Upper density estimates for the marginal law of an stable process and its supremum: From simulation to theory

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From Valentin's work we learned how to use the parametrix method as a two way road between approximations and their limits.

Konakov-Mammen:Local limit theorems for transition densities of Markov chains converging to diffusions (2000) PTRF 551-587

Here we give another different twist at the story for the couple of the supremum and the current value of a stable process.

The goal is to obtain an almost optimal upper bound for the joint density.

Stable process

A general α-stable process is a Lévy process with characteristic function ($\alpha \in (0,2)$)

$$-\log\left(\mathbb{E}\left[e^{i\theta X_t}\right]\right) = ct|\theta|^\alpha(1 - i\,\mathbf{sgn}(\theta)\tan(\pi\alpha(2\rho - 1)/2))$$

•Here, $\rho := \mathbb{P}(X_t \geq 0)$ is the positivity parameter.
•The generator of the Markov process X is (non-local operator)

$$Lf(x) = \gamma f'(x) + \int (f(x+y) - f(x) - f'(x)1_{(-1,1)}(y))\nu(dy)$$
$$\nu(dy) = \frac{dy}{|y|^{1+\alpha}}\left(\rho 1_{(0,\infty)}(y) + (1-\rho)1_{(-\infty,0)}(y)\right).$$

• Our goal: Study of the joint law of $(X_T, \bar{X}_T) \equiv (X_T, \sup_{s\in[0,T]} X_s)$ on the domain $\mathcal{O} := \{(x,y) \in \mathbb{R}^2; y \geq x \vee 0\}$ (including the behavior close to the diagonal and for $x, y \approx 0$ or $x, y \approx \infty$)

•Many previous results: R.A. Doney and M.S. Savov (2010).

$$\partial_y^3 \mathbb{P}(\bar{X}_T \in dy) \overset{y\to\infty}{\approx} Ty^{-\alpha-1}; \quad \partial_y^3 \mathbb{P}(\bar{X}_T \in dy) \overset{y\to 0}{\approx} T^{-\rho} y^{\alpha\rho-1}.$$

$$\partial_x^3 \mathbb{P}(X_T \in dx) \overset{x\to\infty}{\approx} Tx^{-\alpha-1}; \quad \partial_x^3 \mathbb{P}(X_T \in dx) \overset{x\to 0}{\approx} T^{-1/\alpha}.$$

Analytical arguments based on stable meanders (excursion identities). ρ appears in the constants.

• 4 domains required: We will see that the domains are determined by the lines $y = T^{1/\alpha}$ and $y = x + T^{1/\alpha}$, $y \geq x \vee 0$.

• Expansion results for the density of $\bar{X}_T$ have been obtained by Kuznetsov (2011,2013)

Excursion/Fluctuation based equation:

Let X denote a generic Lévy process which is not compound Poisson. S and I will be the associated supremum and infimum processes, we will write L and L^* for the local times at zero of the reflected processes $S - X$ and $X - I$, respectively, and n and n^* will denote the characteristic measures of the excursions away from 0 of these processes. We write ε for a typical excursion, ζ for its lifetime and $\overline{\pi^*}(t) = n^*(\zeta > t)$.

LEMMA 6. There is a constant, $0 < k < \infty$, which depends only on the normalization of L and L^*, such that, for $t, x > 0$,

$$kP\left(S_t > x\right) = \int_0^t n^*\left(\varepsilon_{t-s} > x, \zeta > t - s\right)\left(b + \bar{\pi}(s)\right)ds + \Delta n^*\left(\varepsilon_t > x, \zeta > t\right)$$

where Δ denotes the drift, b the killing rate and $\bar{\pi}(s) = n(\zeta > s)$ the tail of the Lévy measure of the increasing ladder time process T.

Main statement

Let $\mathcal{O} = \{(x, y) \in \mathbb{R}^2 : y > x \vee 0\}$, $n, m \geq 1$, $T > 0$.

Theorem

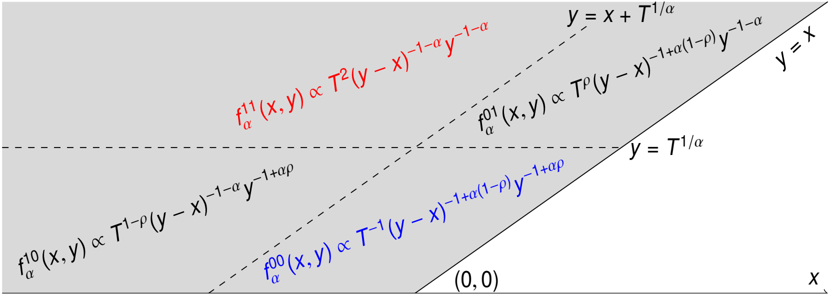
Let $F(x, y) := \mathbb{P}(X_T \leq x, \overline{X}_T \leq y)$ be the law of $(X_T, \overline{X}_T)$. Then $F \in C^\infty(\mathcal{O})$. Moreover, for any fixed $\alpha' \in [0, \alpha)$ there is $C > 0$ s.t. for all $(x, y) \in \mathcal{O}$,

$$|\partial_x^n \partial_y^m F(x, y)| \leq C y^{-m}(y - x)^{1-n-m}(2y - x)^{m-1} \times \min\{f_{\alpha'}^{00}(x, y), f_{\alpha'}^{01}(x, y), f_{\alpha'}^{10}(x, y), f_{\alpha'}^{11}(x, y)\},$$

Density function: $n = m = 1$

$$\min\Big\{T^{-\frac{\alpha'}{\alpha}}(y - x)^{-1+\alpha'(1-\rho)}y^{-1+\alpha'\rho}, T^{\frac{\alpha'}{\alpha}\rho}(y - x)^{-1+\alpha'(1-\rho)}y^{-1-\alpha'}, T^{\frac{\alpha'}{\alpha}(1-\rho)}(y - x)^{-1-\alpha'}y^{-1+\alpha'\rho}, T^{2\frac{\alpha'}{\alpha}}(y - x)^{-1-\alpha'}y^{-1-\alpha'}\Big\}.$$

4 Domains: Behavior at ∞. Behavior at 0. This result is optimal in time and almost optimal in space taking $\alpha' \approx \alpha$. The main reason is that there is some Chebyshev's type argument used. The term $2y - x$ appears due to "weak reflection principle" $\bar{X}_T - (X_T - \bar{X}_T)$



The set $\mathcal{O} = \{(x,y) \in \mathbb{R}^2 : y > \max\{x, 0\}\}$ (shaded in the figure) is the support of the joint density of $(X_T, \overline{X}_T)$. According to our Theorem, the support can be partitioned into 4 sub-regions according to which of the functions f_α^{ij}, $i,j \in \{0,1\}$, is the smallest in the (optimal) case $\alpha' = \alpha$.

Sample of consequences

•Assume that $\alpha \in (0,2)$. Then the distribution function $F(y) := \mathbb{P}(\overline{X}_T \le y) \in C^\infty(0,\infty)$ and, for every $\alpha' \in [0,\alpha)$ and $n \ge 1$, there exists some constant $C > 0$ such that for all $y > 0$ and $T > 0$, we have for $n \ge 1$

$$|\partial_y^n F(y)| \le C y^{-n} \min\{T^{\frac{\alpha'}{\alpha}} y^{-\alpha'}, T^{-\frac{\alpha'}{\alpha}\rho} y^{\alpha'\rho}\}.$$

• Define $\tau_{y_0} := \inf\{t > 0 : X_t > y_0\}$, $y_0 > T^{1/\alpha}$. Then the density of τ_{y_0} is smooth and the following estimate is satisfied for $n \ge 1$:

$$|\partial_T^n \mathbb{P}(\tau_{y_0} \le T)| \le C T^{-\frac{1}{\alpha}-n} \times \min\{T^{\frac{\alpha'}{\alpha}} y_0^{-\alpha'}, 1\}.$$

• If $\alpha(1-\rho) \ge 1$ there is no blow up of the density at the boundary $y = x$. Blow up appears with density derivatives.

• Assume that $\alpha \in (0,2)$ and let $y_0 \ge T^{1/\alpha}$, $x_0 \le 0$. Then for any $\alpha' \in (0,\alpha)$

$$\mathbb{P}(X_T \le x_0, \tau_{y_0} < T) \le C T^{2\frac{\alpha}{\alpha'}} \times \min\{y_0^{-2\alpha'}, y_0^{-\alpha'}(-x_0)^{-\alpha'}\}$$

Corollary

Fix $-x_0 > y_0 > t^{1/\alpha} > 0$, $\alpha \in (1,2)$. *Then for any* $\alpha' \in (0,\alpha)$, *we have*

$$\overline{\lim_{u-t\downarrow 0}}\, \mathbb{P}(X_u \in dx_0, \tau_{y_0} \in dt) \le Ct^{\frac{2+\alpha'}{\alpha}} y_0^{-1-\alpha'} |x_0 - y_0 + t^{1/\alpha}|^{-\alpha}.$$

Classical and non-classical

Malliavin Calculus based on a indep. increment process X uses

$$D_s = \frac{\partial}{\partial \Delta X_s}$$

Are there any other variations of this definition?
If the functional $F(X)$ can be approximated using functions F_n

$$F_n \equiv F_n(E_1, ..., E_n, G_1, ..., G_n, U_1, ..., U_n) \overset{\mathcal{L}}{\Rightarrow} F(X)$$

One may try to do the analysis based on the i.i.d. sequence $\{E_i, G_i, U_i\}_{i\in\mathbb{N}}$ if densities are explicit.
Danger 1: The convergence rate is not good enough. R.N. Bhattacharaya and R. Ranga Rao, Normal Approximation and Asymptotic Expansions. SIAM Classics, 2010

- In our situation the convergence is of exponential order !
- The method will perform integration by parts using the i.i.d. exponentially distributed random variables $\{E_i\}$ and condition on the other r.v.'s $\{G_i, U_i\}_{i\in\mathbb{N}}$.

$$\mathbb{E}[f'(F_n)\Phi_n] = \mathbb{E}[D_E f(F_n)\Phi_n] = \mathbb{E}[f(F_n)H(F_n, \Phi_n)]$$

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Danger 2: The used variables $\{E_i\}_{i\in\mathbb{N}}$ are not explaining the main density behavior of $F(X)$.Because one concentrates on $F(X)$ one may loose track of other relevant path behavior
Danger 3: Need to analyze the explicit expression of $H(F_n, \Phi_n)$. In particular, one needs to see the optimal bounds and Exploding moments may create problems

Malliavin Calculus for jumps

Can it be used?

Bichteler,Gravereaux and Jacod, Nualart-Nualart,Kunita,Ishikawa, Nualart, Bouleau-Denis

• Fournier-Printems, Bally, V. and Caramellino: L. Stochastic Integration by Parts. Advanced Courses in Mathematics - CRM Barcelona, 2016

• Key questions:

1)How to choose the approximations?

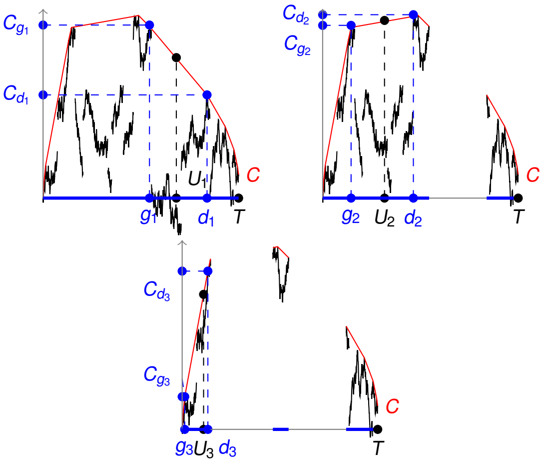
2) Which random variables to use? (Partial Malliavin Calculus)

• Main answers:

1) Convex majorants of Lévy processes (50's$\sim$) with multi-level (2008) will give fast convergence

2) Chambers-Mallows-Stuck decomposition method (recall that explicit stable laws are not available)('76, Kanter ('75)). Base the calculations on exponential r.v.'s $\{E_i\}_{i\in\mathbb{N}}$. That is, the "length" of stable increments.

Convex majorants



Selecting the first three faces of the concave majorant: U_1, U_2, U_3 are uniform with length T, $T - (d_1 - g_1)$ and $T - (d_1 - g_1) - (d_2 - g_2)$, respectively. $C_{g_1} - C_{d_1}$ has the stable distribution with time $g_1 - d_1$ with an associated positive or negative sign.

Mathematical Definition

Exponentially converging stick-breaking process: $\ell = (\ell_k)_{k\geq 1}$ on $[0, T]$, defined using $U_k \sim U[0,1]$

$$\ell_1 = T(1 - U_1)$$
$$\ell_k = TU_1 ... U_{k-1}(1 - U_k)$$

⚠ $$\mathbb{E}[\ell_k^{r/\alpha}] = T^r \left(1 + \frac{r}{\alpha}\right)^{-k}.$$

i.i.d. stable r.v.'s $(S_k)_{k\geq 1}$ with parameters (α, ρ) (i.e. $S_k \overset{d}{=} X_1$).

$$\overline{X}_T = X_+ = \sum_{k=1}^{\infty} \ell_k^{1/\alpha} [S_k]^+$$
$$X_T = X_+ - X_- = X_+ - \sum_{k=1}^{\infty} \ell_k^{1/\alpha} [S_k]^-.$$

Remark: One looses some information about the path!

Chambers-Mallows-Stuck/Kanter

Next step: Probabilistic representation:(Chambers-Mallows-Stuck, $\alpha \in (0,1) \cup \{1\} \cup (1,2)$). E_k: length. G_k: oscillations

$$S_k \overset{\mathcal{L}}{=} E_k^{\,1-1/\alpha} G_k \qquad \text{and} \qquad G_k = g(V_k), \qquad k \in \mathbb{N},$$

for i.i.d. $Exp(1) \sim (E_k)_{k\geq 1} \perp (V_k)_{k\geq 1} \sim \mathrm{U}(-\frac{\pi}{2}, \frac{\pi}{2})$

$$g(x) = \frac{\sin\left(\alpha(x + \pi(\rho - \frac{1}{2}))\right)}{\cos^{1/\alpha}(x)\cos^{1-1/\alpha}\left((1-\alpha)x - \alpha\pi(\rho - \frac{1}{2})\right)}, \qquad x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big).$$

Note that indeed $\mathbb{P}(S_k > 0) = \rho$.

$$\rho = \mathbb{P}(S_k > 0) \in [1 - 1/\alpha, 1/\alpha] \cap (0,1).$$

semi-linear structure in the representation for $(X_T, \bar{X}_T) \equiv (X_+, X_-)$

$$\overline{X}_T = X_+ = \sum_{k=1}^{\infty} \ell_k^{1/\alpha} E_k^{1-1/\alpha} [G_k]^+$$

$$X_T = X_+ - X_- = X_+ - \sum_{k=1}^{\infty} \ell_k^{1/\alpha} E_k^{1-1/\alpha} [G_k]^-.$$

coordinate change and n-th order approx. to $X = (X_+, X_-)$:

$X_n = (X_{+,n}, X_{-,n})$

$a_n := T^{1/\alpha} \kappa^n$ with $\kappa \in (0, 1)$. Let η_+ and $\eta_- \sim Exp(1)$

$$X_{\pm,n} = \sum_{k=1}^{n} \ell_k^{1/\alpha} E_k^{1-1/\alpha} [G_k]^\pm + a_n \eta_+^{1-1/\alpha}.$$

$n = 0$, $X_{\pm,0} \equiv 0$, $[x]^+ = \max\{x, 0\}$,

$$0 \neq X_{\pm,n} - X_{\pm,n-1} = \ell_n^{1/\alpha} E_n^{1-1/\alpha} [G_n]^\pm + (a_n - a_{n-1}) \eta_\pm^{1-1/\alpha}$$

Intuitively, if the sequence $\{a_n\}$ decays too fast, then it will not serve its purpose. In particular, given the assumption below moments estimates will follow.

Assumption[A-κ] $a_n := T^{1/\alpha} \kappa^n$, $n \in \mathbb{N}$ where $\kappa^\alpha \in [\rho \vee (1 - \rho), 1)$.

Reconstructive derivative operator and IBP

$$\mathcal{D}_n^{\pm} = \eta_{\pm}\partial_{\eta_{\pm}} + \sum_{k=1}^{n} E_k \mathbf{1}_{\{[G_k]^{\pm}>0\}}\partial_{E_k},$$

⚠ The factor $(\eta_{\pm}, E_k)$ cancels boundary terms and has a regenerative property for $\mathcal{D}_n^{\pm} X_{\pm,n} = (1 - 1/\alpha) X_{\pm,n}$. This keeps the stable moments controlled! . It also shows that G_k does not need to be differentiated.

$\mathcal{S}_n(\Omega) = \{\Phi \in L^0(\Omega); \exists \phi \in \mathcal{S}_\infty((0,\infty)^{3n+3}; \mathbb{R}), \Phi = \phi(\bar{E}_n, \bar{U}_n, \bar{V}_n, \eta_+, \eta_-)\}$,

Theorem (The approx. but exploding IBP formula)

Fix $n, m \in \mathbb{N}$ with $m \geq n$. Then for any $\Phi \in \mathcal{S}_m(\Omega)$ and f smooth,

$$\mathbb{E}[\partial_{\pm} f(X_n)\Phi] = \mathbb{E}[f(X_n) H_{n,m}^{\pm}(\Phi)],$$

$$H_{n,m}^{\pm}(\Phi) := \frac{\alpha/(\alpha-1)}{X_{\pm,n}}\Big(\Big(\eta_{\pm} - \frac{1}{\alpha} + \sum_{k=1}^{m}(E_k - 1)\mathbf{1}_{\{[G_k]^{\pm}>0\}}\Big)\Phi - \mathcal{D}_m^{\pm}[\Phi]\Big).$$

m: number of variables used for the IBP

n: approximation parameter

⚠ The numerator grows polynomically fast w.r.t. m. → Explosion

⚠ Note that time rescaling is clear in the denominator. Malliavin variance

Multi-level in IBP and iterations

$$H^{\pm,k+1}_{n,m}(\Phi) = H^{\pm}_{n,m}(H^{\pm,k}_{n,m}(\Phi)) \qquad \text{for } k \geq 0, \text{ where} \qquad H^{\pm,0}_{n,m}(\Phi) = \Phi.$$

Theorem (The ∞-dim. IBP formula)

Let $\Phi \in \bigcap_{n\in\mathbb{N}} \mathcal{S}_n(\Omega)$. For any $n \geq 1$, $k_+, k_- \geq 0$ and $f \in C_\epsilon^{k_+ + k_-}(\mathbb{R}^2_+)$ we have

$$\mathbb{E}[\partial_+^{k_+}\partial_-^{k_-} f(X_+, X_-)\Phi] = \mathbb{E}[f(X_n)H^{+,k_+}_{n,n}(H^{-,k_-}_{n,n}(\Phi))] + \sum_{k=n}^{\infty} \mathbb{E}[f(X_{k+1})H^{+,k_+}_{k+1,k+1}(H^{-,k_-}_{k+1,k+1}(\Phi)) - f(X_k)H^{+,k_+}_{k,k+1}(H^{-,k_-}_{k,k+1}(\Phi))].$$

⚠ The convergence in the above sum is geometric in n due to the difference $X_{k+1} - X_k$.

⚠ The weight $H^{+,k_+}_{k,k+1}$ only grows polynomically in n.

⚠ Positive and negative moment estimates are needed!

The magic re-scaling estimates

Let $f(x_+, x_-) = [x_+ - x]^+[x_- - y]^+$ and $Z_m = \eta_+ + \eta_- + \sum_{k=1}^m E_k$

$$\left|f(X_{n+1})H^{+,k_+}_{n+1,n+1}(H^{-,k_-}_{n+1,n+1}(\Phi)) - f(X_n)H^{+,k_+}_{n,n+1}(H^{-,k_-}_{n,n+1}(\Phi))\right|^p$$

$$= \left|\frac{f(X_{n+1})}{X^{k_+}_{+,n+1}X^{k_-}_{-,n+1}} - \frac{f(X_n)}{X^{k_+}_{+,n}X^{k_-}_{-,n}}\right|^p \left|H^{+,k_+}_{n,n+1}(H^{-,k_-}_{n,n+1}(\Phi))X^{k_+}_{+,n}X^{k_-}_{-,n}\right|^p$$

$$\leq \left|\frac{f(X_{n+1})}{X^{k_+}_{+,n+1}X^{k_-}_{-,n+1}} - \frac{f(X_n)}{X^{k_+}_{+,n}X^{k_-}_{-,n}}\right|^p p^{\phi}_{k_+,k_-}(Z_{n+1}, n+1)^p.$$

Here p is a polynomial of order $k_+ + k_-$. Now, recall

$$0 \leq X_{\pm,n} - X_{\pm,n-1} \leq \ell_n^{1/\alpha}[S_n]^+ + (a_n - a_{n-1})\eta_+^{1-1/\alpha}$$

How the moments are used: At infinity, there is an exchange of space variable and random variable. At zero, something more complicated (but similar) happens. For example,

$$\mathbf{1}_{X_+ > x} \leq \frac{X_+^p}{x^p}.$$

Some main points

- Integrate by parts up to obtaining a semi-linear function in order to earn the re-scaling property.
- Average in an orderly fashion, using independence leaving the exponentials at the end.
- Two different situations: positive moments and negative moments
- Geometric rates of convergence are embedded in the stick breaking process. In fact, $\mathbb{E}[\ell_n^{\frac{r}{\alpha}}] = (1 + \frac{r}{\alpha})^{-n}$.
- Inverse moments are obtained using the Gamma function trick:

$$\Gamma(p)\mathbb{E}[X_{+,n}^{-p}] = \int_0^\infty \lambda^{p-1}\mathbb{E}[e^{-\lambda X_{+,n}}]d\lambda$$

  Non trivial estimate for + part of non-symmetric stable law:

$$\mathbb{E}[e^{-\lambda [G]^+}] \leq 1 - \rho + \rho \min\{1, (\alpha\rho\lambda)^{-1}\}.$$

  Many products of this type appear together with $\mathbb{E}[e^{-\lambda a_n \eta_+}]$

Some judgements on Malliavin Calculus

Classical Malliavin Calculus

- It is always sub-optimal
- Its framework is always fixed through the process increments
- No irregular functionals can be treated efficiently
- It is only interesting for hypoelliptic cases
- It is a pathwise approach

Positive conclusions of the methodology used

- Find your own framework which expresses the irregularity of the functional even if it can not describe the whole path
- If the class of processes to be treated is too large you are bound to obtain sub-optimal results
- There is a weak type approach through probabilistic representations.
- This formula does not converge to a “continuous” version. Still, one hopes to obtain continuous IBP formulas for other applications

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