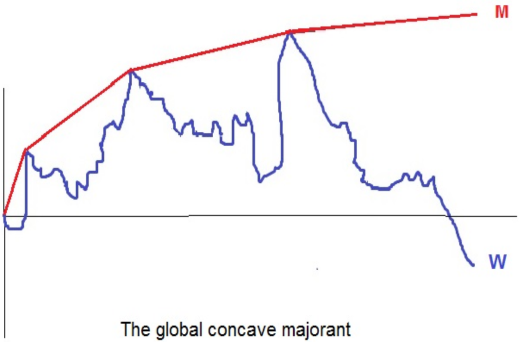


It turns out that, again, (as in the case of bilateral constraints) there is a universal solution independent on the energy function!
We will describe it now.

Concave majorant of the Brownian motion

The global concave majorant of the Brownian motion W is the smallest concave function M on $[0, \infty)$ satisfying

$$M(t) \geq W(t), \qquad t \geq 0.$$



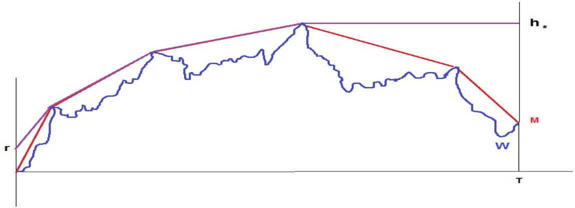
The global concave majorant

Similarly, the *local* concave majorant concerns
$M(t) \geq W(t), \qquad 0 \leq t \leq T.$

Minimal energy function

The following function h_* solves the problems

$$\int_0^T \varphi(h'(t))dt \searrow \min, \quad h(0) = r, h(t) \geq W(t), 0 \leq W(t), 0 \leq t \leq T.$$



h_* has three pieces:

- the first piece is a linear tangent to the local convex majorant (starting from r);
- the second piece is the local convex majorant until $argmax_{[0,T]} W$;
- the third piece is the constant equal to $\max_{[0,T]} W$.

Unilateral constraints: minimal energy behavior

We have understood that the optimal function essentially coincides with the minimal concave majorant. Therefore, a crude heuristic comes:
$W(t) \approx \sqrt{t}$; $M(t) \approx \sqrt{t}$;$h'_*(t) \approx M'(t) \approx t^{-1/2}$;for the energy we have

$$\int_0^T \varphi(h'_*(t))\, dt \approx \int_0^T \varphi\left(t^{-1/2}\right) dt$$

Theorem (ML+S.Nikitin, 24)

For kinetic energy we have (for each fixed $r > 0$)

$$\int_0^T h'_*(t)^2 dt \sim \frac{\log T}{2} \quad a.s., \qquad as\ T \to \infty.$$

Compare this rate with the linear one for bilateral constraints! Yet, according to our heuristics, the logarithmic rate is rather exceptional.

Unilateral constraints: minimal energy behavior

Theorem (ML+S.Nikitin, 24)

For kinetic energy we have (for each fixed $r > 0$)

$$\int_0^T h_*'(t)^2 dt \sim \frac{\log T}{2} \quad a.s., \qquad as\ T \to \infty.$$

The *log*-rate here is rather exceptional. In general case, the behavior of

$$\int_0^T \varphi(h_*'(t))\, dt$$

as $T \to \infty$, is determined by the growth of φ at zero (because $h_*'(t)$ tends to zero at infinity).

Unilateral approximation: non-kinetic energies

Let us replace the quadratic function x^2 in the definition of kinetic energy with another power function $\varphi(x) = |x|^\alpha$.

Theorem (S.Nikitin, 25)

Let $\alpha > 2$. Then for each fixed $r > 0$ we have

$$0 < \lim_{T\to\infty} \int_0^T h_*'(t)^\alpha \, dt < \infty, \qquad a.s.$$

This means that we may approximate Brownian motion as long as we wish using a limited amount of energy.

Theorem (S.Nikitin, 25)

Let $\alpha \in (1,2)$. Then for each fixed $r > 0$

$$0 < \limsup_{T\to\infty} \frac{1}{T^{1-\alpha/2}(\log\log T)^{\alpha/2}} \int_0^T h_*'(t)^\alpha \, dt < \infty, \qquad a.s.$$

Unilateral approximation: non-kinetic energies (continued)

The following result describes the $\liminf$ behavior of the minimal non-kinetic energy. The $\liminf$ behavior is more complicated than the $\limsup$ one.

Theorem (S.Nikitin, 25)

Let $\alpha \in (1,2)$ *and* $g(\cdot)$ *some decreasing function. Then for each fixed* $r > 0$

$$\liminf_{T\to\infty} \frac{1}{T^{1-\alpha/2} g(T)} \int_0^T h_*'(t)^\alpha \, dt = \begin{cases} = \infty, & \textit{iff } \int^\infty g(b)^{1/(2-\alpha)} \frac{db}{b} < \infty, \\ = 0, & \textit{iff } \int^\infty g(b)^{1/(2-\alpha)} \frac{db}{b} = \infty. \end{cases}$$

The critical function here is $g(b) = (\log b)^{-(2-\alpha)}$.

Unilateral adaptive approximation: diffusion strategy

The following result provides an optimal (in a certain class of stationary strategies) adaptive diffusion strategy.

Theorem (ML+S.Nikitin, 24)

The optimal diffusion pursuit strategy is given by

$$x'(t) = (x(t) - W(t))^{-1}.$$

for this strategy it is true that

$$\frac{\int_0^T x'(t)^2 dt}{\ln T} \xrightarrow{a.s.} 1, \qquad as\ T \to \infty.$$

We conclude that this adaptive strategy yields, in a long run, two times larger energy loss than the optimal non-adaptive one.

An extended setting: ”Pursuit under Potential”

Consider a fixed time horizon $[0, T]$, introduce a penalty function (potential) $Q(\cdot)$. Problem: find a pursuit process $X(\cdot)$ such that

$$\mathbb{E} \int_0^T \Big[X'(t)^2 + Q(X(t) - W(t)) \Big] \, dt \searrow \min$$

among all adapted absolutely continuous random functions X.
We also consider an infinite horizon problem stated as

$$\lim_{T \to \infty} T^{-1} \, \mathbb{E} \int_0^T \Big[X'(t)^2 + Q(X(t) - W(t)) \Big] \, dt \searrow \min$$

By appropriate interpretation of Q this setting formally includes the Brownian dog problem, whenever

$$Q(y) := \begin{cases} 0, & |y| \le 1, \\ +\infty, & |y| > 1. \end{cases}$$

A strategy of optimal pursuit

Strategy: $X'(t) := b(X - W, T - t)$.

At every moment we determine the pursuit speed as a prescribed function of two arguments: the current distance from the target W and the remaining time $T - t$. We show that this kind of strategy is the best among all adapted strategies on every finite interval of time provided that the drift function $b(\cdot, \cdot)$ is chosen properly.

Consider the expected penalty function achievable on the time interval of length t when starting at the point $X(0) = y$,

$$F(y,t) := \mathbb{E} \int_0^t \Big[X'(s)^2 + Q(X(s) - W(s)) \Big] \, ds = \mathbb{E} \int_0^t \Big[b(Y(s), t-s)^2 + Q(Y(s)) \Big] \, ds.$$

A version of Feynman–Kac formula leads to an equation

$$\begin{cases} F'_t(y,t) = Q(y) - \frac{1}{4} (F'_y)^2(y,t) + \frac{1}{2} F''_y(y,t) \\ F(y,0) = 0 \end{cases} .$$

Heat equation

Our equation

$$F_t'(y,t) = Q(y) - \frac{1}{4}\,(F_y')^2(y,t) + \frac{1}{2}\,F_y''(y,t)$$

is quite close to Burgers equation. Therefore, one has to use Hopf–Cole transform $F(y,t) := -2\ln V(y,t)$ which leads to some form of heat equation, namely,

$$V_t'(y,t) = \frac{V_{yy}''(y,t)}{2} - \frac{Q(y)\,V(y,t)}{2}\ .$$

with initial condition $V(y,0) = 1$. This is the heat equation up to the additional inhomogeneous term at the end.

From heat equation to survival probability

For that sort of heat equation, a good probabilistic solution is known. We find there

$$V(y,t) = \mathbb{E} \exp\left\{ -\frac{1}{2} \int_0^t Q(W_y(s))\, ds \right\},$$

where W_y stands for a Brownian motion starting at a point y. This is the survival probability until time t for a Wiener process starting at y, if the process is killed at the rate $Q(x)ds/2$ when passing through a point x, independently on its past.
We also have the following expression for the drift function of the optimal diffusion:

$$b(y,t) = \frac{V'_y}{V}(y,t).$$

Recall again that the pursuit strategy with the speed $b(X-W; T-t)$ is the optimal one among all adapted strategies.

Two basic examples

- For the Brownian dog problem we just have $V(y,t) = \mathbb{P}\left(|W_y(s)| \le 1, 0 \le s \le t\right)$ which, for large t, is nothing but small ball probability.
  The distortion $Y = X - W$ of the optimal pursuit coincides with the Brownian motion conditioned to survive under the killing rate Q, which, for specific potential, means the Brownian motion conditioned to stay in the strip $[-1,1]$.
- For quadratic potential $Q(y) = y^2$ we get $b(y,t) = -\mathtt{tanh}(t)\, y \sim -y$ (for large t) which corresponds to the Ornstein – Uhlenbeck process.

Infinite intervals

We search an adapted and absolutely continuous pursuit X minimizing asymptotic energy per unit of time

$$\lim_{T\to\infty} T^{-1}\, \mathbb{E} \int_0^T \Big[X'(t)^2 + Q(X(t) - W(t))\Big]\, dt.$$

Again, a natural candidate for being an optimal pursuit is a process X satisfying $X'(t) := b(X - W)$, where now the speed depends only on the distortion. This strategy is optimal provided that the drift function $b(\cdot)$ is chosen properly.
Optimization arguments and the variable change $b = V'/V$ lead to the eigenvalue problem for 1-dimensional Schrödinger equation

$$V''(y) - Q(y)\, V(y) = -\lambda\, V(y).$$

We conclude that the minimal asymptotic energy in the stationary regime is equal to the minimal eigenvalue of the respective Schrödinger equation, while the optimal speed function $b(y)$ is equal to the log-derivative of the corresponding eigenfunction.

Generalization

Brownian motion $\nearrow$ general process with stationary increments or a stationary process.

Kinetic energy $\nearrow$ general form of energy.

General potential Q $\searrow$ quadratic potential $Q(y) = \alpha y^2$.
This makes possible to consider the L_2 (or wide sense) setting.

Problem setting

Let $(B(t))_{t\in\Theta}$ with $\Theta = \mathbb{Z}$ or $\Theta = \mathbb{R}$ be a wide sense stationary process with discrete or continuous time.

We search for an approximation process X such that

- The pair (B, X) is jointly stationary.
- Smoothness (or finite energy). Process X is M-times differentiable, so that the energy

$$\mathcal{E}[X](t) := \Big| \sum_{m=0}^{M} \ell_m X^{(m)}(t) \Big|^2$$

  is well defined. Basic example: kinetic energy $\mathcal{E}[X](t) := \left|\alpha X'(t)\right|^2$.
- Optimality.

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N \left[|X(t) - B(t)|^2 + \mathcal{E}[X](t) \right] dt \searrow \min .$$

- (optional) Adaptivity: $X(t) \in \overline{\mathrm{span}}\{B(s), s \le t\}$.

Problem setting: continued

If, additionally, the process $X(t) - B(t)$ and the derivative $X'(t)$ are stationary processes in the strict sense, in many situations ergodic theorem applies and the limit

$$\lim_{N\to\infty} \frac{1}{N} \int_0^N \left[|X(t) - B(t)|^2 + \mathcal{E}[X](t) \right] dt$$

is equal to $\mathbb{E}|X(0) - B(0)|^2 + \mathbb{E}\mathcal{E}[X](0)$. Setting aside ergodicity issues, we may solve the problem

$$\mathbb{E}|X(0) - B(0)|^2 + \mathbb{E}\,\mathcal{E}[X](0) \to \min .$$

Spectral theory: reminder

Let $B(t), t \in \mathbb{R}$, be a complex-valued random process. It is called wide sense stationary, if $\mathbb{E}B(t) = const$ and $Cov(B(s), B(t)) := K(s-t)$ depends only on $s - t$.
In the sequel, we assume that $\mathbb{E}B(t) = 0$ and that covariance $K(\cdot)$ is continuous. (Equivalently, the mapping $t \mapsto B(t)$ is L_2-continuous on $\mathbb{R}$).
Then, by a Bochner's theorem, there exists a finite positive measure μ on $\mathbb{R}$ such that

$$K(t) = \int_{\mathbb{R}} e^{itu}\, \mu(du), \qquad t \in \mathbb{R}.$$

Thus, K and μ are essentially Fourier transforms of each other. The measure μ is called spectral measure of B. Many properties of B can be expressed in terms of μ.

The next step is to give a similar representation of the process B itself. For this aim, we need a notion of orthogonal random measure and respective integral.

Spectral theory: reminder (continued)

Let $(\mathcal{R}, \mathcal{A}, \mu)$ be a measure space. Let $\mathcal{A}_0 = \{A \in \mathcal{A} : \mu(A) < \infty\}$. A family of random variables $\{\mathcal{W}(A), A \in \mathcal{A}_0\}$ is called an orthogonal random measure of intensity μ if $\mathbb{E}\mathcal{W}(A) = 0$ for all A and $cov(\mathcal{W}(A), \mathcal{W}(B)) = \mu(A \cap B)$ for all A, B. In particular, the values of $\mathcal{W}$ on disjoint sets are uncorrelated (orthogonal).

It is easy to check that $\mathcal{W}$ is additive: if the sets A_j are disjoint and $\mu\left(\bigcup_j A_j\right) < \infty$, then

$$\mathcal{W}\Big(\bigcup_j A_j\Big) = \sum_j \mathcal{W}(A_j), \qquad \text{a.s.}$$

The next step is to define the integral w.r.t. $\mathcal{W}$. For step functions we set

$$\int_{\mathcal{R}} \Big[\sum_j c_j \mathbf{1}_{A_j}(u)\Big]\, \mathcal{W}(du) := \sum_j c_j \mathcal{W}(A_j)$$

and then extend the integral by continuity to $\int_{\mathcal{R}} f(u)\mathcal{W}(du)$ for all $f \in L_{2,\mathbb{C}}(\mathcal{R}, \mathcal{A}, \mu)$.

Spectral theory: reminder (continued)

The main feature of the integral is its isometric property

$$cov\left(\int_{\mathcal{R}} f\, d\mathcal{W}, \int_{\mathcal{R}} g\, d\mathcal{W}\right) = (f,g)_2 = \int_{\mathcal{R}} f\,\overline{g}\, d\mu.$$

Now we apply this general construction to stationary processes.

Let B be a (wide sense) stationary process with spectral measure μ.Then there exists a unique orthogonal measure $\mathcal{W}$ on $\mathbb{R}$ with intensity μ such that

$$B(t) = \int_{\mathbb{R}} e^{itu}\, \mathcal{W}(du), \qquad t \in \mathbb{R}.$$

Remark: if a process B is real, the corresponding measure $\mathcal{W}$ need not be real ! It must only satisfy the conjugation property $\mathcal{W}(-A) = \overline{\mathcal{W}(A)}$.

Spectral theory: reminder (continued)

Similarly, if a centered process B has (wide sense) stationary increments (e.g. Brownian motion, fractional Brownian motion etc), then it has a similar spectral representation

$$B(t) = \xi\, t + \int_{\mathbb{R}} \left(e^{itu} - 1 \right) \mathcal{W}(du), \qquad t \in \mathbb{R}.$$

where ξ is a square integrable centered random variable.

The only difference is that now the intensity measure μ for $\mathcal{W}$ need not be finite. It should satisfy less restrictive Lévy condition

$$\int_{\mathbb{R}} \min\{u^2, 1\}\, \mu(du) < \infty,$$

which allows accumulation of infinite measure at zero.

Spectral representation of the problem

Recall the spectral representation

$$B(t) = \int_{\mathbb{R}} e^{itu} \mathcal{W}(du)$$

where $\mathcal{W}$ is an orthogonal random measure with $\mathbb{E}|\mathcal{W}(A)|^2 = \mu(A)$, μ being the spectral measure of B.
We search approximation process in the form

$$X(t) = \int_{\mathbb{R}} g(u) e^{itu}\, \mathcal{W}(du)$$

(in adaptive case: $g \in \overline{\mathrm{span}}\{e^{isu}, s \leq 0 | L_2(\mu)\}$.

Spectral representation of the problem (continued)

Having an approximation $X(t) = \int g(u)e^{itu}\,\mathcal{W}(du)$, we first compute its energy. Differentiate m times and obtain

$$X^{(m)}(t) = \int_{\mathbb{R}} g(u)(iu)^m e^{itu}\,\mathcal{W}(du),$$

hence,

$$\begin{aligned}\mathcal{E}[X](t) &= \left|\int_{\mathbb{R}} g(u)\sum_{m=0}^{M}\ell_m(iu)^m e^{itu}\,\mathcal{W}(du)\right|^2 \\ &:= \left|\int_{\mathbb{R}} g(u)\ell(iu)e^{itu}\,\mathcal{W}(du)\right|^2\end{aligned}$$

with the energy polynomial $\ell(z) := \sum_{m=0}^{M}\ell_m z^m$. By isometric property,

$$\mathbb{E}\,\mathcal{E}[X](t) = \int_{\mathbb{R}} |g(u)|^2|\ell(iu)|^2\mu(du),$$

$$\mathbb{E}|B(t) - X(t)|^2 = \mathbb{E}\left|\int(1-g(u))e^{itu}\mathcal{W}(du)\right|^2 = \int|1-g(u)|^2\mu(du).$$

Two error terms

Therefore, the optimization problem takes a spectral form

$$\int_{\mathbb{R}} \left(|1 - g(u)|^2 + |g(u)|^2 |\ell(iu)|^2 \right) \, \mu(du) \searrow \min .$$

Using the identity

$$|1-g|^2 + |g|^2|\ell|^2 = \left| g - \frac{1}{|\ell|^2+1} \right|^2 \left(|\ell|^2 + 1 \right) + \frac{|\ell|^2}{|\ell|^2+1}, \qquad \forall g, \ell \in \mathbb{C},$$

we must minimize

$$\int_{\mathbb{R}} \left| g(u) - \frac{1}{|\ell(iu)|^2+1} \right|^2 \left(|\ell(iu)|^2 + 1 \right) \mu(du) + \int_{\mathbb{R}} \frac{|\ell(iu)|^2}{|\ell(iu)|^2+1} \, \mu(du).$$

The second term is not influenced by optimization of g. This is the error of non-adaptive prediction. The first term is the additional error of adaptive prediction. Its optimization is very similar to solving a prediction problem.

Optimal non-adaptive approximation

For non-adaptive approximation the unique solution of the problem is given by $g(u) = \frac{1}{|\ell(iu)|^2+1}$, thus

$$X(t) = \int_{\mathbb{R}} \frac{e^{itu}}{1+|\ell(u)|^2}\, \mathcal{W}(du)$$

and the corresponding minimum is equal to

$$\int_{\mathbb{R}} \frac{|\ell(iu)|^2}{1+|\ell(iu)|^2}\, \mu(du).$$

Interestingly, the form of the solution does not depend on the spectral measure of B.

Optimal non-adaptive approximation: kinetic energy

In continuous time case the kinetic energy problem

$$\mathbb{E}|X(0) - B(0)|^2 + \alpha^2 \mathbb{E}|X'(0)|^2 \searrow \min$$

corresponds to $\ell(z) = \alpha z$. Using that

$$\frac{1}{1 + |\ell(iu)|^2} = \frac{1}{1 + \alpha^2 u^2} = \frac{1}{2\alpha} \int_{\mathbb{R}} \exp\{-|\tau|/\alpha\} e^{i\tau u} d\tau,$$

We see that

$$\begin{aligned} X(t) &= \frac{1}{2\alpha} \int_{\mathbb{R}} e^{itu} \int_{\mathbb{R}} \exp\{-|\tau|/\alpha\} e^{i\tau u} d\tau \mathcal{W}(du) \\ &= \frac{1}{2\alpha} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(t+\tau)u} \mathcal{W}(du) \exp\{-|\tau|/\alpha\} d\tau \\ &= \frac{1}{2\alpha} \int_{\mathbb{R}} B(t+\tau) \exp\{-|\tau|/\alpha\} d\tau. \end{aligned}$$

is a double sided moving average. This is indeed non-adaptive!

Discrete time theory

When considering discrete time wide sense stationary sequences $B(n), n \in \mathbb{Z}$, we have the same spectral theory except that the spectral parameter runs over $[0, 2\pi)$ or over the unit cercle instead of the real line. That is, we have an isometric spectral representation

$$B(n) = \int_0^{2\pi} e^{inu}\, \mathcal{W}(du), \qquad n \in \mathbb{Z}.$$

The energy-efficience problem may not now use the derivatives. We must replace them with difference operators having the same sense. For example, the discrete time counterpart of kinetic energy minimization is $\mathbb{E}|X(0) - B(0)|^2 + \alpha^2 \mathbb{E}|X(1) - X(0)|^2 \searrow \min$. It is solved by the double-sided series

$$X(n) = \frac{1}{\sqrt{1+4\alpha^2}} \left(B(n) + \sum_{k=1}^{\infty} \beta^{-k} \left(B(n+k) + B(n-k) \right) \right)$$

with $\beta = 1 + \frac{1+\sqrt{1+4\alpha^2}}{2\alpha^2}$ (the golden section while $\alpha = 1$).

Adaptive approximation: problem setting

We are back to continuous time case. Let $B(t), t \in \mathbb{R}$, be a stationary process (a target). We search for an approximating process X such that

- The pair (B, X) is jointly stationary.
- Adaptivity.
$$X(t) \in \overline{\text{span}}\{B(s), s \leq t\}.$$
- Smoothness (or finite energy). Process X is M-times differentiable, so that the energy
$$\mathcal{E}[X](t) := \Big| \sum_{m=0}^{M} \ell_m X^{(m)}(t) \Big|^2$$
is well defined.
- Optimality.
$$\mathbb{E}|X(0) - B(0)|^2 + \mathbb{E}\,\mathcal{E}[X](0) \searrow \min .$$

Adaptive approximation: reduction to prediction problem

Recall that our problem is to minimize

$$\int_{\mathbb{R}} \left| g(u) - \frac{1}{|\ell(iu)|^2+1} \right|^2 \left(|\ell(iu)|^2 + 1 \right) \mu(du) + \int_{\mathbb{R}} \frac{|\ell(iu)|^2}{|\ell(iu)|^2+1} \mu(du)$$

over $g \in \text{past}$. The second term does not admit any optimization. Compare: our remaining problem

$$\int \left| g(u) - \frac{1}{|\ell(iu)|^2+1} \right|^2 \left(|\ell(iu)|^2 + 1 \right) \mu(du) \searrow \min$$

and the classical prediction problem

$$\int \left| g(u) - e^{itu} \right|^2 \mu(du) \searrow \min, \qquad t > 0.$$

Since the prediction problem solution is linear in its argument, we may use it for solving our problem.

Adaptive approximation: solution via prediction problem

To solve

$$\int_{\mathbb{R}} \left| g(u) - \frac{1}{|\ell(iu)|^2+1} \right|^2 \left(|\ell(iu)|^2 + 1 \right) \mu(du) \searrow \min$$

factorize

$$|\ell(iu)|^2 + 1 = \lambda_\ell(u)\, \overline{\lambda_\ell(u)} = |\lambda_\ell(u)|^2, \qquad u \in \mathbb{R}.$$

Then we have to minimize

$$\int_{\mathbb{R}} \left| \lambda_\ell(u)\, g(u) - \frac{1}{\overline{\lambda_\ell(u)}} \right|^2 \mu(du).$$

Assume that

$$\frac{1}{\overline{\lambda_\ell(u)}} = \int_0^\infty e^{i\tau u} \nu_\ell(d\tau), \qquad u \in \mathbb{R},$$

with some finite complex measure ν_ℓ depending on the energy polynomial $\ell(\cdot)$. Then we have a mix of classical prediction problems.

Adaptive approximation: solution via prediction problem (continued)

For solving $\int \left|\lambda_\ell(u)\, g(u) - \frac{1}{\overline{\lambda_\ell(u)}}\right|^2 \mu(du) \searrow \min$ over $g \in \text{past}$ under

$$\frac{1}{\overline{\lambda_\ell(u)}} = \int_0^\infty e^{i\tau u} \nu_\ell(d\tau), \qquad u \in \mathbb{R},$$

we let $\widehat{q}_*^{(\tau,\mu)}$ be the solution of the classical prediction problem:

$$\int \left|q(u) - e^{i\tau u}\right|^2 \mu(du) \searrow \min \qquad \tau \geq 0,$$

over $q \in \text{past}$. Then the function

$$\widehat{q}_*^{(\ell,\mu)}(u) := \int_0^\infty \widehat{q}_*^{(\tau,\mu)}(u) \nu_\ell(d\tau), \qquad u \in \mathbb{R},$$

solves the problem $\int \left|q(u) - \frac{1}{\overline{\lambda_\ell(u)}}\right|^2 \mu(du) \searrow \min$ over $q \in \text{past}$ and our problem is solved by letting $g_* := \widehat{q}_*^{(\ell,\mu)}/\lambda_\ell$. (why $g_* \in \text{past}$?)

Adaptive approximation: solution via prediction problem (concluding remarks)

- The good factorization $|\ell(iu)|^2 + 1 = \lambda_\ell(u)\,\overline{\lambda_\ell(u)}$, is possible for any energy polynomial $\ell(iu)$.
- For the presented method, we need to know the solutions of the classical prediction problems.Yet we may apply the methods of prediction theory directly to our problem and solve it in a straight way.

A good factorization for energy polynomials

Let $\ell(\cdot)$ be a polynomial of degree M with complex coefficients.Then we must factorize

$$1 + |\ell(iu)|^2 = 1 + \ell(iu)\overline{\ell(iu)} := \mathcal{P}(u),$$

where $\mathcal{P}$ is a polynomial of degree $2M$ with real coefficients and having no real roots. Thus we may write

$$\mathcal{P} = C \prod_{m=1}^{M} (u - \beta_m)(u - \overline{\beta_m}),$$

where $C > 0$ and $Im(\beta_m) > 0$. Then the good factorization is given by

$$1 + |\ell(iu)|^2 = 1 + \ell(iu)\overline{\ell(iu)} := \mathcal{P}(u) = \lambda_\ell(u)\,\overline{\lambda_\ell(u)},$$

$$\lambda_\ell(u) = C^{1/2} \prod_{m=1}^{M} (u - \beta_m).$$

A good factorization for energy polynomials (continued)

The factorization with $\lambda_\ell(u) = C^{1/2} \prod_{m=1}^{M} (u - \beta_m)$ is good because the desired representation

$$\frac{1}{\overline{\lambda_\ell(u)}} = \int_0^\infty e^{i\tau u}\, \nu_\ell(d\tau), \qquad u \in \mathbb{R},$$

or equivalently

$$\frac{1}{\lambda_\ell(u)} = \int_0^\infty e^{-i\tau u}\, \overline{\nu_\ell}(d\tau), \qquad u \in \mathbb{R},$$

essentially follows from the representations of every factor

$$\frac{1}{u - \beta_m} = i \int_0^\infty e^{-i\tau(u-\beta_m)} d\tau, \qquad u \in \mathbb{R}.$$

One example: Ornstein–Uhlenbeck process

The Ornstein – Uhlenbeck process $(B(t))_{t\in\mathbb{R}}$, is a centered (Gaussian) stationary process with covariance $K_B(t) = e^{-|t|/2}$ and the spectral measure $\mu(du) := \frac{2du}{\pi(4u^2+1)}$. We solve the problem related to kinetic energy

$$\mathbb{E}|X(0) - B(0)|^2 + \alpha^2\, \mathbb{E}|X'(0)|^2 \searrow \min .$$

The optimal non-adaptive approximation is given by

$$X(t) = \frac{1}{2\alpha} \int_{-\infty}^{\infty} B(t+\tau)\, e^{-|\tau|/\alpha}\, d\tau.$$

The error of non-adaptive approximation is $\frac{\alpha}{2+\alpha}$. The optimal adaptive approximation is given by

$$X(t) = \frac{2}{(2+\alpha)\alpha} \int_{-\infty}^{0} B(t+\tau)\, e^{-|\tau|/\alpha}\, d\tau.$$

The error of adaptive approximation is $\frac{\alpha}{2+\alpha} + \frac{2\alpha}{(2+\alpha)^2} = \frac{4\alpha+\alpha^2}{(2+\alpha)^2}$.

Adaptive approximation: discrete time theory

The theory is essentially the same. Instead of

$$\frac{1}{\overline{\lambda_\ell(u)}} = \int_0^\infty e^{i\tau u}\, \nu_\ell(d\tau), \qquad u \in \mathbb{R},$$

one needs

$$\frac{1}{\overline{\lambda_\ell(u)}} = \sum_{\tau=0}^{\infty} \nu_\ell(\tau) e^{i\tau u}, \qquad u \in [0, 2\pi).$$

For example, for discrete-time analog of kinetic energy $\ell(z) = \alpha(z-1)$ the factorization is

$$1 + |\ell(e^{iu})|^2 = \lambda_\ell(u)\, \overline{\lambda_\ell()}, \qquad u \in [0, 2\pi),$$

with $\lambda_\ell(u) := \frac{\alpha}{\sqrt{\beta}}\,(e^{-iu} - \beta)$ and $\beta = 1 + \frac{1+\sqrt{1+4\alpha^2}}{2\alpha^2}$, as before.In particular,

$$\frac{1}{\overline{\lambda_\ell(u)}} = \frac{-1}{\alpha\sqrt{\beta}} \sum_{\tau=0}^{\infty} \beta^{-\tau} e^{i\tau u}, \qquad u \in [0, 2\pi).$$

Application of the prediction technique

If the solution of prediction problem is not available, the previous approach does not work. Yet we can still apply the prediction technique. We only consider discrete time here.
Introduce some notation. Let Λ denote Lebesgue measure on $[0, 2\pi)$.
Let

$$L_{\le 0} := \overline{span}\left\{e^{i\tau u}, \tau \le 0, \tau \in \mathbb{Z}\right\} \subset L_2([0, 2\pi), \Lambda);$$
$$L_{>0} := \overline{span}\left\{e^{i\tau u}, \tau > 0, \tau \in \mathbb{Z}\right\} \subset L_2([0, 2\pi), \Lambda);$$

A function $\gamma \in L_2([0, 2\pi), \Lambda)$ is called an outer function if

$$\overline{span}\left\{\gamma(u)e^{i\tau u}, \tau \le 0, \tau \in \mathbb{Z}\right\} = L_{\le 0}.$$

We will assume that the spectral measure of our process has a density: $\mu(du) = f(u)du$.

Application of the prediction technique

The classical prediction technique suggests using factorization

$$f(u) = \gamma_f(u)\,\overline{\gamma_f(u)}, \qquad u \in [0, 2\pi),$$

where γ_f is an outer function. This factorization exists if Kolmogorov regularity condition

$$\int_0^{2\pi} |\log f(u)| du < \infty$$

is satisfied. We will use this factorization along with the former energy function factorization

$$1 + |\ell(e^{iu}|^2 + 1 = \lambda_\ell(u)\,\overline{\lambda_\ell(u)}, \qquad u \in [0, 2\pi)],$$

Thus we split the properties of the process and those of the energy form.

Application of the prediction technique: solution

Theorem

Let γ_f and λ_ℓ be the functions from two factorizations given above. Denote Q the orthogonal projection of $\gamma_f/\overline{\lambda_\ell}$ to $L_{>0}$ in $L_2([0,2\pi),\Lambda)$. Then the optimal adaptive approximation is given by

$$X(t) = \int_0^{2\pi} g_*(u)\mathcal{W}(du),$$

where

$$g_*(u) = \frac{1}{|\lambda_\ell(u)|^2} - \frac{Q}{\lambda_\ell\, \gamma_f}\,.$$

The additional adaptive approximation error is given by $||Q||^2_{2,\Lambda}$.

Application of the prediction technique: solution

For discrete kinetic energy $\ell(z) = \alpha(z-1)$ we may give an explicit representation of the additional adaptive approximation error via spectral density

$$\frac{2\pi}{\beta^2\sqrt{1+4\alpha^2}} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{\beta^2-1}{\beta^2+1-2\beta\cos u} \log f(u)\, du \right\}.$$

Here $\beta = (2\alpha^2+1+\sqrt{1+4\alpha^2}/(2\alpha^2)$.
This reminds very much the classical formula for one step prediction error

$$2\pi \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log f(u)\, du \right\}.$$

A paralel continuous time theory is also available.

An example: autoregressive sequence

Consider an autoregressive sequence given by equation $B(t) = \rho B(t-1) + \xi(t)$ where $|\rho| < 1$ and $\xi(t)$ are centered uncorrelated r.v. with $\mathbb{E}|\xi(t)|^2 := \sigma^2$. The spectral density is

$$f(u) = \frac{\sigma^2}{2\pi|1-\rho e^{-iu}|^2} = \gamma_f(u)\overline{\gamma_f(u)}$$

with $\gamma_f(u) = \frac{\sigma}{\sqrt{2\pi}}(1-\rho e^{-iu})^{-1}$. Then the optimal approximation is given by another autoregressive sequence,

$$\begin{aligned} g_*(u) &= \frac{1}{\alpha^2(\beta-\rho)} \sum_{j=0}^{\infty} \beta^{-j} e^{-iju}, \qquad \text{which means} \\ X(t) &= \frac{1}{\alpha^2(\beta-\rho)} \sum_{j=0}^{\infty} \beta^{-j} \xi(t-j), \end{aligned}$$

with additional adaptivity error $\frac{\sigma^2}{\sqrt{1+4\alpha^2(\beta-\rho)^2}}$.

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