Kolmogorov SPDEs and applications to stochastic filtering

Andrea Pascucci[1]
(joint work with Antonello Pesce)

22 September 2021

– *LSA Autumn meeting* –

[1]Dipartimento di Matematica, Università di Bologna, Italy.

Plan of the talk

- statement of the problem:
  parametrix construction for filtering SPDEs
- uniformly parabolic SPDEs
- SPDEs under the weak Hörmander condition
- applications to filtering theory

Statement of the problem: parametrix for filtering SPDEs

Motivation: SPDEs from filtering theory

- Let (X, Z) be a diffusion
- extract information about X from $\mathcal{F}_t^Z = \sigma(Z_s,\, s \le t)$

  $$E\left[f(X_t) \mid \mathcal{F}_t^Z\right] = \int p_t(x) f(x) dx$$

  $p_t(x)$ is the **conditional density** of X_t **given** $\mathcal{F}_t^Z$
- $$dp_t(x) = \mathbf{L}_t p_t(x) dt + \mathbf{G}_t p_t(x) dW_t$$

  where
  - $\mathbf{L}_t$ second-order, $\mathbf{G}_t$ first-order operators
  - the coefficients depend on t, x, Z_t and are therefore random and not smooth
- if X and Z are independent then $\mathbf{G}_t = 0$ and $\mathbf{L}_t$ is the Fokker-Planck (**forward Kolmogorov**) operator

Cauchy problem for parabolic SPDEs

- L^p **theory**
  Pardoux (1975), Rozovskii and Krylov (1977), Flandoli (1990), Krylov (1999) ...
- **Hölder theory**
  Rozovskii (1975), Shimizu (1982), Chow and Jiang (1992)

Analytical techniques for SPDEs

- Duhamel principle: Chow-Jiang (1992), Mikulevicius (2000), Kleptsyna-Piatnitski-Popier (2020)
- Schauder estimates: Du-Liu (2019), Zhang-Zhang (2021)
- Moser's iteration: Denis, Matoussi and Stoica (2005)
- Hörmander's theorem: Krylov (2015), Qiu (2018)

Constant coefficients: forward heat SPDE

$$du_t(x) = \frac{\mathbf{a}}{2}\partial_{xx}u_t(x)dt + \sigma\partial_x u_t(x)dW_t$$

$$u_t(x) = u_0(x) + \frac{\mathbf{a}}{2}\int_0^t \partial_{xx}u_s(x)ds + \sigma\int_0^t \partial_x u_s(x)dW_s$$

Stochastic characteristics: $u_t(x) = U(t, x + \sigma W_t)$ then

$$\partial_t U = \frac{\mathbf{a} - \sigma^2}{2}\partial_{xx}U$$

Stochastic fundamental solution: for $t > s$

$$p(s, y; t, x) := \frac{1}{\sqrt{2\pi(\mathbf{a} - \sigma^2)(t - s)}}\exp\left(-\frac{(x - y + \sigma(W_t - W_s))^2}{2(\mathbf{a} - \sigma^2)(t - s)}\right)$$

- damping effect of noise: coercivity condition $\mathbf{a} - \sigma^2 > 0$
- distinctive asymptotic behaviour at pole (**Sowers**, 1994)
- $t \mapsto p(s, y; t, x)$ is **adapted** and Hölder continuous

Uniformly parabolic SPDEs with Hölder coefficients

Functional setting

- "Standard" Hölder space: $C^{\alpha}_{t,T}$

  measurable functions $f = f_s(x)$ on $[t,T] \times \mathbb{R}^d$ s.t.

$$\sup_{\substack{s\in[t,T]\\ x\neq y}} \frac{|f_s(x)-f_s(y)|}{|x-y|^{\alpha}} < \infty$$

- Stochastic Hölder space: $\mathbf{C}^{\alpha}_{t,T}$

  predictable processes $f = f_s(x;\omega)$ on $[t,T] \times \mathbb{R}^d \times \Omega$

  such that $f_s(x;\cdot) \in C^{\alpha}_{t,T}$ almost surely

- Similarly higher orders

Stochastic fundamental solution

$p(s, y; \cdot, \cdot) \in \mathbf{C}^2_{s,t}$ is a "classical" solution to the **forward** SPDE

- for $t > s$

$$p(s, y; t, x) = \delta_y + \int_s^t \mathbf{L}_\tau p(s, y; \tau, x) d\tau + \int_s^t \mathbf{G}_\tau p(s, y; \tau, x) dW_\tau$$

Operators in the SPDE:

$$\mathbf{L}_t = \frac{1}{2}\mathbf{a}_t^{ij}(x)\partial_{x_i x_j} + \mathbf{b}_t^i(x)\partial_{x_i} + \mathbf{c}_t(x) \qquad \mathbf{G}_t = \sigma_t^i(x)\partial_{x_i} + \nu_t(x)$$

Parametrix method for SPDEs: two problems

- lack of the Duhamel principle (cf. **Sowers** (1998))
- roughness of the coefficients (only measurable in time)

Parametrix method and Duhamel principle

$$\begin{cases} \partial_t u_t = \frac{1}{2}\partial_{xx} u_t + \mathbf{g} \\ u_0 = 0 \end{cases} \iff u_t(x) = \int_0^t \int_{\mathbb{R}} \mathbf{g}(s,y) p^{\text{heat}}(s,y;t,x) dy ds$$

The fundamental solution solves $\begin{cases} \partial_t p(0,\xi;t,\cdot) = \mathbf{L} p(0,\xi;t,\cdot) \\ p(0,\xi;0,\cdot) = \delta_\xi \end{cases}$

therefore we look for p in the form

$$p(0,\xi;t,x) = p^{\text{heat}}(0,\xi;t,x) + \int_0^t \int_{\mathbb{R}} \mathbf{g}(0,\xi;s,y) p^{\text{heat}}(s,y;t,x) dy ds$$

where $\mathbf{g}$ has to be determined by imposing the PDE

Naïve Parametrix for SPDEs

Also for the heat SPDE

$$du_t(x) = \frac{1}{2}\partial_{xx}u_t(x)dt + \left(\sigma\partial_x u_t(x) + \mathbf{g}_t(x)\right)dW_t$$

by the Itô formula, we have the Duhamel formula

$$u_t(x) = \int_0^t \int_{\mathbb{R}} \mathbf{g}_s(y)p^{\mathrm{heat}}(s, y; t, x)\,dy\,dW_s$$

but...

$$p^{\mathrm{heat}}(s, y; t, x) := \frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(x+\sigma(W_t - W_s) - y)^2}{2(t-s)}\right)$$

either

- $\sigma = 0$ as in **Mikulevicius** (2000) and others
- $\mathbf{g} \equiv 0$, but $\mathbf{g}$ is needed in the parametrix method

Another approach: the Itô-Wentzell formula

$$du_t(x) = \frac{\mathbf{a}_t^{ij}(x)}{2}\partial_{ij}u_t(x)dt + \sigma_t^i(x)\partial_i u_t(x)dW_t$$

Consider the stochastic flow of diffeomorphisms of $\mathbb{R}^d$

$$X_t(x) = x - \int_0^t \sigma_s(X_s(x))dW_s, \qquad t \geq 0$$

Itô-Wentzell change of coordinates

$$\hat{u}_t(x) = u_t(X_t(x))$$

Watanabe (1994): backward SPDE with $\mathbf{a} = \sigma\sigma^*$

Itô-Wentzell: from SPDEs to random PDEs

u solves the **SPDE**

$$du_t(x) = \frac{\mathbf{a}_t^{ij}(x)}{2}\partial_{ij}u_t(x)dt + \sigma_t^i(x)\partial_i u_t(x)dW_t$$

if and only if $\hat{u}_t(x) = u_t(X_t(x))$ solves the **PDE with random coefficients**

$$d\hat{u}_t(x) = \Big(\frac{a_t^{ij}(x)}{2}\partial_{ij}\hat{u}_t(x) + b_t^j(x)\partial_j\hat{u}_t(x)\Big)dt$$

where

$$a_t = \nabla X_t\left(\hat{\mathbf{a}}_t - \hat{\sigma}_t\hat{\sigma}_t^*\right)(\nabla X_t)^* \qquad b_t = \hat{\mathbf{a}}_t\nabla^2 X_t$$

Assumption: $\mathbf{a}_t - \sigma_t\sigma_t^*$ is uniformly (in x) positive definite

Problem: estimate $\nabla X_t(x)$ uniformly w.r.t. x

A special case

If σ_t is independent of x then

$$X_t(x) = x - \int_0^t \sigma_s dW_s$$

and the gradient is simply

$$\nabla X_t(x) = \mathrm{Id}$$

This is easy but not interesting at all!

General case:

$$X_t(x) = x - \int_0^t \sigma_s(X_s(x)) dW_s$$
$$\nabla X_t(x) = \mathrm{Id} - \int_0^t (\nabla \sigma_s)(X_s(x)) \nabla X_s(x) dW_s$$

Coercivity of the flow

$$\nabla X_t(x) = \exp\left(\int_0^t \underbrace{\nabla \sigma_s(X_s(x))}_{F_s(x)} dW_s + \cdots \right)$$

General question. How to estimate a x-dependent Itô integral

$$\Big| \int_0^t F_s(x) dW_s \Big| \approx \Big| \sum_k F_{t_k}(x) \left(W_{t_{k+1}} - W_{t_k} \right) \Big| \leq Z$$

with a random variable Z *independent of* x?

Theorem. There exist positive random matrices c, C such that

$$c \leq \nabla X_t(x) \leq C \qquad x \in \mathbb{R}^d,\ P\text{-a.s.}$$

Coercivity of the flow

$$I_t(x;\cdot) := \int_0^t \nabla\sigma_s(X_s(x))dW_s$$

- $L^p(\Omega)$-estimates for the flow X from assumption ($\varepsilon > 0$)

$$\sup_{\substack{s\in[0,T]\\ x\in\mathbb{R}^d}} |x|^\varepsilon|\partial^\beta\sigma_s(x)| < \infty \quad \text{for } 1 \le |\beta| \le 2,\ P\text{-a.s.}$$

- Doob's and Burkholder's inequalities give

$$E\left[\|I_t(\cdot;\omega)\|^p_{W^{1,p}(\mathbb{R}^d)}\right] \lesssim t^{\frac{p}{2}}$$

- Sobolev's embedding + Kolmogorov continuity

$$\sup_{x\in\mathbb{R}^d} |I_t(x;\omega)| \le c(\omega)t^\alpha \qquad a.e.\ \omega \in \Omega$$

- Sobolev embedding is independent of $\omega \in \Omega$

Schilling (2000): Sobolev implies Kolmogorov

Kolmogorov's continuity theorem is an *analytic result*

It follows from Sobolev's embedding theorem (for Bessel potential spaces)

Time-dependent parametrix

Apply Itô-Wentzell and fix the trajectory $\omega \in \Omega$:

$$L_t = \frac{1}{2} a_t(x)\partial_{xx} + b_t(x)\partial_x + c_t(x)$$

uniformly elliptic PDO with **measurable in time** coefficients

The parametrix is the Gaussian solution of

$$L_{t,\bar{x}} = \frac{1}{2} a_t(\bar{x})\partial_{xx} + b_t(\bar{x})\partial_x + c_t(\bar{x})$$

Existence of a fundamental solution

$$du_t(x) = \mathbf{L}_t u_t(x)dt + \mathbf{G}_t u_t(x)dW_t$$

Gaussian estimates

$$\frac{1}{\mu_2}\Gamma^{\frac{1}{\mu_1}}(t-s, X_{s,t}^{-1}(x) - y) \le p(s,y;t,x) \le \mu_2 \Gamma^{\mu_1}(t-s, X_{s,t}^{-1}(x) - y)$$

$$|\partial_{x_i} p(s,y;t,x)| \le \frac{\mu_2}{\sqrt{t-s}}\Gamma^{\mu_1}(t-s, X_{s,t}^{-1}(x) - y)$$

$$\left|\partial_{x_i x_j} p(\tau,\xi;t,x)\right| \le \frac{\mu_2}{t-s}\Gamma^{\mu_1}(t-s, X_{s,t}^{-1}(x) - y)$$

where

- μ_1 and μ_2 positive random variables
- Γ^{μ} is the fundamental solution of the heat operator $\frac{\mu}{2}\Delta - \partial_t$

Derivation of the filtering SPDE

X signal, Z observation:

$$E\left[f(X_t^{s,y}) \mid \mathcal{F}_t^Z\right] = \int p(s,y;t,x) f(x)dx$$

- Classical approach (from 1980):

  Pardoux, **Kallianpur**, **Krylov, Rozovsky**, **Kunita**

- "Direct" approaches:
  - **Krylov-Zatezalo** (2000): forward SPDE

    $$dp(s,y;t,x) = \mathbf{L}_t^* p(s,y;t,x)dt + \mathbf{G}_t^* p(s,y;t,x)dW_t$$

  - **Veretennikov** (1994): backward SPDE

    $$-dp(s,y;t,x) = \mathbf{L}_s p(s,y;t,x)ds + \mathbf{G}_s(s,y;t,x) \star dW_s$$

SPDEs under the weak Hörmander condition

Model operator: Kolmogorov (1934)

Langevin model (Ornstein-Ulhenbeck)

$$\begin{cases} dX_t = V_t dt & \text{position / drift / transport} \\ dV_t = \sigma dW_t & \text{velocity / diffusion} \end{cases}$$

Fokker-Planck (forward) PDE

$$(\partial_t + v\partial_x)f = \frac{\sigma^2}{2}\partial_{vv} f \qquad (t, x, v) \in \mathbb{R}^3$$

<u>**Applications**</u>

- **Kinetic theory:** also non-linear (Boltzmann-Landau)
- **Finance:** Asian options, path-dependent models

Eqs with constant coefficients: the Lie group structure

Kolmogorov Eqs and linear SDEs

$$dX_t = BX_t dt + \sigma dW_t$$

W d-dimensional Brownian motion

B constant $N \times N$ matrix

σ constant $N \times d$ matrix

Solution:

$$X_t = e^{tB}\left(x + \int_0^t e^{-sB}\sigma dW_s\right), \qquad x \in \mathbb{R}^N$$

X_t is a Gaussian process:

- Mean

$$E[X_t] = e^{tB}x$$

- Covariance matrix

$$\mathcal{C}(t) = \int_0^t e^{sB}\sigma\left(e^{sB}\sigma\right)^* ds$$

Equivalent non-degeneracy conditions

SDE	$dX_t = BX_t dt + \sigma dW_t$
PDE	$\partial_t f + \langle Bx, \nabla f \rangle = \frac{1}{2}(\sigma\sigma^*)_{ij}\partial_{x_i x_j} f$

- **Probability:** the covariance matrix $\mathcal{C}(t)$ is positive definite
- **PDE theory:** <u>weak</u> Hörmander condition

$$\text{rank Lie}\,(\partial_{x_1}, \ldots, \partial_{x_d}, \mathbf{Y}) = N + 1, \qquad \mathbf{Y} = \partial_t + \langle Bx, \nabla \rangle$$

- **Control theory:** Kalman condition

$$\text{rank}\left(\sigma, B\sigma, B^2\sigma, \cdots, B^{N-1}\sigma\right) = N$$

Transition density / fundamental solution

SDE	$dX_t = BX_t dt + \sigma dW_t$
PDE	$\partial_t f + \langle Bx, \nabla f \rangle = \frac{1}{2}\sigma\sigma^* \partial_{xx} f$

If the covariance matrix $\mathcal{C}(t)$ **is <u>positive definite</u>** then

- **Gaussian** fundamental solution

$$p_0(t,x) = \frac{1}{\sqrt{(2\pi)^N \det \mathcal{C}(t)}} \exp\left(-\frac{1}{2}\langle \mathcal{C}^{-1}(t)x, x\rangle\right)$$

- X has transition density

$$p(s,y;t,x) = p_0\left(t-s, x - e^{(t-s)B}y\right), \qquad t > s$$

Analysis on Lie groups

$$\mathbf{K} = \frac{\sigma^2}{2}\partial_{vv} - v\partial_x - \partial_t$$

is invariant w.r.t. the **left-$*$-translations** (non-commutative)

$$(T, X, V) * (t, x, v) = (T + t, X + x + tV, V + v)$$

and homogeneous of degree two w.r.t. the **dilations**

$$\delta_\lambda(t, x, v) = (\lambda^2 t, \lambda^3 x, \lambda v) \qquad \lambda > 0$$

$(\mathbb{R}^3, *, \delta)$ is a Lie group with **δ-homogeneous norm**

$$|(t, x, v)|_{\mathbf{K}} = |t|^{\frac{1}{2}} + |x|^{\frac{1}{3}} + |v|$$

and **distance**

$$d_{\mathbf{K}}\left((t, x, v), (T, X, V)\right) = |(T, X, V)^{-1} * (t, x, v)|_{\mathbf{K}}$$

New problems

- Itô-Wentzell modifies the drift from linear to **non-linear**: freezing the coefficients is not sufficient

- how to handle **space-time regularity** on Lie groups

Intrinsic Hölder continuity

Semigroup approach: distributional solutions

Anisotropic **spatial** α-Hölder condition

$$|f(X,V) - f(x,v)| \lesssim |X - x|^{\frac{\alpha}{3}} + |V - v|^{\alpha}$$

- homogeneous norm + Euclidean translations
- introduced by **Da Prato-Lunardi** (1995)
- used by **Lorenzi** (2005), **Priola** (2009), **Menozzi** (2010) and many others
- no smoothing in time: Schauder estimates
- suitable for the stochastic setting
- Lévy SDEs: **Hao-Wu-Zhang** (2020), **Marino** (2021)

Lie group approach: classical solutions

Intrinsic α-Hölder condition

$$|f(T,X,V) - f(t,x,v)| \lesssim |T-t|^{\frac{\alpha}{2}} + |X-x-tV|^{\frac{\alpha}{3}} + |V-v|^{\alpha}$$

- homogeneous norm + intrinsic translations
- introduced by **Lanconelli-Polidoro** (1994)
- used by **Manfredini** (1997), **Di Francesco-P.** (2005), **Imbert-Mouhot** (2020) and many others
- joint space-time regularity: Schauder estimates
- **Pagliarani-P.-Pignotti** (2016): optimal definition of Hölder regularity and intrinsic Taylor formula
- restrictive in the stochastic framework: space-time regularity cannot be decoupled

So close, no matter how far

Points that are **far** from each other in the Euclidean sense, can be very **close** in the intrinsic sense:

$$P = \left(t, x, \frac{X - x}{t - T}\right), \qquad Q = \left(T, X, \frac{X - x}{t - T}\right)$$

have intrinsic distance equal to $|T - t|^{\frac{1}{2}}$ for any $x, X \in \mathbb{R}$

If $f = f(x)$ is Hölder in the intrinsic sense then

$$|f(x) - f(X)| \lesssim |t - T|^{\frac{\alpha}{2}}$$

f is necessarily constant

Intrinsic solution (P.-Pesce, 2021)

Let $\mathbf{Y} = v\partial_x + \partial_t$ and $\gamma_t(x,v)$ its integral curve from (x,v)

Langevin PDE	$\mathbf{Y}f = a\partial_{vv}f$
SPDE	$d_{\mathbf{Y}}f = a\partial_{vv}f\,dt + \sigma\partial_v f\,dW_t$

Solution: continuous process $f = f_t(x,v)$ s.t. $\partial_{vv}f$ exists and

$$f_t(\gamma_t(x,v)) = f_0(x,v) + \int_0^t a_s(\gamma_s(x,v))\partial_{vv}f_s(\gamma_s(x,v))ds + \int_0^t \sigma_s(\gamma_s(x,v))\partial_v f_s(\gamma_s(x,v))dW_s$$

Parametrix construction: main steps

- Itô-Wentzel: from SPDEs to random PDEs
- time-dependent parametrix:

  convergence as in **Delarue-Menozzi** (2010)

  - Fleming transform: a logarithmic transform of the remainder is seen as the value function of a stochastic optimization problem
  - stochastic control techniques

References

- P.-Pesce: The parametrix method for parabolic SPDEs. **Stochastic Process. Appl.** (2020)
- P.-Pesce: Backward and forward filtering under the weak Hormander condition. To appear in **Stoch. Partial Differ. Equ. Anal. Comput.** (2021)
- P.-Pesce: On stochastic Langevin and Fokker-Planck equations: the two-dimensional case. Preprint (2021)

Thank you!

Thank you!

General results for Kolmogorov PDEs

Kolmogorov equations with Hölder coefficients

$$L = \sum_{i,j=1}^{d} a_{ij}(t,x)\partial_{x_i x_j} + \underbrace{\langle Bx, \nabla \rangle + \partial_t}_{\mathbf{Y}} + \sum_{i=1}^{d} a_i(t,x)\partial_{x_i} + a(t,x)$$

- $(t, x) \in \mathbb{R} \times \mathbb{R}^N$
- $\mu^{-1} I_{\mathbb{R}^d} \leq (a_{ij}) \leq \mu I_{\mathbb{R}^d}$ intrinsic. Hölder continuous
- $K = \triangle_{\mathbb{R}^d} + \mathbf{Y}$ is hypoelliptic ($\mathcal{C} > 0$)
- a_i, a are bounded and intrinsic Hölder continuous

Parametrix method

- Existence of a fundamental solution
  Weber (1951), **Polidoro** (1995) **Di Francesco-P.** (2005)

- Gaussian **upper** bounds of Γ and derivatives:

$$\Gamma(t,x) \le \frac{C^+}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-c^+ \langle \mathcal{C}^{-1}(t)x, x\right)$$

$$\left|\partial_{x_i x_j}\Gamma(t,x)\right| + |\mathbf{Y}\Gamma(t,x)| \le \frac{C^+}{t\sqrt{\det \mathcal{C}(t)}} \exp\left(-c^+ \langle \mathcal{C}^{-1}(t)x, x\right)$$

- Harnack inequality, **lower** bounds: **Polidoro** (1997)

$$\Gamma(t,x) \ge \frac{C^-}{\sqrt{\det \mathcal{C}(t)}} \exp\left(-c^- \langle \mathcal{C}^{-1}(t)x, x\right)$$

- Well-posedness of the martingale problem and density estimates: **Delarue-Menozzi** (2010), **Menozzi** (2018)

Non-local case

The Brownian part can be replaced with a Lévy-type part

Imbert-Silvestre (2019)
Hao-Wu-Zhang (2020)
Marino (2021),
Manfredini-Pagliarani-Polidoro (2021)