Lattice approximations of control dynamics

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Classical control problem

Dynamics:

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)),$$
$$t \in [0, T], \quad x(t) \in \mathbb{T}^d, \quad u(t) \in U.$$

Here $\mathbb{T}^d$ is the d-dimensional torus, $\mathbb{T}^d \triangleq \mathbb{R}^d/\mathbb{Z}^d$.

Payoff:

$$\sigma(x(T)) \to \min .$$

Assumptions

- U is a metric compact;
- f, σ are continuous;
- f is Lipschitz continuous w.r.t. x.

Measurable controls

Any measurable function $t \mapsto u(t) \in U$ is a control.
$x(\cdot, t_0, x_0, u(\cdot))$ is a solution of the initial value problem:

$$\frac{d}{dt}x(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0.$$

Differential inclusion

$$\frac{d}{dt}x(t) \in \operatorname{co}\{f(t, x(t), u) : u \in U\}, \quad x(t_0) = x_0.$$

Denote the set of solutions of differential inclusion by $\mathcal{X}(t_0, x_0)$.

If $x_*(\cdot) \in \mathcal{X}(t_0, x_0)$, then there exists a sequence of measurable controls $\{u_n(\cdot)\}_{n=1}^{\infty}$ such that

$$\|x_*(\cdot) - x(\cdot, t_0, x_0, u_n(\cdot))\| \to 0 \text{ as } n \to \infty.$$

Value function

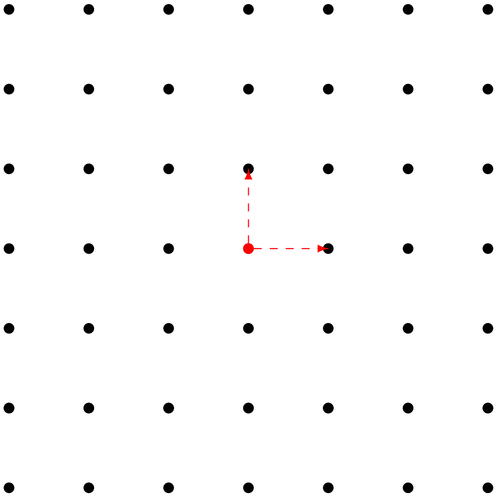
$$\begin{aligned}\mathrm{Val}(t_0, x_0) &\triangleq \inf\{\sigma(x(T, t_0, x_0, u(\cdot))) : u(\cdot) \text{ is measurable}\} \\ &= \min\{\sigma(x(T)) : x(\cdot) \in \mathcal{X}(t_0, x_0)\}.\end{aligned}$$

Bellman equation

$$\frac{\partial \varphi}{\partial x} + H(t, x, \nabla \varphi) = 0, \quad \varphi(T, x) = \sigma(x).$$

- A function $\varphi : [0, T] \times \mathbb{T}^d \to \mathbb{R}$ is a value function of the control problem if and only if φ solves the Bellman equation in the viscosity (minimax) sense.
- In the general case, there is no smooth solution of the Bellman equation.
- The definition of the viscosity solution involves such concept of nonsmooth analysis as sub-/super-differential and directional derivatives.

Lattice approximation



Markov decision problem

- state space: $\mathcal{S} \subset \mathbb{T}^d$, $|\mathcal{S}| < \infty$;
- the dynamics is determined by the controlled Markov chain with the Kolmogorov matrix $Q(t,u) = (Q_{\bar{x},\bar{y}}(t,u))_{\bar{x},\bar{y} \in \mathcal{S}}$;
- the purpose of the control is

$$\mathbb{E}\sigma(X(T)) \to \min,$$

where $X(\cdot)$ is the stochastic process describing the state of the system.

Strategies

1. Feedback relaxed strategies, i.e., we assume that for each $\bar{x} \in \mathcal{S}$ a relaxed control $\gamma_{\bar{x}} : [0, T] \to \mathcal{P}(U)$ is chosen.
2. $\gamma_{\mathcal{S}}(\cdot) = (\gamma_{\bar{x}}(\cdot))_{\bar{x}\in\mathcal{S}}$ is the feedback relaxed strategy.
3. Kolmogorov matrix: if $\gamma_{\mathcal{S}} = (\gamma_{\bar{x}})_{\bar{x}\in\mathcal{S}}$, where $\gamma_{\bar{x}} \in \mathcal{P}(U)$, then $\mathcal{Q}(t, \gamma_{\mathcal{S}}) = (\mathcal{Q}_{\bar{x},\bar{y}}(t, \gamma_{\bar{x}}))_{\bar{x},\bar{y}\in\mathcal{S}}$ is defined by the rule

$$\mathcal{Q}_{\bar{x},\bar{y}}(t, \gamma_{\bar{x}}) \triangleq \int_U Q_{\bar{x},\bar{y}}(t, u)\gamma_{\bar{x}}(du).$$

Value function

$$\begin{aligned}\mathsf{Val}^{Q}_{\bar{x}_0}(t_0) \triangleq \min\{\mathbb{E}\sigma(X(T)) :\ & \\ & \gamma_{\mathcal{S}}(\cdot) \text{ is a feedback relaxed strategy;} \\ & X(\cdot) \text{ is the stochastic process} \\ & \qquad\qquad \text{generated by } \mathcal{Q}(t, \gamma_{\mathcal{S}}(t)), \\ & X(t_0) = \bar{x}_0, \quad P\text{-a.e.}\}.\end{aligned}$$

Dynamic programming for Markov decision process

Let $t \in [0, T]$, $\bar{x} \in \mathcal{S}$, $\phi \in \mathbb{R}^{|\mathcal{S}|}$, set

$$H_{\bar{x}}^{Q}(t,\phi) \triangleq \min_{u \in U} \sum_{\bar{y} \in \mathcal{S}} Q_{\bar{x},\bar{y}}(t,u)\phi_{\bar{y}} = \min_{\gamma \in \mathcal{P}(U)} \sum_{\bar{y} \in \mathcal{S}} \mathcal{Q}_{\bar{x},\bar{y}}(t,\gamma)\phi_{\bar{y}},$$

$$H_{\mathcal{S}}^{Q}(t,\phi) \triangleq (H_{\bar{x}}^{Q}(t,\phi))_{\bar{x} \in \mathcal{S}}.$$

Bellman equation:

$$\frac{d}{dt}\varphi_{\mathcal{S}}(t) = -H_{\mathcal{S}}^{Q}(t,\varphi_{\mathcal{S}}), \quad \varphi_{\bar{x}}(T) = \sigma(\bar{x}).$$

Here $\varphi_{\mathcal{S}}(t) = (\varphi_{\bar{x}}(t)) \in \mathbb{R}^{|\mathcal{S}|}$.

Dynamic programming for Markov decision process

Bellman equation:

$$\frac{d}{dt}\varphi_{\mathcal{S}}(t) = -H^{Q}_{\mathcal{S}}(t, \varphi_{\mathcal{S}}), \quad \varphi_{\bar{x}}(T) = \sigma(\bar{x}).$$

Here $\varphi_{\mathcal{S}}(t) = (\varphi_{\bar{x}}(t)) \in \mathbb{R}^{|\mathcal{S}|}$.

Property: A function $\varphi_{\mathcal{S}}$ is the value function of the Markov decision problem iff it solves the Bellman equation.

Subtraction on $\mathbb{T}^d$

Let $\ell : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}^d$ be a measurable function assigning to a pair of elements $x, y \in \mathbb{T}^d$ a vector $z' \in x - y$ of the minimal norm.

Approximation condition

$$\max_{x \in \mathbb{T}^d} \min_{\bar{y} \in \mathcal{S}} \|x - \bar{y}\| \leq \varepsilon;$$

$$\max_{t \in [0,T], \bar{x} \in \mathcal{S}, u \in U} \Bigg\| f(t, \bar{x}, u) - \sum_{\bar{y} \in \mathcal{S}, \bar{y} \neq \bar{x}} \ell(\bar{y}, \bar{x}) Q_{\bar{x},\bar{y}}(t, u) \Bigg\| \leq \varepsilon,$$

$$\max_{t \in [0,T], \bar{x} \in \mathcal{S}, u \in U} \sum_{\bar{y} \in \mathcal{S}} \|\bar{y} - \bar{x}\|^2 Q_{\bar{x},\bar{y}}(t, u) \leq \varepsilon^2.$$

Theorem. Approximation of control problem

If Val^Q is the value function for the Markov decision problem, while Val is the value function for the deterministic control problem, then, for any $t_0 \in [0, T]$, $x_0 \in \mathbb{T}^d$

$$|\,\mathsf{Val}(t_0, x_0) - \mathsf{Val}^Q_{\bar{x}_0}(t_0)| \leq C\varepsilon.$$

Here $\bar{x}_0 \in \mathcal{S}$ is such that $\|x_0 - \bar{x}_0\| \leq \varepsilon$, C is a constant that does not depend on Q and $\mathcal{S}$.

Lattice Markov chain

Let

- $h > 0$ be such that $1/h \in \mathbb{N}$
- $\mathcal{S}_h \triangleq h\mathbb{Z}^d \cap \mathbb{T}^d$.
- $f(t,x,u) = (f_1(t,x,u), \ldots, f_d(t,x,u))$,
- e^i stand for the i-th coordinate vector.

$$Q^h_{\bar{x},\bar{y}}(t,u) \triangleq \begin{cases} \frac{1}{h}|f_i(t,x,u)|, & \bar{y} = \bar{x} + h \cdot \operatorname{sgn}(f_i(t,x,u))e^i, \\ -\frac{1}{h}\sum_{j=1}^d |f_j(t,x,u)|, & \bar{x} = \bar{y}, \\ 0, & \text{otherwise.} \end{cases}$$

Distance between lattice Markov chain and original system

If the matrix Q is the lattice Markov chain defined as above, then it approximates the origanl system with

$$\varepsilon = \sqrt{h} \cdot \max\left\{\sqrt{R}\sqrt[4]{d}, \frac{\sqrt[4]{d}}{\sqrt{2}}\right\},$$

where

$$R \triangleq \sup\{\|f(t,x,u)\| : t \in [0,T],\, x \in \mathbb{T}^d\, u \in U\}.$$

Mean field type control problem

- dynamics of the probability $m(t)$ obeys

$$\frac{\partial}{\partial t}m(t) + \mathrm{div}(f(t,x,m(t),u(t,x))m(t)) = 0,$$
$$u(t,x) \in U;$$

- payoff function is

$$g(m(T)) \to \min .$$

Mean field type control system

- infinitely many agents interacting via an external media;
- dynamics of each agent is given by

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t, x(t))), \quad u(t, x) \in U;$$

- $m(t)$ is a probability on the phase space describing the distribution of all agents at time t;
- $m(t, Y)$ is a fraction of agents being at time t in the set Y.

Push-forward measure

If (Ω_1, Σ_1), (Ω_2, Σ_2) are measurable spaces, m is a probability on (Ω_1, Σ_1), $h : \Omega_1 \to \Omega_2$ is a measurable function, then the push-forward measure $h\sharp m$ is defined as follows: for any $E \in \Sigma_2$,

$$(h\sharp m)(E) = m(h^{-1}(E)).$$

Space of probabilities

Let (X, ρ_X) be a metric space.

$\mathcal{P}^p(X)$ denotes the set of probabilities m on X such that, for some $x_* \in X$,

$$\int_X \rho_X^p(x, x_*) m(dx) < \infty.$$

Wasserstein metric

For $m_1, m_2 \in \mathcal{P}^p(X)$,

$$W_p(m_1, m_2) \triangleq \inf \left[\left\{ \int_{X \times X} \rho_X^p(x_1, x_2) \pi(d(x_1, x_2)) : \pi \in \Pi(m_1, m_2) \right\} \right]^{1/p}.$$

Here $\Pi(m_1, m_2)$ is the set of probabilities π on $X \times X$ such that its marginal distributions of π are equal to m_1 and m_2 respectively, i.e.

$$\Pi(m_1, m_2) = \{\pi \in \mathcal{P}(X \times X) : \text{ for any measurable } E \subset X \quad \pi(E \times X) = m_1(E), \quad \pi(X \times E) = m_2(E)\}.$$

Space of trajectories of the sample agent

- $\mathcal{C}_{t_0} \triangleq C([t_0, T], \mathbb{T}^d)$.
- If $t \in [t_0, T]$, $x(\cdot) \in \mathcal{C}_{t_0}$, then

$$e_t(x(\cdot)) \triangleq x(t).$$

Mean field type control problem

- dynamics of the probability $m(t)$ obeys

$$\frac{\partial}{\partial t}m(t) + \mathrm{div}(f(t,x,m(t),u(t,x))m(t)) = 0,$$
$$u(t,x) \in U;$$

- payoff function is

$$g(m(T)) \to \min.$$

Mean field type control system

- infinitely many agents;
- dynamics of each agent is given by

$$\frac{d}{dt}x(t) = f(t, x(t), m(t), u(t, x(t))), \quad u(t, x) \in U;$$

- $m(t)$ is a probability on the phase space;
- $m(t, Y)$ is a fraction of agents being at time t in the set $Y \subset \mathbb{T}^d$.

Assumptions

- U is a compact subset of a metric space;
- f, g are continuous;
- f is Lipschitz continuous w.r.t. x and m.

Dynamics of sampling agent

Let

- $t \mapsto m(t)$ be a distribution of agent (for a while);
- t_0 be an initial time, x_0 be an initial state.

The motion of the sampling player satisfies the differential inclusion

$$\frac{d}{dt}x(t) \in \mathrm{co}\{f(t, x(t), m(t), u) : u \in U\}, \quad x(t_0) = x_0.$$

Any such motion can be approximated by trajectories generated by measurable controls.

Dynamics of the flow of probabilities

Let t_0 be an initial time, m_0, $m_0 \in \mathcal{P}^2(\mathbb{T}^d)$ be an initial distribution of agents.

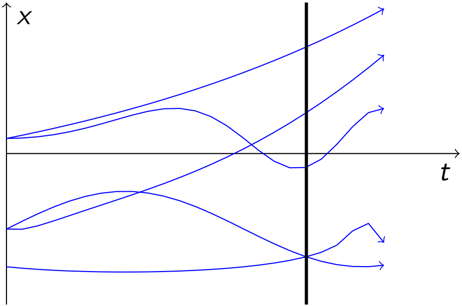
We say that the flow of probabilities $[t_0, T] \ni t \mapsto m(t) \in \mathcal{P}^2(\mathbb{T}^d)$ is a motion of the mean field type control system and write $m(\cdot) \in \mathcal{X}(t_0, m_0)$ if, there exists a $\chi \in \mathcal{P}^2(\mathcal{C}_{t_0})$ such that

- $m(t_0) = m_0$;
- $m(t) = e_t \sharp \chi$;
- χ-a.e. $x(\cdot) \in \mathcal{C}_{t_0}$ satisfies the differential inclusion

$$\frac{d}{dt} x(t) \in \mathrm{co}\{f(t, x(t), m(t), u) : u \in U\}.$$

Distribution of trajectories

Let $\chi \in \mathcal{P}^2(\mathcal{C}_{t_0})$ be a distribution of curves in the phase space.
$m(t) \triangleq e_t \sharp \chi \in \mathcal{P}^2(\mathbb{T}^d)$ is a distribution on the phase space.



Equivalent formulation

Theorem (Jimenez, Marigonda, Quincampoix)

The flow of probabilities $m(\cdot) \in \mathcal{X}(t_0, m_0)$*, if and only if,*

- $m(t_0) = m_0$*;*
- *there exists a velocity field* $v(t,x)$ *such that*

$$v(t,x) \in \mathrm{co}\{f(t,x,m(t),u) : u \in U\},$$
$$\text{for a.e. } t \in [t_0, T] \text{ and } m(t)\text{-a.e. } x \in \mathbb{T}^d,$$

and the equality

$$\frac{\partial m(t)}{\partial t} + \mathrm{div}(v(t,x)m(t)) = 0$$

holds in the sense of distributions.

Mean field type optimal control problem

Given t_0, m_0, minimize

$$g(m(T))$$

over the set $\mathcal{X}(t_0, m_0)$.

Existence of the optimal control

Theorem

There exists at least one optimal flow of probabilities.

Proof is by compactness arguments.

Value function

$$\mathsf{Val}(t_0, m_0) = \min\{g(m(T)) : m(\cdot) \in \mathcal{X}(t_0, m_0)\}.$$

Markov chains

Let

- $\mathcal{S}$ be a finite set;
- $\mathcal{S} \subset \mathbb{T}^d$;
- Σ be a simplex on $\{1, \ldots, |\mathcal{S}|\}$:

$$\Sigma \triangleq \left\{ \mu = (\mu_{\bar{x}})_{\bar{x} \in \mathcal{S}} : \mu_{\bar{x}} \geq 0, \sum_{x \in \mathcal{S}} \mu_{\bar{x}} = 1 \right\};$$

- $\mathbb{1}_{\bar{y}} = (\mathbb{1}_{\bar{y},\bar{x}})_{\bar{x} \in \mathcal{S}}$ be a pure state; here

$$\mathbb{1}_{\bar{y},\bar{x}} = \begin{cases} 1, & \bar{x} = \bar{y}, \\ 0, & \bar{x} \neq \bar{y}. \end{cases}$$

Σ vs $\mathcal{P}(\mathcal{S})$

- $\Sigma \subset \mathbb{R}^{|\mathbb{S}|}$;
- $\mu^1 = (\mu^1_{\bar{x}})_{\bar{x}\in\mathcal{S}}, \mu^2 = (\mu^2_{\bar{x}})_{\bar{x}\in\mathcal{S}} \in \mathbb{R}^{|\mathcal{S}|}$,

$$\|\mu^1 - \mu^2\|_p \triangleq \left[\sum_{x\in\mathcal{S}} |\mu^1_{\bar{x}} - \mu^2_{\bar{x}}|^p\right]^{1/p};$$

- Isomorphism between Σ and $\mathcal{P}(\mathcal{S})$

$$(\mu_{\bar{x}})_{\in\mathcal{S}} = \mu \mapsto \tilde{\mu} = \sum_{\bar{x}\in\mathcal{S}} \mu_{\bar{x}}\delta_{\bar{x}}.$$

Σ vs $\mathcal{P}(\mathcal{S})$

There exists constants C_1 and C_2 such that

$$\|\mu^1 - \mu^2\|_p \leq C_1 W_p(\widetilde{\mu^1}, \widetilde{\mu^2}),$$

$$W_p(\widetilde{\mu^1}, \widetilde{\mu^2}) \leq C_2(\|\mu^1 - \mu^2\|_p)^{1/p}.$$

Mean field type finite state control problem

- a decision maker controls infinitely many agents;
- distribution of agents $\mu = (\mu_{\bar{x}})_{\bar{x}\in\mathcal{S}} \in \Sigma$;
- dynamics of each agents is given by the Markov chain with the Kolmogorov matrix $Q(t,\mu,u) = (Q_{\bar{x},\bar{y}}(t,\mu,u))_{\bar{x},\bar{y}\in\mathcal{S}}$, $u \in U$;
- The decision maker tries to minimize

$$\hat{g}(\mu(T)) \triangleq g(\widetilde{\mu(T)}) = g\left(\sum_{\bar{x}\in\mathcal{S}} \mu_{\bar{x}}\delta_{\bar{x}}\right).$$

Kolmogorov equation

$$\frac{d}{dt}\mu(t) = \mu(t)Q(t, \mu(t), u(t)).$$

Conditions on Q

- for every $(t, \mu, u) \in [0, T] \times \Sigma \times U$, $Q_{\bar{x},\bar{y}}(t, \mu, u) \geq 0$ when $\bar{x} \neq \bar{y}$ and
$$\sum_{\bar{y} \in \mathcal{S}} Q_{\bar{x},\bar{y}}(t, \mu, u) = 0;$$
- the functions $Q_{\bar{x},\bar{y}}(t, \mu, u)$ are continuous;
- there exists a constant L' such that for any $t \in [0, T]$, $\bar{x}, \bar{y} \in \mathcal{S}$, $\mu^1, \mu^2 \in \Sigma$, $u \in U$,
$$|Q_{\bar{x},\bar{y}}(t, \mu^1, u) - Q_{\bar{x},\bar{y}}(t, \mu^2, u)| \leq L' \|\mu^1 - \mu^2\|_2.$$

Feedback strategies

A function $[s, r] \ni t \mapsto \gamma_{\mathcal{S}}(t) = (\gamma_{\bar{x}}(t))_{\bar{x} \in \mathcal{S}} \in \mathcal{P}(U)^{|\mathcal{S}|}$ is a relaxed feedback control for the mean field type finite state control system provided that each function $\gamma_{\bar{x}}(\cdot)$ is weakly measurable.

In this case we regard $\gamma_{\bar{x}}(t)$ as the instantaneous control acting upon the agent who occupy the state $\bar{x}$ at time t.

Relaxed transition rates:

$$\mathcal{Q}_{\bar{x},\bar{y}}(t, \mu, \gamma_{\bar{x}}) \triangleq \int_U Q_{\bar{x},\bar{y}}(t, \mu, u)\gamma_{\bar{x}}(du).$$

Kolmogorov matrix:

$$\mathcal{Q}(t, \mu, \gamma_{\mathcal{S}}) = (\mathcal{Q}_{\bar{x},\bar{y}}(t, \mu, \gamma_{\bar{x}})_{\bar{x},\bar{y} \in \mathcal{S}}.$$

Control problem

- control $t \mapsto \gamma_{\mathcal{S}} = (\gamma_{\bar{x}})_{\bar{x} \in \mathcal{S}} \subset \mathcal{P}(U)^{\mathcal{S}}$;
- dynamics:

$$\frac{d}{dt}\mu_{\bar{y}}(t) = \sum_{\bar{x} \in \mathcal{S}} \mu_{\bar{x}}(t) \mathcal{Q}_{\bar{x},\bar{y}}(t, \mu(t), \gamma_{\bar{x}}(t)), \quad \bar{y} \in \mathcal{S}$$

  or in the vector form

$$\frac{d}{dt}\mu(t) = \mu(t)\mathcal{Q}(t, \mu(t), \gamma_{\mathcal{S}}(t)),$$

- payoff function

$$\hat{g}(\mu(T)) \triangleq g(\widetilde{\mu(T)}) = g\left(\sum_{\bar{x} \in \mathcal{S}} \mu_{\bar{x}} \delta_{\bar{x}}\right).$$

Value function of the mean field type Markov decision problem

$$\begin{aligned}\mathrm{Val}^Q(t_0,\mu_0) \triangleq \min\{g(\mu(T)) :\ & \mu(\cdot) \text{ satisfying the Kolmogorov equation} \\ & \text{with the strategy } \gamma_{\mathcal{S}}(\cdot), \\ & \mu(t_0)=\mu_0\}.\end{aligned}$$

Approximation condition

$$\max_{x\in\mathbb{T}^d}\min_{\bar{y}\in\mathcal{S}}\|x-\bar{y}\|\leq\varepsilon;$$

$$\max_{t\in[0,T],\bar{x}\in\mathcal{S},\mu\in\Sigma,u\in U}\Big\|f(t,\bar{x},\tilde{\mu},u) - \sum_{\bar{y}\in\mathcal{S},\bar{y}\neq\bar{x}}\ell(\bar{y},\bar{x})Q_{\bar{x},\bar{y}}(t,\mu,u)\Big\|\leq\varepsilon,$$

$$\max_{t\in[0,T],\bar{x}\in\mathcal{S},\mu\in\Sigma,u\in U}\sum_{\bar{y}\in\mathcal{S}}\|\bar{y}-\bar{x}\|^2 Q_{\bar{x},\bar{y}}(t,\mu,u)\leq\varepsilon^2.$$

Proximal elements

For $m \in \mathcal{P}^2(\mathbb{T}^d)$, denote by $\mathrm{pr}_{\mathcal{S}}(m)$ an element of Σ such that $\mathrm{pr}_{\mathcal{S}}(m)$ is a proximal to m element of $\mathcal{P}^2(\mathcal{S})$, i.e., $\mathrm{pr}_{\mathcal{S}}(m)$ minimize the function

$$\Sigma \ni \mu = (\mu_{\bar{x}})_{\bar{x}\in\mathcal{S}} \mapsto W_2(\tilde{\mu}, m) = W_2\left(\sum_{\bar{x}\in\mathcal{S}} \mu_{\bar{x}}\delta_{\bar{x}}, m\right).$$

Theorem. Approximation of mean field type control problem

Assume that

- Val is the value function for the deterministic control problem;
- Val^Q is the value function for the mean field type Markov decision problem.

Then, for any $t_0 \in [0, T]$, $m_0 \in \mathcal{P}^2(\mathbb{T}^d)$

$$|\,\mathrm{Val}(t_0, m_0) - \mathrm{Val}^Q(t_0, \mathrm{pr}_{\mathcal{S}}(m_0))| \leq C\varepsilon.$$

Here C is a constant that does not depend on Q and $\mathcal{S}$.

Lattice Markov chain

Let

- $h > 0$ be such that $1/h \in \mathbb{N}$
- $\mathcal{S}_h \triangleq h\mathbb{Z}^d \cap \mathbb{T}^d$.
- $f(t, x, m, u) = (f_1(t, x, m, u), \dots, f_d(t, x, m, u))$,
- e^i stand for the i-th coordinate vector.

$$Q^h_{\bar{x},\bar{y}}(t, \mu, u) \triangleq \begin{cases} \frac{1}{h}|f_i(t, x, \tilde{\mu}, u)|, & \bar{y} = \bar{x} + h \\ & \cdot\, \mathrm{sgn}(f_i(t, x, \tilde{\mu}, u))e^i, \\ -\frac{1}{h}\sum_{j=1}^{d}|f_j(t, x, \tilde{\mu}, u)|, & \bar{x} = \bar{y}, \\ 0, & \text{otherwise.} \end{cases}$$

Distance between lattice Markov chain and original system

Let

$$\|f(t,x,m_1,u) - f(t,x,m_2,u)\| \leq L'' W_1(m_1,m_2)$$

for some constant L''.

If the matrix Q is the lattice Markov chain defined as above, then it approximates the original system with

$$\varepsilon = \sqrt{h} \cdot \max\left\{\sqrt{R}\sqrt[4]{d}, \frac{\sqrt[4]{d}}{\sqrt{2}}\right\}.$$

Thank you for your attention!