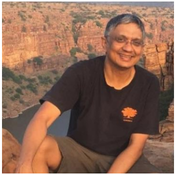
Skew fractional Brownian motion and stochastic sewing with controls

Oleg Butkovsky

Joint work with



Siva Athreya

Khoa Le

Leonid Mytnik

Weierstrass institute

22 September 2021

Plan of the talk

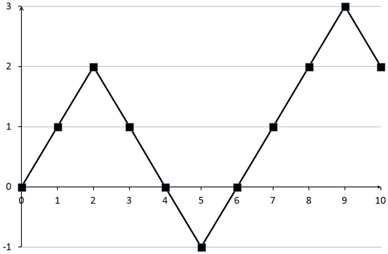
- Skew random walk and skew Brownian motion
- Skew fractional Brownian motion and skew stochastic heat equation
- Sewing of Gubinelli and sewing with controls of Friz–Zhang
- Veretennikov–Zvionkin transformation vs stochastic sewing with controls

Skew random walk and skew Brownian motion

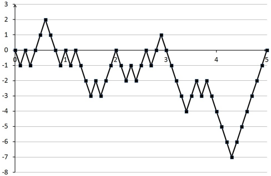
α-skew simple random walk

Fix $\alpha \in [0, 1]$ and consider the following simple Markov process $(S_n)_{n\in\mathbb{Z}_+}$:

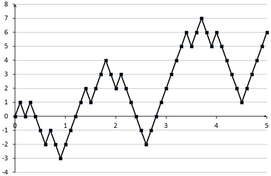
- $S_0 = 0$;
- $\mathrm{P}(S_{n+1} = i + 1|S_n = i) = \mathrm{P}(S_{n+1} = i - 1|S_n = i) = 1/2$ if $i \neq 0$;
- $P(S_{n+1} = 1|S_n = 0) = \alpha$;
- $P(S_{n+1} = -1|S_n = 0) = 1 - \alpha$.



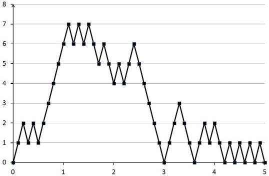
α-skew simple random walk (α-SSRW)



$\alpha = 0.2$



$\alpha = 0.5$



$\alpha = 0.8$

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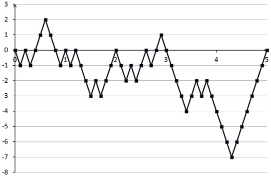

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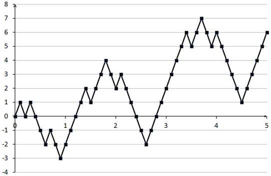

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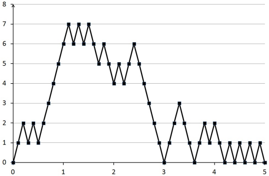

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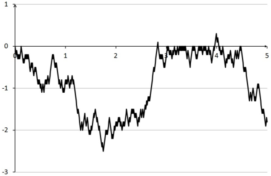
- Harisson and Shepp (1981) showed that a properly rescaled α-SSRW converges to an *α-skew Brownian motion*



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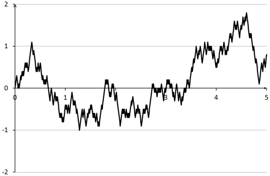

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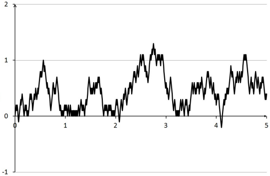

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More on α-skew Brownian motion (α-SBM)

- Harisson and Shepp: $n^{-1/2} S_{\lfloor n \cdot \rfloor} \to X^{(\alpha)}$, where $X^{(\alpha)}$ is α-SBM.
- One can construct an α-SSRW in the following way: take SRW and change independently the sign of its excursions: each excursion is positive with probability α and negative with probability $1 - \alpha$.

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- One can construct an α-SSRW in the following way: take SRW and change independently the sign of its excursions: each excursion is positive with probability α and negative with probability $1 - \alpha$.
- The same is true for an α-SBM (Ito, Mckean, 1965): take BM and change independently the sign of its excursions: each excursion is positive with probability α and negative with probability $1 - \alpha$.
- In particular, if $\alpha = 0.5$, $X^{(\alpha)}$ is a standard BM; if $\alpha = 1$, $X^{(\alpha)}$ is a reflected BM.

More on α-skew Brownian motion (α-SBM)

- It is clear that α-SBM is not a martingale and not Gaussian for $\alpha \neq 0.5$.
- However, it is still a diffusion process! It satisfies the following SDE:

$$X^{(\alpha)}(t) = \beta \int_0^t \delta(X_s^{(\alpha)})\, ds + W_t, \qquad (*)$$

  where $\beta := 2\alpha - 1$.

Theorem (Harisson,Shepp, 1981)

If $\beta \in [-1, 1]$ SDE $(*)$ has a unique strong solution. This solution has the same law as α-SBM.

SDE for skew BM

$$X^{(\alpha)}(t) = \beta \int_0^t \delta(X_s^{(\alpha)})\, ds + W_t, \qquad (*)$$

- Since δ is not a function but just a distribution, $\delta(X_t)$ is a priori not well-defined.
- We say that a process X solves SDE $(*)$ if $X = W + \psi$ and one has

  $$\sup_{t\in[0,1]} \left| \psi_t - \beta \int_0^t p_\varepsilon(X_s)\, ds \right| \to 0,$$

  in probability as $\varepsilon \to 0$.
- Here p_ε is a Gaussian kernel with mean 0 and variance ε, however the choice of a sequence of functions approximating δ is not important.

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- Here p_ε is a Gaussian kernel with mean 0 and variance ε, however the choice of a sequence of functions approximating δ is not important.
- What if $|\beta| > 1$? Maybe $(*)$ would still have a solution?

Theorem (Harisson,Shepp, 1981)

If $|\beta| > 1$ then SDE $(*)$ does not have even a weak solution.

Warm-up (reminder)

$$X^{(\alpha)}(t) = \beta \int_0^t \delta(X_s^{(\alpha)})\, ds + W_t, \qquad (*)$$

- Let's show that this SDE has a weak solution for $\beta = 1$.
- Take $X_t := |B_t|$, where B is a Brownian motion. Then by Ito's formula (informally)

  $$dX_t = d|B_t| = \mathrm{sign}(X_t) dB_t + \delta(X_t) dt = dM_t + \delta(X_t) dt,$$

  where $M_t := \int_0^t \mathrm{sign}(X_t) dB_t$. Since $\langle M \rangle_t = t$, we see that M is a BM.
- Thus the pair $(|B|, M)$ is indeed a weak solution to $(*)$.

SDE for Skew BM

$$X^{(\alpha)}(t) = \beta \int_0^t \delta(X_s^{(\alpha)})\, ds + W_t, \qquad (*)$$

- In the general case $|\beta| \neq 1$ Harisson–Shepp applied the Veretennikov-Zvonkin method.
- They construct a nice function F such that the process $Y_t := F(X_t)$, does not have a drift and satisfy a nice SDE

  $$dY(t) = G(Y_s)\, dW_s,$$

  where G is not too bad. Then from uniqueness/non-existence of solutions to this SDE one can show uniqueness/non-existence of solutions to the original equation.

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  where G is not too bad. Then from uniqueness/non-existence of solutions to this SDE one can show uniqueness/non-existence of solutions to the original equation.
- This method was pioneered by Veretennikov, Zvonkin in 1970ies for showing well-posedness of SDEs with bounded drift, and later developed by Flandolli, Gubinelli, Priola, Bass, etc.
- To apply this method one needs Ito's formula!

Skew fractional Brownian motion and skew stochastic heat equation

Skew fractional Brownian motion (SfBM)

- Consider now **fractional** Brownian motion W^H, $H \in (0,1)$.
- Recall that W^H is a Gaussian process with mean 0 and covariance $\mathrm{E} W_t^H W_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.
- Its trajectories are Hölder with the exponent $H-$ a.s.
- For $H = 1/2$ fBM is just BM; for $H \neq 1/2$ it is not a Markov process nor a semimartingale.
- Ito's formula is not available here.
- We want to define a skew fractional Brownian motion by analogy with SBM.

Skew fractional Brownian motion (SfBM)

- We cannot define SfBM as a limit of a random walk model.
- We cannot define SfBM by flipping its excursions, because fBM is not a Markov process.
- Thus, the only hope is to define SfBM via the corresponding SDE.

Definition

The unique strong solution to the following SDE will be called SfBM:

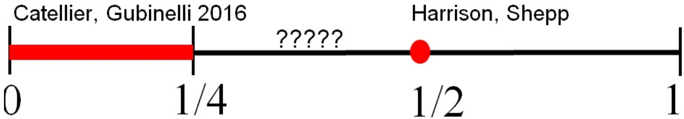
$$X(t) = \beta \int_0^t \delta(X_s)\, ds + W_t^H. \qquad (**)$$

- Does sFBM exist?

Skew fractional Brownian motion (SfBM)

$$X(t) = \beta \int_0^t \delta(X_s)\, ds + W_t^H. \qquad (**)$$

- It is known that $(**)$ has a unique strong solution in the red intervals of the plot below; $1/4$ is **not** included.

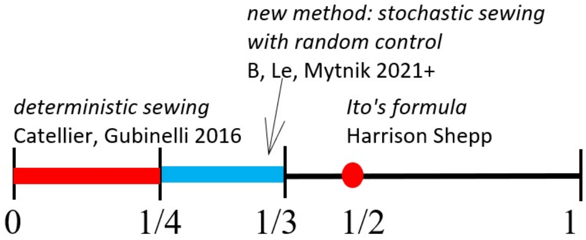


- Heuristically, the more irregular the driving noise, the rougher the drift can be.
- The bound $1/4$ does not have any special meaning; it is known that if $H < 1/4$ then SDE $(**)$ has a unique strong solution for any drift in $\mathcal{C}^{-1}$.
- But δ is also a measure!

Main result and conjecture

$$X(t) = \beta \int_0^t \delta(X_s)\, ds + W_t^H. \qquad (**)$$

- The gap appears because Ito's formula is not available for $H \neq 1/2$ and one has to develop other methods to solve this problem.



new method: stochastic sewing with random control
B, Le, Mytnik 2021+

deterministic sewing
Catellier, Gubinelli 2016

Ito's formula
Harrison Shepp

- Conjecture: equation $(**)$ has a unique strong solution if $H < 1/2$ and no weak solutions if $H > 1/2$.

Skew stochastic heat equation

- The fact that now we are able to cover the case $H = 1/4$ allows to show that the *skew stochastic heat equation* is well–defined.
- This process was conjectured to exist by Bounebache, Zambotti, 2011.

$$\partial_t u = \partial_{xx} u + \beta\delta(u) + \dot{W}, \quad t \geqslant 0,\ x \in \mathbb{R},$$

where $\dot{W}$ is a space-time white-noise.

Skew stochastic heat equation

$$\partial_t u = \partial_{xx} u + \beta\delta(u) + \dot{W}, \quad t \geqslant 0,\ x \in \mathbb{R},$$

As usual, we say that u solves this equation if

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} \beta p_{t-s}(x-x')\delta(u(s,x'))\,dx'ds + V(t,x)$$

where p is the standard heat kernel and

$$V(t,x) := \int_0^t \int_{\mathbb{R}} p_{t-s}(x-x')\,W(ds,dx').$$

- For fixed $x \in \mathbb{R}$ the process $V(t)$ "behaves like" fBM $1/4$. Thus, the following theorem holds.

Theorem (ABLM, 2021)

For any $\beta \in \mathbb{R}$ skew stochastic heat equation has a unique strong solution.

Sewing of Gubinelli and sewing with controls of Friz–Zhang

Proof idea: big picture

- To fix the ideas consider 1D SDE with "bad" drift $b \in \mathcal{C}^\gamma$, $\gamma < 1$.
$$dX_t = b(X_t)dt + dW_t.$$
- Let us try to prove strong uniqueness of solutions to this equation. Let X and $\widetilde{X}$ be two solutions to this equation. Denote $Z := \widetilde{X} - X$. We have
$$\|Z_t\|_{L_p} = \Big\| \int_0^t [b(\widetilde{X}_s) - b(X_s)]ds \Big\|_{L_p}$$

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- At least we want to show that
$$\Big\| \int_0^t [b(W_s + z) - b(W_s)]ds \Big\|_{L_p} \leqslant Ct^\rho |z|.$$
Recall that for us $b = \delta$.

Sewing lemma of Gubinelli

- Let $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$. Then it is well–known that $\int fdg$ exists and can be defined as a limit of Riemann sums if $\alpha + \beta > 1$.
- One way to prove it, is Gubinelli's sewing lemma (2004).

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- Suppose we are given a continuous (deterministic) process $A_{s,t}$, indexed by $0 \leqslant s \leqslant t \leqslant 1$.
- Define for $0 \leqslant s \leqslant u \leqslant t$ $\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$.

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Theorem (Gubinelli)

Assume that $|\delta A_{s,u,t}| \leqslant N|t-s|^{1+\varepsilon}$. *Then the following process exists*

$$\mathcal{A}_t := \lim \sum A_{t_i, t_{i+1}},$$

and $|\mathcal{A}_t - \mathcal{A}_s| \leqslant |A_{s,t}| + CN|t-s|^{1+\varepsilon}$.

- For the Young case we take $A_{s,t} := f_s(g_t - g_s)$. Then $|\delta A_{s,u,t}| = |f_s(g_t - g_s) - f_s(g_u - g_s) - f_u(g_t - g_u)| = |(f_s - f_u)(g_t - g_u)| \leqslant |t-s|^{\alpha+\beta}$.

Sewing lemma with controls of Friz–Zhang

- Take $f \in \mathcal{C}^\alpha$, where $\alpha > 0$ is very small, and $g(t) := t^\beta$, $\beta > 0$ is also very small. Then $g \in \mathcal{C}^\beta$, $\alpha + \beta < 1$, yet $\int_0^t f(s)\,ds^\beta = N \int_0^t f(s)s^{\beta-1}\,ds$ exists.

Sewing lemma with controls of Friz–Zhang

- Take $f \in \mathcal{C}^\alpha$, where $\alpha > 0$ is very small, and $g(t) := t^\beta$, $\beta > 0$ is also very small. Then $g \in \mathcal{C}^\beta$, $\alpha + \beta < 1$, yet $\int_0^t f(s)\, ds^\beta = N \int_0^t f(s) s^{\beta - 1}\, ds$ exists.
- Following Friz–Zhang, we say that a nonnegative continuous function $\lambda(s,t)$, where $0 \leqslant s \leqslant t \leqslant 1$ is a *control* if

$$\lambda(s,u) + \lambda(u,t) \leqslant \lambda(s,t), \quad \text{for any } 0 \leqslant s \leqslant u \leqslant t \leqslant 1.$$

Theorem (Friz, Zhang, 2017)

Assume that $|\delta A_{s,u,t}| \leqslant N|t-s|^\rho \lambda(s,t)$, $\rho > 0$. *Then the following process exists*

$$\mathcal{A}_t := \lim \sum A_{t_i, t_{i+1}},$$

and $|\mathcal{A}_t - \mathcal{A}_s| \leqslant |A_{s,t}| + C|t-s|^\rho \lambda(s,t)$.

How do we gain with sewing with controls?

Demo mode

- Suppose we want to get a good bound on $\mathcal{A}_t := \int_0^t f(s)\, ds^\beta$, $f \in \mathcal{C}^\alpha$, $\alpha + \beta < 1$.
- Set $A_{s,t} := f_s(t^\beta - s^\beta)$.

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- Then $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t} = (f_t - f_s)(t^\beta - s^\beta)$.

$$|\delta A_{s,u,t}| \leqslant$$

.

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$$|\delta A_{s,u,t}| \leqslant |t-s|^{\alpha} |t-s|^{\beta}$$

.

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- Then $\delta A_{s,u,t} = A_{s,t} - A_{s,u} - A_{u,t} = (f_t - f_s)(t^\beta - s^\beta)$.

$$\begin{aligned} |\delta A_{s,u,t}| &\leqslant |t-s|^\alpha \, \xcancel{|t-s|^\beta} \\ &\leqslant |t-s|^\alpha \lambda(s,t) \end{aligned}$$

where $\lambda(s,t) := t^\beta - s^\beta$ is indeed a control.

Stochastic sewing with random controls

$$X(t) = \beta \int_0^t \delta(X_s)\, ds + W_t^H;$$
$$X = \psi + W^H.$$

A couple of observations:

- ψ is increasing; thus $\lambda(s,t) := \psi_t - \psi_s$ is a *random* control (i.e. $\lambda(s,u) + \lambda(u,t) \leqslant \lambda(s,t)$ whenever $s \leqslant u \leqslant t$).
- But $\|\psi_t - \psi_s\|_{L_p(\Omega)}$ is **NOT** a control for $p > 1$:-(.
- So one has to be very careful in extending Friz–Zhang to the stochastic setting so that the result is still useful.
- Recall: a very useful extension of Gubinelli to the stochastic setup is due to Le, 2019.

Stochastic sewing with random controls

- We say that the process $\lambda(s,t,\omega)$ is a random control if $\lambda(s,u,\omega)+\lambda(u,t,\omega)\leqslant\lambda(s,t,\omega)$ a.s. whenever $s\leqslant u\leqslant t$.

Theorem (B., Mytnik, 2020)

Let $A_{s,t}$ be an $\mathcal{F}_t$-measurable random variable. Assume that for some $p\geqslant 2$, $\rho>0$ one has

$$\begin{aligned}&\|\delta A_{sut}\|_{L_p}\leqslant K_1|t-s|^{1/2+\varepsilon}\\&|\mathsf{E}[\delta A_{sut}|\mathcal{F}_u]|\leqslant K_2|t-s|^{\rho}\lambda(s,t)\ \textit{a.s.}.\end{aligned}$$

Then there exists a process B_{st} and a constant $C>0$, such that the following holds:

$$\|\mathcal{A}_t-\mathcal{A}_s\|_{L_p}\leqslant\|A_{st}\|_{L_p}+CK_1|t-s|^{1/2+\varepsilon}+CK_2|t-s|^{\rho}\|\lambda(s,t)\|_{L_p}.$$

Stochastic sewing with random controls: application (sketch)

$$X(t) = \beta \int_0^t \delta(X_s)\, ds + W_t^H;$$
$$X = \psi + W^H.$$

- We need to show that $\|\psi_t - \psi_s\|_{L_p} \leqslant |t-s|^\gamma$.
- Recall that if $H = 1/2$, b is bounded, then this step is immediate; but already for $H = 1/2$, $b \in L_p$ this is not trivial.

Stochastic sewing with random controls: application (sketch)

$$X(t) = \beta \int_0^t \delta(X_s)\, ds + W_t^H;$$

$$X = \psi + W^H.$$

- We need to show that $\|\psi_t - \psi_s\|_{L_p} \leqslant |t-s|^\gamma$.
- Recall that if $H = 1/2$, b is bounded, then this step is immediate; but already for $H = 1/2$, $b \in L_p$ this is not trivial.
- Take $A_{s,t} := \int_s^t \delta(W_r^H + \psi_s)\, dr$.
- Then $\delta A_{sut} = \int_u^t \delta(W_r^H + \psi_u) - \delta(W_r^H + \psi_s)\, ds$.
- Hence $|\mathsf{E}[\delta A_{sut}|\mathcal{F}_u]| \leqslant |t-s|^\gamma |\psi_u - \psi_s|$ - which is very good!

Further directions: conjectures/paradoxes/open problems

Open problems

- Well-posedness of $dX_t = \delta_0(X_t)dt + dW_t^H$, $H \in (1/3, 1/2)$.
- Weak existence and uniquness?
- Numerics for SDEs driven by α-stable processes. We are planning to improve Mikulevicius–Xu, 2016.

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- Numerics for SDEs driven by α-stable processes. We are planning to improve Mikulevicius–Xu, 2016.
- B., Le, Zambotti (work in progress). Consider SPDE on $[0,1]$ with Dirichlet BC

  $$\partial_t u = \frac{1}{2}\partial_{xx} u + \nabla f(u) + \dot{W}.$$

  It's invariant measure is the Gibbs measure given by

  $$\pi(A) := \frac{1}{Z}\int_A \int_0^1 e^{-f(x(z))} dz\, \mu(dx),$$

  where μ is the law of the Brownian Bridge $0 \to 0$.
- Note that π is well-defined even if ∇f is a distribution! Can we take $\nabla f = \delta_0$?

Summary

- Harrison and Shepp using Veretennikov–Zvonikin technique showed that skew Brownian motion is well-defined.
- Extension of this to the fractional BM case is not easy! No Ito's formula.
- Catellier, Gubinelli used deterministic sewing to show that SfBM is well posed for $H < 1/4$.
- Inspired by sewing with controls of Friz–Zhang and stochastic sewing of Le, we developed stochastic sewing with controls.
- This allows to show well–posedness of SfBM for $H < 1/3$ and well-posedness of skew stochastic heat equation, thus resolving a conjecture of Bounebache–Zambotti.
- The proofs are based on stochastic sewing and its variations, which we believe to be a very flexible and useful tool. We hope that one day it will become as popular as the Zvonkin–Veretennikov transform.