A Kramers' type law for the collision time of two self-interacting diffusion processes and of their related particle approximation

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Problem: Establishing a Kramers' type for the small viscosity/zero-noise limit of the first collision time:

$$C(\sigma) = \inf\{t \geq 0 \,:\, X_t = Y_t\},$$

and related collision location $X_{C(\sigma)}$ between two independent self-stabilizing diffusion processes:

$$X_t = x + \sigma B_t - \int_0^t \Big\{\nabla V(X_s) + \alpha(X_s - \mathbb{E}[X_s])\Big\}\, ds, \quad t \geq 0,$$

$$Y_t = y + \sigma \tilde{B}_t - \int_0^t \Big\{\nabla V(Y_s) + \alpha(Y_s - \mathbb{E}[Y_s])\Big\}\, ds, \quad t \geq 0,$$

whenever $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ are two independent $\mathbb{R}^d$-Brownian motions and the potential V admits two possible attractors and each diffusion is driven towards one of these particular attractors.

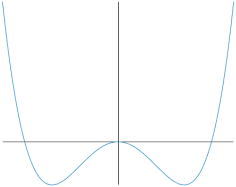
Analog problem for the related particle systems:

$$X_t^{i,N} = x + \sigma B_t^i - \int_0^t \Big\{\nabla V(X_s^{i,N}) + \alpha\big(X_s - \frac{1}{N}\sum_{j=1}^N X_s^{j,N}\big)\Big\}\, ds, \quad t \geq 0,$$

$$Y_t^{i,N} = y + \sigma \tilde{B}_t^i - \int_0^t \Big\{\nabla V(Y_s^{i,N}) + \alpha\big(Y_s^{i,N} - \frac{1}{N}\sum_{j=1}^N Y_s^{j,N}\big)\Big\}\, ds, \quad t \geq 0.$$

Prototypical case

Double wells landscape: $d = 1$, $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$.



At the zero-noise limit $\sigma = 0$, $X_t \to \phi_t(x)$ and $Y_t \to \phi_t(y)$ for

$$\phi_t(z) = z - \int_0^t \phi_s(z)^3 - \phi_s(z)\, ds,\ t \geq 0.$$

Assuming further $x < -1$ and $1 < y$, then collisions between the two self-stabilizing diffusions are due to the (join) effort of the Brownian motions.
As the noise vanishes the collision time is expected to follow a Kramers' type law of the form:

$$\frac{\sigma^2}{2} \log(C(\sigma)) \xrightarrow{\mathbb{P}} H$$

and the collision location $X_{C(\sigma)}$ should persists in some region in the space or be ∞.

Assumptions

(**A**) $-$ (*i*) V admits two distinct minima located in λ_1 and λ_2.

(**A**) $-$ (*ii*) The starting points x and y lie in a respective basin of attraction $\mathcal{G}(\lambda_1)$ and $\mathcal{G}(\lambda_2)$ of the wells of V and, for $\phi_t(z) = z - \int_0^t \nabla V(\phi_s(z))\,ds$, the gradient flow related to V, it holds

$$2\epsilon_0 := \inf_{t \geq 0} \left(|\phi_t(x) - \phi_t(y)|\right) > 0.$$

(**A**) $-$ (*iii*) $V : \mathbb{R}^d \to \mathbb{R}$ is of class $\mathcal{C}^2$, locally Lipschitz, convex at infinity (namely $\inf_{|x| \geq R'} \nabla^2 V(x)$ is positive definite for some $R' > 0$) and is such that ∇V grows at a $2n$-polynomial rate on $\mathbb{R}^d$:

$$\sup_{x \in \mathbb{R}^d} \left\{(1 + |x|^{2n})^{-1} |\nabla V(x)|\right\} < \infty\,.$$

(**A**) $-$ (*iv*) Synchronization condition: $\alpha I_d + \nabla^2 V$ is definite positive on $\mathbb{R}^d$.

Freidlin-Wentzell's exit time problem [FW98]

Perturbed dynamical systems:

$$z_t^\sigma = z_0 + \sigma \mathcal{W}_t - \int_0^t \nabla U\left(z_s^\sigma\right) ds\,, t \geq 0.$$

As $\sigma \to 0$: $z_t^\sigma \to \Psi_t(z_0)$ solution to

$$\Psi_t(z_0) = z_0 - \int_0^t \nabla U\left(\Psi_s(z_0)\right) ds\,, t \geq 0.$$

Large deviation principle: For any finite arbitrary time horizon T, and for any $\delta > 0$,

$$\lim_{\sigma \to 0} \frac{\sigma^2}{2} \log \mathbb{P}\left\{\sup_{t \in [0;T]} |z_t^\sigma - \Psi_t(z_0)| > \delta\right\}$$
$$= -\inf_{\Phi}\left\{I_T(\Phi)\,:\,\Phi \in \mathcal{C}^1([0,T];\mathbb{R}^d),\,\Phi(0) = x_0 \text{ and } \max_{0 \leq t \leq T} |\Phi(t) - \Psi_t(x_0)| > \delta\right\},$$

the action functional I_T being given by

$$I_T(\Phi) := \int_0^T \left|\frac{d\Phi}{dt} + \nabla U(\Phi(t))\right|^2 dt.$$

Definition

Let $\mathcal{G}$ be a subset of $\mathbb{R}^d$ and let ∇U be a Lipschitz vector field on $\mathbb{R}^d$. We say that the domain $\mathcal{G}$ is stable by ∇U if the orbit $\{\Psi_t(z_0)\,;\,t \in \mathbb{R}_+\}$ is included in $\mathcal{G}$ for all $z_0 \in \mathcal{G}$.

Freidlin-Wentzell's exit time problem [FW98]

Exit-time estimate:

Theorem

Let $\mathcal{G}$ be an open bounded set of $\mathbb{R}^d$, stable by ∇U, where U of class $\mathcal{C}^2$ and convex. Assume also that for all $x_0 \in \partial\mathcal{G}$, $\Psi_t(x_0)$ converges to a unique point a_0 at large time. Then, for any z_0 in $\mathcal{G}$, and

$$\tau_{\mathcal{G}}(\sigma) = \inf\{t \geq 0 \,:\, z_t^\sigma \notin \mathcal{G}\}\,,$$

we have:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H} - \delta\right)\right] < \tau_{\mathcal{G}}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H} + \delta\right)\right]\right\} = 1\,. \tag{1}$$

The value $\underline{H}$ representing the main cost of exiting $\mathcal{G}$ is given by

$$\underline{H} = \inf_{z \in \partial\mathcal{G}} \inf_{T > 0} \inf_{\Phi} \left\{I_{T,z}(\Phi)\right\} = \inf_{x \in \partial\mathcal{G}} \Big(U(x) - U(a_0)\Big)$$

Additionally:

(i) For all z_0 in $\mathcal{G}$, $\lim_{\sigma\to 0} \frac{\sigma^2}{2} \log\Big(\mathbb{E}\{\tau_{\mathcal{G}}(\sigma)\}\Big) = \underline{H}$;

(ii) Exit location: If $\inf_{x\in\partial\mathcal{G}}\Big(U(x) - U(a_0)\Big)$ is achieved in a unique point $z_\star$ in $\partial\mathcal{G}$, then, for all $\delta > 0$, $z_0 \in \mathcal{G}$,

$$\lim_{\sigma\to 0} \mathbb{P}\left\{|z^\sigma_{\tau_{\mathcal{G}}(\sigma)} - z_\star| < \delta\right\} = 1.$$

A very short over-view on self-stabilizing diffusion processes

• Benachour, Roynette, Talay and Vallois [BRTV98]: Existence of an invariant distribution for the one dimensional model:

$$X_t = X_0 + B_t - \frac{1}{2}\int_0^t \int \beta(X_s - y)\mu_s(dy)\, ds,\ \mu_t = \mathrm{Law}(X_t),\ t \geq 0$$

with $\beta : \mathbb{R} \to \mathbb{R}$ odd, increasing and satisfying

$$|\beta(x) - \beta(y)| \leq C|x-y|\big(1 + |x|^r + |y|^r\big),\ \text{for some}\ r \in \mathbb{N} \setminus \{0\},$$

$$\big(\beta(x) - \beta(y)\big) \geq \alpha_1(x-y) + \alpha_2,\ \alpha > 0,\ \text{for all}\ x \geq y.$$

• Hermann, Imkeller and Peithmann [HIP08]: Long time behaviour of

$$X_t = y + \sigma B_t - \int_0^t \Big\{ U(X_s) + \int \Phi(X_s - y)\mu_s(dy)\Big\}\, ds,\ \mu_t = \mathrm{Law}(X_t), \qquad t \geq 0,$$

with U and Φ relatively smooth and (globally) convex functions.The authors further established a Kramers' type law in the case $V = \nabla U$ and $\Phi = \nabla b$, the exit cost being given by

$$\underline{H} = \inf_{z\in\partial G} \inf_{T,\Psi} \frac{1}{2}\int_0^T |\Psi_t(z) - V(\Psi_t(z)) - \Phi(x - z^\star)|^2\, ds = \inf_{z\in\partial G}\big(U(z) + b(z - z_\star) - U(z_\star)\big).$$

• Tugaut 2007-2021: Kramers' type law in the case of a double wells landscape and other globally non-convex situation (e.g. [T21] for the case of the granular media equation).

Strategy: Given the multi-dimensional self-stabilizing diffusions:

$$X_t = x + \sigma B_t - \int_0^t \left\{ \nabla V(X_s) + \alpha(X_s - \mathbb{E}[X_s]) \right\} ds, \quad t \geq 0,$$

$$Y_t = y + \sigma \tilde{B}_t - \int_0^t \left\{ \nabla V(Y_s) + \alpha(Y_s - \mathbb{E}[Y_s]) \right\} ds, \quad t \geq 0,$$

introduce the ϵ-approximation:

$$C_\epsilon(\sigma) = \inf \{ t \geq 0 \,:\, |X_t - Y_t| \leq 2\epsilon \} \,, \epsilon < \epsilon_0 \,,$$

reformulate this time in terms of an exit time

$$\begin{aligned} C_\epsilon(\sigma) &= \inf_{\lambda \in \mathbb{R}^d} \beta_{\lambda,\epsilon}(\sigma), \\ \beta_{\lambda,\epsilon}(\sigma) &:= \inf \{ t \geq 0 \,:\, X_t \in B(\lambda, \epsilon),\ Y_t \in B(\lambda, \epsilon) \} \\ &= \inf \left\{ t \geq 0 \,:\, X_t \notin (\mathbb{R}^d \setminus B(\lambda, \epsilon)),\ Y_t \notin (\mathbb{R}^d \setminus B(\lambda, \epsilon)) \right\} , \end{aligned}$$

and apply a particular coupling between (X_t, Y_t) and linearized version:

$$x_t^\sigma = x + \sigma B_t - \int_0^t \left\{ \nabla V(x_s^\sigma) + \alpha(x_s^\sigma - \lambda_1) \right\} ds, \quad t \geq 0,$$

$$y_t^\sigma = y + \sigma \tilde{B}_t - \int_0^t \left\{ \nabla V(y_s^\sigma) + \alpha(y_s^\sigma - \lambda_2) \right\} ds, \quad t \geq 0.$$

Preliminaries: First collision time of two stochastic gradient flows

Models:

$$x_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla \Psi_1 \left(x_s^\sigma \right) ds \, , t \geq 0,$$

and

$$y_t^\sigma = y_0 + \sigma \widetilde{B}_t - \int_0^t \nabla \Psi_2 \left(y_s^\sigma \right) ds \, , t \geq 0.$$

Assumptions:

- Ψ_1 and Ψ_2 are of class $\mathcal{C}^2$, strictly convex and admits λ_1 and λ_2 as their respective minimizers.
- $\inf_{t \geq 0} |\phi_t^1(x) - \phi_t^2(y)| > 2\epsilon_0$ for $\phi_t^i(z) = z - \int_0^t \nabla \Psi_i(\phi_s^i(z)) \, ds, \ t \geq 0, \ i = 1, 2.$

ϵ **approximation of the first collision time**:

$$c_\epsilon(\sigma) := \inf \left\{ t \geq 0 \ : \ |x_t^\sigma - y_t^\sigma| \leq 2\epsilon \right\} , \, \epsilon > 0.$$

Strategy: For $\epsilon < \epsilon_0$,

$$c_\epsilon(\sigma) = \inf_\lambda \beta_{\lambda,\epsilon}(\sigma) \, ,$$

$$\begin{aligned} \beta_{\lambda,\rho}(\sigma) &= \inf\{t \geq 0 \, : \, (x_t^\sigma, y_t^\sigma) \in B(\lambda, \rho) \times B(\lambda, \rho)\} \\ &= \inf\{t \geq 0 \, : \, (x_t^\sigma, y_t^\sigma) \notin (\mathbb{R}^d \setminus B(\lambda, \rho)) \times (\mathbb{R}^d \setminus B(\lambda, \rho))\} \end{aligned}$$

and approximate $\beta_{\lambda,\rho}(\sigma)$ by $\widehat{\beta}_{\lambda,\rho}(\sigma)$ where $\widehat{\beta}_{\lambda,\rho}(\sigma)$ is the first entering time of $\mathcal{D}^1(\lambda, \rho) \times \mathcal{D}^2(\lambda, \rho)$ for

$$\mathcal{D}^i(\lambda, \rho) := \{\phi_t^{i,+}(z) \, : \, t \geq 0, \, z \in B(\lambda, \rho)\}, \qquad \phi_t^{i,+}(z) = z + \int_0^t \nabla \Psi_i(\phi_s^{i,+}(z)) \, ds, \ t \geq 0 \, .$$

Note:

- Whenever λ is too close to one of the two infimum, say $|\lambda - \lambda_1| = \tilde{\epsilon}$ with $\tilde{\epsilon} < \rho$, then, $B(\lambda,\rho)$ is an attractive set for $\phi^{1,+}$ and so $\mathcal{D}^1_{\lambda,\rho} = \mathbb{R}^d$. If $\tilde{\epsilon}$ is (strictly) smaller than ϵ_0, this reduces $\widehat{\beta}_{\lambda,\rho}(\sigma)$ to the first exit-time from $\mathcal{D}^2_{\lambda,\rho}$.
- In the opposite case, $\min_{i=1,2}(|\lambda - \lambda_i|) > \rho$, $\left(\mathbb{R}^d \setminus \mathcal{D}^1_{\lambda,\rho}\right) \times \left(\mathbb{R}^d \setminus \mathcal{D}^2_{\lambda,\rho}\right)$ is stable by $(-\nabla\Psi^1, -\nabla\Psi^2)$.
- The cases $|\lambda - \lambda_i| = \rho$ are singular and require a rescaling.
- Outside these "degenerated" cases, the exit-costs of $\mathbb{R}^d \setminus \mathcal{D}^i_{\lambda,\rho}$ and of $\mathbb{R}^d \setminus B(\lambda,\rho)$ are the same:

$$\inf_{x \in \partial B(\lambda,\rho)} \Big(\Psi_i(x) - \Psi_i(\lambda_i)\Big) = \inf_{x \in \partial \mathcal{D}^i_{\lambda,\rho}} \Big(\Psi_i(x) - \Psi_i(\lambda_i)\Big),\ i = 1, 2.$$

Last approximation: For $\epsilon < \epsilon_0$, $0 < \rho < 1$,

$$\widehat{\beta}^{\rho}_{\lambda,\epsilon}(\sigma) := \inf\left\{t \geq 0 \,:\, (x^\sigma_t, y^\sigma_t) \in \mathcal{O}_{\lambda,\epsilon,\rho}\right\} \tag{2}$$

where the domain $\mathcal{O}_{\lambda,\epsilon,\rho}$ is given by

$$\mathcal{O}_{\lambda,\epsilon,\rho} := \begin{cases} \partial\mathcal{D}^1(\lambda,\rho\epsilon) \times \partial\mathcal{D}^2(\lambda,\epsilon) \text{ if } |\lambda - \lambda_1| = \epsilon, \\ \partial\mathcal{D}^1(\lambda,\epsilon) \times \partial\mathcal{D}^2(\lambda,\rho\epsilon) \text{ if } |\lambda - \lambda_2| = \epsilon, \\ \partial\mathcal{D}^1(\lambda,\epsilon) \times \partial\mathcal{D}^2(\lambda,\epsilon) \text{ otherwise.} \end{cases}$$

Applying classical Kramers' type law:

Lemma

For any λ in $\mathbb{R}^d$ and for any $\delta > 0$,

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H_\epsilon^\rho(\lambda)-\delta\right)\right] < \widehat{\beta}^\rho_{\lambda,\epsilon}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(H_\epsilon^\rho(\lambda)+\delta\right)\right]\right\} = 1\,,$$

for

$$H_\epsilon^\rho(\lambda) = \begin{cases} \inf_{x\in\partial B(\lambda;\rho\epsilon)} (\Psi_1(x)-\Psi_1(\lambda_1)) + \inf_{y\in\partial B(\lambda;\epsilon)} (\Psi_2(y)-\Psi_2(\lambda_2)) \; if |\lambda-\lambda_1| = \epsilon, \\ \inf_{x\in\partial B(\lambda;\epsilon)} (\Psi_1(x)-\Psi_1(\lambda_1)) + \inf_{y\in\partial B(\lambda;\rho\epsilon)} (\Psi_2(y)-\Psi_2(\lambda_2)) \; if |\lambda-\lambda_2| = \epsilon, \\ \inf_{x\in B(\lambda;\epsilon)} (\Psi_1(x)-\Psi_1(\lambda_1)) + \inf_{y\in B(\lambda;\epsilon)} (\Psi_2(y)-\Psi_2(\lambda_2)) \; otherwise. \end{cases}$$

Moreover, we have: for any $\delta > 0$,

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\min\left(dist\left(x^\sigma_{\widehat{\beta}^\rho_{\lambda,\epsilon}(\sigma)}, B(\lambda,\rho)\right), dist\left(y^\sigma_{\widehat{\beta}^\rho_{\lambda,\epsilon}(\sigma)}, B(\lambda;\rho)\right)\right) \leq \delta\right\} = 1\,.$$

for $dist(x, B(\lambda;\rho)) := \inf_{z\in B(\lambda;\rho)} |x-z|$.

Lemma

The same Kramers' type law holds for

$$\beta^{\rho}_{\lambda,\epsilon}(\sigma) = \begin{cases} \inf\{t \geq 0 \,:\, (x^{\sigma}_t, y^{\sigma}_t) \in \partial B(\lambda, \rho\epsilon) \times \partial B(\lambda, \epsilon)\} & \mathit{if} |\lambda - \lambda_1| = \epsilon, \\ \inf\{t \geq 0 \,:\, (x^{\sigma}_t, y^{\sigma}_t) \in \partial B(\lambda, \epsilon) \times \partial B(\lambda, \rho\epsilon)\} & \mathit{if} |\lambda - \lambda_2| = \epsilon, \\ \inf\{t \geq 0 \,:\, (x^{\sigma}_t, y^{\sigma}_t) \in \partial B(\lambda, \epsilon) \times \partial B(\lambda, \epsilon)\} & \mathit{otherwise}. \end{cases}$$

Asymptotic $\rho \to 1$:

Lemma

For any $\lambda \in \mathbb{R}^d$, *and for any* $\delta > 0$*:*

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \exp\left[\frac{2}{\sigma^2}\left(H_\epsilon(\lambda) - \delta\right)\right] < \beta_{\lambda,\epsilon}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(H_\epsilon(\lambda) + \delta\right)\right]\right\} = 1\,.$$

for

$$H_\epsilon(\lambda) = \inf_{x \in \partial B(\lambda,\epsilon)} (\Psi_1(x) - \Psi_1(\lambda_1)) + \inf_{y \in \partial B(\lambda,\epsilon)} (\Psi_2(y) - \Psi_2(\lambda_2))\,.$$

Moreover,

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \min\left(\mathit{dist}\left(x^{\sigma}_{\beta_{\lambda,\epsilon}(\sigma)}, B(\lambda,\rho)\right), \mathit{dist}\left(y^{\sigma}_{\beta^{\rho}_{\lambda,\epsilon}(\sigma)}, B(\lambda;\rho)\right)\right) \leq \delta \right\} = 1\,.$$

Kramers' type laws for $c_\epsilon(\sigma) = \inf_\lambda \beta_{\lambda,\epsilon}(\sigma)$

Theorem

For any $\delta > 0$:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \exp\left[\frac{2}{\sigma^2} \left(\underline{H}_\epsilon - \delta \right) \right] < c_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2} \left(\underline{H}_\epsilon + \delta \right) \right] \right\} = 1 \,,$$

where

$$\underline{H}_\epsilon = \inf_{\lambda \in \mathbb{R}^d} H_\epsilon(\lambda).$$

In addition, for $\mathcal{H}_\epsilon$ the set of all minimizers λ_ϵ of $\lambda \mapsto H_\epsilon(\lambda)$, and for ϵ small enough:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \inf_{\lambda_\epsilon \in \mathcal{H}_\epsilon} \max\left(|x^\sigma_{c_\epsilon(\sigma)} - \lambda_\epsilon|, |y^\sigma_{c_\epsilon(\sigma)} - \lambda_\epsilon| \right) \leq \delta \right\} = 1 \,.$$

Note: The exit-cost $\underline{H_\epsilon}$ can be achieved in more than one points. Nevertheless,

$$\forall \lambda \in \mathbb{R}^d, \lim_{\epsilon \to 0} H_\epsilon(\lambda) = H_0(\lambda) := \left(\Psi_1(\lambda) - \Psi_1(\lambda_1) \right) + \left(\Psi_2(\lambda) - \Psi_2(\lambda_2) \right)$$

and

$$\lim_{\sigma \to 0} \underline{H}_\epsilon = \underline{H}_0.$$

Corollary

For λ_0 *the minimizer of* $\lambda \mapsto H_0(\lambda) = \big(\Psi_1(\lambda) - \Psi_1(\lambda_1)\big) + \big(\Psi_2(\lambda) - \Psi_2(\lambda_2)\big)$, *and* $\underline{H_0} := H_0(\lambda_0)$, *we have: for any* $\delta > 0$*:*

$$\lim_{\epsilon \to 0} \lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H}_0 - \delta\right)\right] < c_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H}_0 + \delta\right)\right]\right\} = 1$$

and

$$\lim_{\epsilon \to 0} \lim_{\sigma \to 0} \mathbb{P}\left\{\max\left(|x^\sigma_{c_\epsilon(\sigma)} - \lambda_0|, |y^\sigma_{c_\epsilon(\sigma)} - \lambda_0|\right) \le \delta\right\} = 1\,.$$

Under the assumptions $(A)-(i)$ to $(A)-(iv)$, [HIP08]: the dynamics $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are wellposed (in the pathwise sense) and

$$\sup_{t\geq 0} \mathbb{E}[|X_t|^p + |Y_t|^p] < \infty, \ \forall p \in \mathbb{Z}.$$

Moreover [T21]: Given $\kappa > 0$, there exists a finite (non-random) time T_κ and a critical thereshold σ_κ such that

$$\max_{\sigma \leq \sigma_\kappa, t \geq T_\kappa} \mathbb{E}[|X_t - \lambda_1|^2] + \mathbb{E}[|Y_t - \lambda_2|^2] \leq \kappa.$$

Corollary (Coupling estimate)

For

$$\begin{aligned} x_t^\sigma &= x_1 + \sigma B_t - \int_0^t \{\nabla V(x_s^\sigma) + \nabla F(x_s^\sigma - \lambda_1)\}\, ds, \\ y_t^\sigma &= x_2 + \sigma \tilde{B}_t - \int_0^t \{\nabla V(y_s^\sigma) + \nabla F(y_s^\sigma - \lambda_2)\}\, ds, \end{aligned}$$

and any $\kappa > 0$, there exists $T_\kappa, \sigma_\kappa > 0$ such that

$$\lim_{\sigma \to 0} \mathbb{P}\left\{ \max_{t \geq T_\kappa} |X_t - x_t^\sigma|^2 + |Y_t - y_t^\sigma|^2 \geq \kappa \right\} = 0.$$

From the Kramers' type law for $(x_t^\sigma, y_t^\sigma)_{t\geq 0}$ to the Kramers' law for $(X_t, Y_t)_{t\geq 0}$:

Proposition

Define

$$\widehat{\tau}_{\lambda,\epsilon}(\sigma) := \inf\left\{t \geq 0 \ : \ (X_t, Y_t) \in \mathbb{B}(\lambda;\epsilon)^2\right\},$$

the first time the nonlinear diffusion $(X_t, Y_t)_{t\geq 0}$ enters in the domain $B(\lambda;\epsilon)\times B(\lambda;\epsilon)$. Then, for any $\lambda \in \mathbb{R}^d$, for any $\delta > 0$, we have

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H_\epsilon(\lambda) - \delta\right)\right] < \widehat{\tau}_{\lambda,\epsilon}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(H_\epsilon(\lambda) + \delta\right)\right]\right\} = 1, \tag{3}$$

where

$$H_\epsilon(\lambda) = \inf_{x\in\partial B(\lambda,\epsilon)}\left(V(x) + F(x-\lambda_1) - V(\lambda_1)\right) + \inf_{y\in\partial B(\lambda,\epsilon)}\left(V(y) + F(y-\lambda_2) - V(\lambda_2)\right).$$

Theorem

Given $\epsilon > 0$, let $\lambda(\epsilon)$ be an arbitrary minimizer of H_ϵ. Then, for any $\delta > 0$, we have

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H_\epsilon(\lambda(\epsilon)) - \delta\right)\right] < C_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(H_\epsilon(\lambda(\epsilon)) + \delta\right)\right]\right\} = 1.$$

In addition, for $\mathcal{H}_\epsilon$ the set of all minimizers λ_ϵ of $\lambda \mapsto H_\epsilon(\lambda)$, and for ϵ small enough:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\inf_{\lambda_\epsilon\in\mathcal{H}_\epsilon} \max\left(|X_{C_\epsilon(\sigma)} - \lambda_\epsilon|, |Y_{C_\epsilon(\sigma)} - \lambda_\epsilon|\right) \leq \delta\right\} = 1.$$

Corollary

For any $\delta > 0$*, we have*

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0}-\delta\right)\right] < C_\epsilon(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0}+\delta\right)\right]\right\} = 1\,.$$

where

$$\underline{H_0} = \min H_0(\lambda),\ H_0(\lambda) = \lim_{\epsilon\to 0} H_\epsilon(\lambda) = \big(V(\lambda) + F(\lambda-\lambda_1) - V(\lambda_1)\big)$$

Moreover, for

$$\lambda_0 = argmin_\lambda H_0(\lambda) = \Big(\nabla V + \alpha I_d\Big)^{-1}(\alpha(\lambda_1+\lambda_2)/2)\,,$$

it holds

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{|X_{C_\epsilon(\sigma)}-\lambda_0|\leq\delta\right\} = 1 = \lim_{\epsilon\to 0}\lim_{\sigma\to 0}\mathbb{P}\left\{|Y_{C_\epsilon(\sigma)}-\lambda_0|\leq\delta\right\}\,.$$

Example: In the prototypical case: $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$,

$$\mathrm{argmin}\underline{H}_0 = \alpha + \frac{1}{2},\quad \mathrm{argmin} H_\epsilon \in \left\{\alpha + \frac{1}{2}, 4\epsilon^4 - 2\epsilon^2 + \alpha(2\epsilon\pm 1)^2 + \frac{1}{2}\right\}\,.$$

Collision time for the particle systems

Zero-noise limit of

$$C_\epsilon(\sigma)^N = \inf_{1\le i\le N} \inf\left\{t \ge 0 \,:\, |X_t^{i,N} - Y_t^{i,N}| \le \epsilon\right\}$$

where

$$X_t^{i,N} = x + \sigma B_t^i - \int_0^t \left\{\nabla V(X_s^{i,N}) + \alpha\big(X_s^{i,N} - \frac{1}{N}\sum_{j=1}^N X_s^{j,N}\big)\right\} ds, \quad t \ge 0,$$

$$Y_t^{i,N} = y + \sigma \tilde{B}_t^i - \int_0^t \left\{\nabla V(Y_s^{i,N}) + \alpha\big(Y_s^{i,N} - \frac{1}{N}\sum_{j=1}^N Y_s^{j,N}\big)\right\} ds, \quad t \ge 0,$$

for $(B_t^1)_{t\ge 0}, \cdots, (B_t^N)_{t\ge 0}$, and $(\tilde{B}_t^1)_{t\ge 0}, \cdots, (\tilde{B}_t^N)_{t\ge 0}$, two families of independent Brownian motions.

Anticipated exit cost: The potential related to each family of particles

$$\Upsilon_N(\mathbf{x}^N) = \sum_{i=1}^N V(x_i) + \frac{\alpha}{2N}\sum_{i,j=1}^N |x_i - x_j|^2, \, \mathbf{x}^N = (x_1, \cdots, x_N) \in \mathbb{R}^{Nd},$$

and the exit-cost is given by

$$\begin{aligned} &\lim_{\epsilon\to 0} \inf_{\lambda\in\mathbb{R}^{dN}} \inf_{\mathbf{x}^N \in \partial B^N(\lambda,\epsilon)} \Upsilon_N(\mathbf{x}^N) - \Upsilon_N(\lambda_1, \cdots, \lambda_1) \\ &+ \lim_{\epsilon\to 0} \inf_{\lambda\in\mathbb{R}^{dN}} \inf_{\mathbf{y}^N \in \partial B^N(\lambda,\epsilon)} \Upsilon_N(\mathbf{y}^N) - \Upsilon_N(\lambda_2, \cdots, \lambda_2), \end{aligned}$$

for

$$B^N(\lambda, \epsilon) = \left\{\mathbf{x}^N = (x_1, \cdot, x_N) \in \mathbb{R}^{dN}\,;\, \forall i,\, x_i \notin \mathbb{B}(\lambda, \epsilon)\right\}.$$

Preliminary note

• As long as (A) hold the particle systems and their "linear" analog:

$$x_t^{i,\sigma} = x_1 + \sigma B_t^i - \int_0^t \nabla V(x_s^{i,\sigma})ds - \int_0^t \nabla F(x_s^{i,\sigma} - \lambda_1)ds\,,$$

$$y_t^{i,\sigma} = x_2 + \sigma \widetilde{B}_t^i - \int_0^t \nabla V(y_s^{i,\sigma})ds - \int_0^t \nabla F(y_s^{i,\sigma} - \lambda_2)ds\,.$$

are well-posed in the pathwise sense. Moreover, for all $1 \leq i \leq N$, T finite and $1 \leq p < \infty$

$$\max_{t\in[0,T]} \mathbb{E}[|X_t^{i,N}|^p + |Y_t^{i,N}|^p] < \infty.$$

• Propagation of chaos: For $(X_t^1)_{t\geq 0}, \cdots, (X_t^N)_{t\geq 0}$ and $(Y_t^1)_{t\geq 0}, \cdots, (Y_t^N)_{t\geq 0}$, N-copies of $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ driven respectively by $(B_t^1)_{t\geq 0}, \cdots, (B_t^N)_{t\geq 0}$ and $(\tilde{B}_t^1)_{t\geq 0}, \cdots, (\tilde{B}_t^N)_{t\geq 0}$,

$$\mathbb{E}[\max_{0\leq t\leq T} |X_t^{i,N} - X_t^i|^p + \max_{0\leq t\leq T} |Y_t^{i,N} - Y_t^i|^2] \leq \frac{C(\sigma,T)}{N}.$$

Non-uniform propagation of chaos $\Rightarrow$ We cannot rely on the Kramers' law established in the mean-field limit situation to deal with the particle case.

$\Rightarrow$ **Start over and apply a strategy analog to the mean-field case**. Namely: Establish the Kramers' type law from a coupling between the particle systems and their linear analogs.

Lemma

For any $\kappa > 0$ and for all $N > 0$ large enough, there exists a finite (deterministic) time $0 \leq T_\kappa$, uniform with respect to σ, such that $\overline{X}_t^N = \frac{1}{N}\sum_{j=1}^N X_t^{j,N}$ and $\overline{Y}_t^N = \frac{1}{N}\sum_{j=1}^N Y_t^{j,N}$, it holds

$$\lim_{\sigma\to 0} \mathbb{P}\left\{ \max_{t\in\left[T_\kappa, \exp[\frac{2}{\sigma^2}(H_0(\lambda_0)+2)]\right]} \left(||\overline{X}_t^N - \lambda_1|| + ||\overline{Y}_t^N - \lambda_2|| \right) \leq 2\kappa \right\} = 1\,.$$

Proposition

For any $\xi > 0$, there exists a finite time T_κ such that

$$\lim_{\sigma\to 0} \mathbb{P}\left(\sup_{T_\kappa \leq t \leq \exp[\frac{2}{\sigma^2}(H_0(\lambda_0)+2)]} \left\{ ||X_t^{i,N} - x_t^{i,\sigma}|| + ||Y_t^{i,N} - y_t^{i,\sigma}|| \right\} \geq \xi \right) = 0\,,$$

and

$$\lim_{\sigma\to 0} \mathbb{P}\left(\sup_{T_\kappa \leq t \leq \exp[\frac{2}{\sigma^2}(H_0(\lambda_0)+2)]} ||Y_t^{i,N} - y_t^{i,\sigma}|| \geq \xi \right) = 0\,,$$

provided that κ and σ are small enough while N is large enough.

Theorem

Let $\lambda_0(\epsilon)$ be a minimizer of H_ϵ. Then, for any $\delta > 0$ and for N large enough:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_\epsilon}-\delta\right)\right] < \mathcal{C}_{\epsilon,N}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_\epsilon}+\delta\right)\right]\right\} = 1\,.$$

Moreover, the collision persists near the minimizers of H_ϵ in the sense: for $\mathcal{H}_\epsilon$ the set of all minimizers λ_ϵ of $\lambda \mapsto H_\epsilon(\lambda)$, and for ϵ small enough:

$$\lim_{\sigma\to 0} \mathbb{P}\left\{\inf_{\lambda_\epsilon\in\mathcal{H}_\epsilon} \max\left(|X^{i,N}_{C^N_\epsilon(\sigma)}-\lambda_\epsilon|, |Y^{i,N}_{C^N_\epsilon(\sigma)}-\lambda_\epsilon|\right) \leq \delta\right\} = 1\,.$$

Corollary

For any $\delta > 0$, we have, for N large enough:

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0}-\delta\right)\right] < \mathcal{C}_{\epsilon,N}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0}+\delta\right)\right]\right\} = 1\,.$$

Moreover, for any $1 \leq i \leq N$,

$$\lim_{\epsilon\to 0}\lim_{\sigma\to 0} \mathbb{P}\left\{|X^{i,N}_{\mathcal{C}_{\epsilon,N}(\sigma)}-\lambda_0| \leq \delta\right\} = 1 = \lim_{\epsilon\to 0}\lim_{\sigma\to 0} \mathbb{P}\left\{|Y^{i,N}_{\mathcal{C}_{\epsilon,N}(\sigma)}-\lambda_0| \leq \delta\right\}\,.$$

Note on the one-dimensional case

In this situation, one can deal more directly with the true collision times:

$$C(\sigma) = \inf\left\{t \geq 0 \,:\, X_t = Y_t\right\}, \quad C_N(\sigma) = \inf_{1 \leq i \leq N} \inf\left\{t \geq 0 \,:\, X_t^{i,N} = Y_t^{i,N}\right\}.$$

Theorem

For any $\delta > 0$:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} - \delta\right)\right] < C(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} + \delta\right)\right]\right\} = 1\,.$$

Moreover, for λ_0 the minimizer of H_0,

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\left|X_{C(\sigma)} - \lambda_0\right| \leq \delta\right\} = 1 = \lim_{\sigma \to 0} \mathbb{P}\left\{\left|Y_{C(\sigma)} - \lambda_0\right| \leq \delta\right\}\,.$$

Theorem

For any $\delta > 0$, and N sufficiently large:

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} - \delta\right)\right] < C_N(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(\underline{H_0} + \delta\right)\right]\right\} = 1\,,$$

and, for all $1 \leq i \leq N$

$$\lim_{\sigma \to 0} \mathbb{P}\left\{|X^{i,N}_{C_N(\sigma)} - \lambda_0| \leq \delta\right\} = 1 = \lim_{\sigma \to 0} \mathbb{P}\left\{|Y^{i,N}_{C_N(\sigma)} - \lambda_0| \leq \delta\right\}\,.$$

Generalizations

• Random initial conditions: As long as (x_0^σ, y_0^σ) or (X_0, Y_0) are a.s. bounded, at a $2\epsilon_0$-distance from each others, and the law of X_0 and Y_0 have full support on different basin of attraction of V, our main results still hold true.

• Regular multi-wells confining potential. For instance if V admits m wells located at $\lambda_1, \cdots, \lambda_m$ then, again, the Kramers'law for $C_\epsilon(\sigma)$, $C_{\epsilon,N}(\sigma)$, $C(\sigma)$ and $C_N(\sigma)$ hold and the collision λ_0 is located at

$$\Big(\sum_{l=1}^m \nabla\Psi_l\Big)^{-1}\big(\alpha\sum_{l'=1}^m \lambda_{l'}\big), \qquad \Psi_l(x) = V(x) + \frac{\alpha}{2}||x-\lambda_l||^2 \,.$$

• Further self-stabilizing forces: Provided that F is a smooth function such that $F(x) = G(||x||)$ where $G:\mathbb{R}\to\mathbb{R}$ is a even polynomial function G, with a degree larger than 2, satisfying $G(0) = 0$ (i.e. framework of [HT10, T20]) then the self-stabilizing force derive can be extended into more general kernel

$$\int F(x-y)\mu(dy).$$

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