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On positive recurrence of ergodic diffusions with switching

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S. Molchanov 80 & V. Konakov 75

Congratulations & many years of fruitful research to both!

It is unbelievable for me to hear these figures 80 and 75 above. Because I remember both jubilees much younger. I steel from my talk a few minutes to recall two episodes.

S.Molchanov was one of my tutors in my MSU studentship. In my very first International conference in Vilnius 1978 (S.A. was just about 37!) I gave a short talk. After my talk S.A. approached me and said: "Shame on you!" I did not understand: why, what I did wrong?! "You used Greeks in your talk, and Kiyoshy Ito who was sleeping in the first raw had to wake up each time you pronounced "eta" [i:ta]!

Once I was walking quietly in the centre near Notre Dame de Paris, and suddenly met Valentin ($\sim$50y.o.) who was also quietly walking. What did we do then? We stopped and we were talking, I think, for an hour if not more about maths.

Abstract; Svetlana Anulova

Diffusion & switching

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Positive recurrence of d-dimensional diffusion with an additive Wiener process and with switching and with one recurrent and one transient regime is established under suitable conditions on the drift in both regimes and on the intensities of switching. These intensities may depend on the diffusion component at each moment of time. The paper is a continuation of the earlier publication [Veretennikov A. (2021) On Positive Recurrence of One-Dimensional Diffusions with Independent Switching. In: Shiryaev A.N., Samouylov K.E., Kozyrev D.V. (eds) Recent Developments in Stochastic Methods and Applications. ICSM-5 2020. Springer Proceedings in Mathematics & Statistics, vol 371, 242 - 252. Springer, Cham] as well as of several earlier joint papers with S. Anulova. **The paper above is in memory of her.**

The setting

Let us consider the process (X_t, Z_t) with a continuous component $X \in \mathbb{R}^d$ and a discrete one $Z \in \{0, 1\}$ described by the stochastic differential equation

$$dX_t = b(X_t, Z_t)\,dt + dW_t, \quad t \geq 0, \quad X_0 = x,\ Z_0 = z, \tag{1}$$

for the component X, while Z_t is a continuous-time conditionally Markov process given X on the state space $S = \{0, 1\}$ with positive intensities of respective transitions $\lambda_{01}(x) =: \lambda_0(x)$, & $\lambda_{10}(x) =: \lambda_1(x)$; here the variable x signifies a certain (arbitrary Borel measurable) dependence on the component X; the trajectories of Z are assumed to be càdlàg; the probabilities of jumps for Z are conditionally independent given the trajectory of the component X (see the precise description in what follows).

b and λ

Denote

$$\begin{aligned} b(x,0) &= b_-(x), \quad & b(x,1) &= b_+(x), \\ \overline{\lambda}_0 &:= \sup_{x,z} \lambda_0(x), \quad & \underline{\lambda}_0 &:= \inf_{x,z} \lambda_0(x), \\ \overline{\lambda}_1 &:= \sup_{x,z} \lambda_1(x), \quad & \underline{\lambda}_1 &:= \inf_{x,z} \lambda_1(x). \end{aligned}$$

It is assumed that

$$0 < \underline{\lambda}_0 \wedge \underline{\lambda}_1 \le \overline{\lambda}_0 \vee \overline{\lambda}_1 < \infty. \tag{2}$$

These conditions along with the boundedness of the function b in x suffice for the process (X_t, Z_t) to be well-defined.

Rigorous construction

A rigorous construction of the system (X, Z) of this type may be given by the SDE system

$$dX_t = b(X_t, Z_t)\,dt + dW_t, \ \ t \ge 0, \ \ X_0 = x \in \mathbb{R}^d,$$

$$dZ_t = 1(Z_t = 0)d\pi^0_t - 1(Z_t = 1)d\pi^1_t, \ \ Z_0 \in \{0, 1\}, \qquad (3)$$

where π^i_t, $i = 0, 1$, are two Poisson processes with intensities $\lambda_i(X_t)$, $i = 0, 1$, respectively. More precisely,

$$\pi^i_t = \bar{\pi}^i_{\phi_i(t)},$$

where $\bar{\pi}^i_t, i = 0, 1$, are, in turn, two standard Poisson processes with a constant intensity one, independent of the Wiener process (W_t) and of each other, and the time changes

$$t \mapsto \phi_i(t) := \int_0^t \lambda_i(X_s)ds, \quad i = 0, 1,$$

are applied to each of them, respectively.

Strong solution exists

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By virtue of the assumption (2) the equation between the jumps only concerns the diffusion part of the SDE (3), for which it is well-known since 1979 that the equation has a pathwise unique strong solution. The jump moments are stopping times with respect to the filtration $(\mathcal{F}_t = \mathcal{F}_t^{W,\pi^0,\pi^1},\ t \geq 0)$, and the position of the system after any jump (X_τ, Z_τ) is uniquely determined by the left limiting values $(X_{\tau-}, Z_{\tau-})$: X is continuous, while Z changes its position,

$$X_\tau = X_{\tau-}, \quad Z_\tau = 1(Z_{\tau-} = 0).$$

Generator of Markov process

After any such jump, the diffusion part of the SDE is solved starting from the position X_τ until the next jump, say, τ', of the component Z, and the moment of this next jump is determined by the trajectories of π^0_t and (or) of π^1_t and by the intensity $\lambda_{Z_s}(X_s)$, $s < \tau'$. Since there might be only a finite numbers of jumps on any bounded interval of time, then pathwise (and, hence, also weak) uniqueness follows on $[0, \infty)$. Therefore, the process (X, Z) exists and is markovian. Its generator has a form

$$Lh(x,z) = \frac{1}{2}\Delta_{x,z}h(x,z) + b_z(x)\nabla_{x,z}h(x,z) \\ +\lambda_z(x)\left(h(x,\bar z) - h(x,z)\right),$$

where $\bar z := 1(z = 0)$ (that is, $\bar z$ is **not a** z, the **other state** from the set $\{0, 1\}$).

PDE system

For any $t > 0$ fixed let us define the function

$$v(s,x,z) := \mathbb{E}_{s,x,z} f(X_t, Z_t).$$

The vector-function $v(s,x) = (v(s,x,0), v(s,x,1))$ satisfies the system of PDEs

$$\begin{aligned} v_s(s,x,0) + L^0 v(s,x,0) + \lambda_0(x)\left(v(s,x,1) - v(s,x,0)\right) &= 0, \\ v(t,x,0) &= f(x,0), \\ v_s(s,x,1) + L^1 v(s,x,1) + \lambda_1(x)\left(v(s,x,0) - v(s,x,1)\right) &= 0, \\ v(t,x,1) &= f(x,1), \end{aligned}$$

where

$$L^i = \frac{1}{2}\Delta + \langle b(x,i), \nabla_{x,z} \rangle, \quad i = 0, 1.$$

Feller process; strong Markov property

Due to the classical results in [A.V.Solonnikov 1965, Theorem 5.5] its solution is continuous in the variable s for any bounded and continuous f. Hence, the process is Feller's (that is, $\mathbb{E}_{x,z}h(X_t)$ is continuous in x and, of course, bounded for any $h \in C_b$ and any $t > 0$). Since the process is Markov and càdlàg, then it is also strong Markov according to the Feller sufficient condition which guarantees that in this case any Markov process is also strong Markov. This is important for the rest of the talk because allows to use stopping times.

Actually, Solonnikov's results allow to use Ito-Krylov's formula applied to the solution with the process (X_t, Z_t) substituted in it, which is also of a great importance in principle; however, in this talk we will not use it.

Some references

The SDE solution is assumed ergodic under the regime $Z = 0$ and transient under $Z = 1$. We are looking for sufficient conditions for positive recurrence of the strong Markov process (X_t, Z_t). Such a problem was considered by Cloez, B. & Hairer, M. 2015 for the exponentially recurrent case; for other references see Khasminskii 2012, Mao, Yin & Yuan 2007, Shao & Yuan 2019, and the references therein. Under weak ergodic and transient conditions the setting was earlier investigated in A.V.2021 for the case of the constant intensities λ_0, λ_1. Here we tackle the general case, not assuming any continuity, where the lengths of intervals between successive jumps of the discrete component are not independent of W, and, hence, not independent of each other. This difficulty will be overcome with the help of certain comparison arguments.

Main theorem

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Theorem

Let the drift $b = (b_+, b_-)$ be bounded and Borel measurable, and let there exist $r_-, r_+, M > 0$ such that

$$0 < \underline{\lambda}_0 \wedge \underline{\lambda}_1 \le \bar{\lambda}_0 \vee \bar{\lambda}_1 < \infty, \tag{4}$$

$$\boxed{xb_-(x) \le -r_-, \quad xb_+(x) \le +r_+}, \quad \forall\, |x| \ge M, \tag{5}$$

$$2r_- > d \quad \& \quad \boxed{\underline{\lambda}_1(2r_- - d) > \bar{\lambda}_0(2r_+ + d)}. \tag{6}$$

Then the process (X, Z) is positive recurrent; moreover, there exists $C > 0$ such that for all M_1 large enough and all $x \in \mathbb{R}$ and for $z = 0, 1$

$$\mathbb{E}_{x,z}\tau_{M_1} \le C(x^2 + 1), \tag{7}$$

where $\boxed{\tau_{M_1} := \inf(t \ge 0 : \ |X_t| \le M_1).}$

Proof preliminaries: embedding Markov chain

by stopping (switching) times

Denote $\|b\| = \sup_{x,z} |b(x,z)|$. Let $M_1 \gg M$ (the value M_1 will be specified later). Let

$$T_0 := \inf(t \ge 0 : Z_t = 0),$$

and

$$0 \le T_0 < T_1 < T_2 < \ldots,$$

where for each $n > 1$ is defined by induction as

$$T_n := \inf(t > T_{n-1} : Z_{T_n} - Z_{T_n-} \neq 0).$$

Let

$$\tau := \inf(T_n \ge 0 : |X_{T_n}| \le M_1).$$

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(6) $\underline{\lambda}_1(2r_- - d) > \overline{\lambda}_0(2r_+ + d)$ implies $\exists \epsilon > 0$ s.t. $\underline{\lambda}_1(2r_- - d - \epsilon) > \overline{\lambda}_0(2r_+ + d + \epsilon)$; Lemma 1

To prove the theorem it suffices to evaluate from above the value $\mathbb{E}_{x,z}\tau$ because $\tau \geq \tau_{M_1}$. Let $\epsilon > 0$ be a positive value solving the equation

$$\overline{\lambda}_0(2r_+ + d + \epsilon) = q\underline{\lambda}_1(2r_- - d - \epsilon) \qquad (8)$$

with some $q < 1$ (see (6)). It suffices to assume $|x| > M$.

Lemma (1)

Under the assumptions of the theorem for any $\delta > 0$ there exists M_1 such that

$$\max\left[\sup_{|x|>M_1} \mathbb{E}_{x,z}\left(\int_0^{T_1} 1(\inf_{0\le s\le t}|X_s| \le M)dt | Z_0 = 0\right),\right.$$
$$\left.\sup_{|x|>M_1} \mathbb{E}_{x,z}\left(\int_0^{T_0} 1(\inf_{0\le s\le t}|X_s| \le M)dt | Z_0 = 1\right)\right] < \delta. \qquad (9)$$

Auxiliary diffusions (without switching)

We skip the proof of this lemma because, as I hope, its claim is intuitively evident. *(If not, it will be commented during the talk.)*

Let us denote by X_t^i, $i = 0, 1$ the solutions of the equations

$$dX_t^i = b(X_t^i, i)\, dt + dW_t, \quad t \geq 0, \quad X_0^i = x. \tag{10}$$

Lemma 2

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Lemma (2)

If M_1 is large enough, then under the assumptions of the theorem for any $|x| > M_1$ for any $k = 0, 1, \ldots$

$$\begin{aligned}
\mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) &\le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) \\
&-1(\tau > T_{2k})\mathbb{E}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau|Z_0 = 0, \mathcal{F}_{T_{2k}})((2r_- - d) - \epsilon) \\
&\le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k}}) - 1(\tau > T_{2k})\bar{\lambda}_0^{-1}((2r_- - d) - \epsilon),
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_{x,z}(X^2_{T_{2k+2}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) &\le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) \\
&+ 1(\tau > T_{2k+1})\mathbb{E}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) \\
&\times((2r_- + d) + \epsilon) \\
&\le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k+1}}) \\
&+1(\tau > T_{2k+1})\underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon).
\end{aligned}$$

Corollary 1

Corollary (1) (under the assumptions of the theorem)

If M_1 is large enough, then $\forall\,|x| > M_1$ *and* $\forall\,k = 0, 1, \ldots$

$$\begin{aligned}
&\mathbb{E}_{x,0} X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_{x,0} X^2_{T_{2k}\wedge\tau} \\
&\le -\mathbb{E}_{x,0} 1(\tau > T_{2k}) \mathbb{E}_{x,0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau | \mathcal{F}_{T_{2k}}) \\
&\qquad \times((2r_- - d) - \epsilon) \\
&= -\mathbb{E}_{x,0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau)((2r_- - d) - \epsilon) \\
&\le -\mathbb{E}_{x,0} 1(\tau > T_{2k}) \bar{\lambda}_0^{-1}((2r_- - d) - \epsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{x,1} X^2_{T_{2k+2}\wedge\tau} - \mathbb{E}_{x,1} X^2_{T_{2k+1}\wedge\tau} \\
&\le \mathbb{E}_{x,1} 1(\tau > T_{2k+1})(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)((2r_- + d) + \epsilon) \\
&= \mathbb{E}_{x,1}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)((2r_- + d) + \epsilon) \\
&\le \mathbb{E}_{x,1} 1(\tau > T_{2k+1}) \underline{\lambda}_1^{-1}((2r_+ + d) + \epsilon).
\end{aligned}$$

Lemma 3

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Lemma (3) (under the assumptions of the theorem)

If M_1 is large enough, then for any $k = 0, 1, \ldots$

$$\mathbb{E}_{x,z}(X^2_{T_{2k+2}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}}) \le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}})$$
$$+1(\tau > T_{2k+1})\mathbb{E}_{x,z}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau|Z_0 = 0, \mathcal{F}_{T_{2k+1}}))$$
$$\times((2r_+ + 1) + \epsilon))$$
$$\le \mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 0, \mathcal{F}_{T_{2k+1}}) + 1(\tau > T_{2k+1})\underline{\lambda}_1^{-1}$$
$$\times((2r_+ + 1) + \epsilon)),$$

and

$$\mathbb{E}_{x,z}(X^2_{T_{2k+1}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}}) \le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}})$$
$$+1(\tau > T_{2k})\mathbb{E}_{x,z}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau|Z_0 = 0, \mathcal{F}_{T_{2k}}))$$
$$\le \mathbb{E}_{x,z}(X^2_{T_{2k}\wedge\tau}|Z_0 = 1, \mathcal{F}_{T_{2k}}) - 1(\tau > T_{2k})\overline{\lambda}_0^{-1}((2r_- - 1) - \epsilon)).$$

Corollary 2

Corollary (2) under the assumptions of the theorem)

If M_1 is large enough, then for any $k = 0, 1, \ldots$

$$
\begin{aligned}
&\mathbb{E}_{x,0} X^2_{T_{2k+2}\wedge\tau} - \mathbb{E}_{x,0} X^2_{T_{2k+1}\wedge\tau} \\
&\le \mathbb{E}_{x,0} 1(\tau > T_{2k+1})(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)((2r_+ + 1) + \epsilon) \\
&\le \mathbb{E}_{x,0} 1(\tau > T_{2k+1}) \underline{\lambda}_1^{-1}((2r_+ + 1) + \epsilon)),
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_{x,1} X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_{x,1} X^2_{T_{2k}\wedge\tau} \\
&\le \mathbb{E}_{x,1} 1(\tau > T_{2k})(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau) \\
&\le -\mathbb{E}_{x,1} 1(\tau > T_{2k}) \overline{\lambda}_0^{-1}((2r_- - 1) - \epsilon)).
\end{aligned}
$$

Proof of the theorem, idea

The idea based on the lemmata and two corollaries is as follows:

- Instead of $\mathbb{E}\tau_{M_1}$ for $\tau_{M_1} := \inf(t \ge 0 : |X_t| \le M_1)$ we evaluate $\mathbb{E}\tau$ for $\tau := \inf(T_n \ge 0 : |X_{T_n}| \le M_1)$;
- $$1(\tau > T_{2k}) \ge 1(\tau > T_{2k+1});$$
- According to the lemmata and their corollaries, on average the increase of $\mathbb{E}X^2_{T_n}$ on the next step is dominated by its decrease on the previous one;
- Hence, $\mathbb{E}\tau$ is estimated from above by X_0^2 by virtue of the standard technique (which will be shown on the next slides after the proofs of the lemmata);
- The next (and the last) lemma in this talk is the last preparation to the proof of the theorem.

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Lemma 4

Lemma (4) (under the assumptions of the theorem)

For any $k = 0, 1, \dots$

$$1(\tau > T_{2k+1})\mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$\le 1(\tau > T_{2k+1})\underline{\lambda}_1^{-1},$$

and

$$1(\tau > T_{2k})\mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$\ge 1(\tau > T_{2k})\bar{\lambda}_0^{-1}.$$

Corollary 3

Corollary (3): under the assumptions of the theorem)

For any $k = 0, 1, \dots$

$$\underline{\lambda}_1 \mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \le 1,$$

and

$$\bar{\lambda}_0 \mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \ge 1.$$

In particular,

$$\underline{\lambda}_1 \mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \le \bar{\lambda}_0 \mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

or

$$\boxed{\mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \le \frac{\bar{\lambda}_0}{\underline{\lambda}_1} \mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)}$$

Proof of lemma 2

(I will only show the beginning of the proof; lemma 3 is established similarly)

1. If $Z_0 = 0$ then $T_0 = 0$, and the process X_t coincides with X_t^0 until T_1. Hence, we have on $[0, T_1]$ by Ito's formula

$$dX_t^2 - 2X_t dW_t = (2X_t b_-(X_t) + d)\,dt \le (-2r_- + d)dt,$$

on the set $(|X_t| > M)$ due to the assumptions (5). We get

$$\int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt = \int_0^{T_1\wedge\tau} 2X_t b_-(X_t) 1(|X_t| > M)dt + \int_0^{T_1\wedge\tau} 2X_t b_-(X_t) 1(|X_t| \le M)dt$$

$$\le -2r_- \int_0^{T_1\wedge\tau} 1(|X_t| > M)dt + \int_0^{T_1\wedge\tau} 2M\|b\| 1(|X_t| \le M)dt$$

$$= -2r_- \int_0^{T_1\wedge\tau} 1dt + \int_0^{T_1\wedge\tau} (2M\|b\| + 2r_-) 1(|X_t| \le M)dt$$

$$\le -2r_- \int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-) \int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt.$$

Proof of lemma 2, ctd

Thus, always for $|x| > M_1$,

$$\mathbb{E}_{x,z}\int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt \le -2r_-\mathbb{E}_{x,z}\int_0^{T_1\wedge\tau} 1dt$$

$$+(2M\|b\| + 2r_-)\mathbb{E}_{x,z}\int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt$$

$$= (2M\|b\| + 2r_-)\mathbb{E}_{x,z}\int_0^{T_1\wedge\tau} 1(|X_t| \le M)dt$$

$$-2r_-\mathbb{E}\int_0^{T_1\wedge\tau} 1dt \le -2r_-\mathbb{E}\int_0^{T_1\wedge\tau} 1dt$$

$$+(2M\|b\| + 2r_-)\mathbb{E}_{x,z}\int_0^{T_1} 1(|X_t| \le M)dt$$

$$\le -2r_-\mathbb{E}\int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\delta.$$

(Repeat the latter inequality)

$\mathbb{E}_{x,z}\int_0^{T_1\wedge\tau} 2X_t b_-(X_t)dt \le -2r_- E\int_0^{T_1\wedge\tau} 1dt + (2M\|b\| + 2r_-)\delta$

For our fixed $\epsilon > 0$ let us choose $\delta = \bar\lambda_0^{-1}\epsilon/(2M\|b\| + 2r_-)$. Then, since $|x| > M_1$ implies $T_1 \wedge \tau = T_1$ on $(Z_0 = 0)$, and since

$$\bar\lambda_0^{-1} \le \mathbb{E}_{x,0} T_1 \le \underline{\lambda}_0^{-1}, \qquad (11)$$

we get with $z = 0$

$$\mathbb{E}_{x,z}X^2_{T_1\wedge\tau} - x^2 \le -(2r_- - d)\mathbb{E}_{x,z}\int_0^{T_1} dt + \bar\lambda_0^{-1}\epsilon$$

$$= -(2r_- - d)\mathbb{E}_{x,z}T_1 + \bar\lambda_0^{-1}\epsilon \overset{!}{\le} -\bar\lambda_0^{-1}((2r_- - d) - \epsilon).$$

Substituting here x by $X_{T_{2k}}$ and writing $\mathbb{E}_{x,z}(\cdot|\mathcal{F}_{T_{2k}})$ instead of $\mathbb{E}_{x,z}(\cdot)$, and multiplying by $1(\tau > T_{2k})$, we obtain the first two bounds of the lemma 2, as required. *The other two bounds of the lemma can be established similarly.* QED

The double bound (11) explained

Note that the bound (11) itself follows straightforwardly from

$$\mathbb{E}_{x,0} T_1 = \int_0^\infty \mathbb{P}_{x,0}(T_1 \geq t) dt = \int_0^\infty \mathbb{E}_{x,0} \mathbb{P}_{x,0}(T_1 \geq t | \mathcal{F}_t^{X^0}) dt$$

$$= \mathbb{E}_{x,0} \int_0^\infty \exp(-\int_0^t \lambda_0(X_s^0) ds) dt \leq \int_0^\infty \exp(-\int_0^t \underline{\lambda}_0 ds) dt$$

$$= \int_0^\infty \exp(-t\underline{\lambda}_0) dt = \underline{\lambda}_0^{-1},$$

and similarly

$$\mathbb{E}_{x,0} T_1 = \int_0^\infty \mathbb{E}_{x,0} \exp(-\int_0^t \lambda_0(X_s^0) ds) dt$$

$$\geq \int_0^\infty \exp(-\int_0^t \overline{\lambda}_0 ds) dt = \int_0^\infty \exp(-t\overline{\lambda}_0) dt = \overline{\lambda}_0^{-1}.$$

Proof of lemma 4

Both corollaries follow from the corresponding lemmata just by taking expectations.

Proof of lemma 4
On the set $\tau > T_{2k+1}$ we have,

$$\mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$
$$= \mathbb{E}_{X_{T_{2k+1}},1}(T_{2k+2} - T_{2k+1}) \in [\bar{\lambda}_1^{-1}, \underline{\lambda}_1^{-1}].$$

Similarly, on the set $\tau > T_{2k}$

$$\mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$
$$= \mathbb{E}_{X_{T_{2k}},0}(T_{2k+1} - T_{2k}) \in [\bar{\lambda}_0^{-1}, \underline{\lambda}_0^{-1}].$$

Lemma 4 follows. QED

Proof of theorem (now the real start)

Consider the case $Z_0 = 0$ where $T_0 = 0$. We have, since $T_n \uparrow \infty$ a.s.

$$\tau \wedge T_n = T_0 + \sum_{m=0}^{n-1} ((T_{m+1} \wedge \tau) - (T_m \wedge \tau))$$

and

$$\mathbb{E}_{x,z}(\tau \wedge T_n) = \mathbb{E}_{x,z} T_0 + \mathbb{E}_{x,z} \sum_{m=0}^{n-1} ((T_{m+1} \wedge \tau) - (T_m \wedge \tau)),$$

As $T_n \uparrow \infty$, we get by the monotone convergence theorem

$$\begin{aligned} \mathbb{E}_{x,z}\tau &= \mathbb{E}_{x,z} T_0 + \sum_{m=0}^{\infty} \mathbb{E}_{x,z}((T_{m+1} \wedge \tau) - (T_m \wedge \tau)) \\ &= \mathbb{E}_{x,z} T_0 + \sum_{k=0}^{\infty} \mathbb{E}_{x,z}((T_{2k+1} \wedge \tau) - (T_{2k} \wedge \tau)) \qquad (12) \\ &\quad + \sum_{k=0}^{\infty} \mathbb{E}_{x,z}((T_{2k+2} \wedge \tau) - (T_{2k+1} \wedge \tau)). \end{aligned}$$

According to the corollary 1,

$$\mathbb{E}_{x,0} X^2_{T_{2k+1}\wedge\tau} - \mathbb{E}_{x,0} X^2_{T_{2k}\wedge\tau}$$

$$\le -((2r_- - d) - \epsilon)\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau),$$

and due to the corollary 2,

$$\mathbb{E}_{x,0} X^2_{T_{2k+2}\wedge\tau} - \mathbb{E}_{x,0} X^2_{T_{2k+1}\wedge\tau}$$

$$\le ((2r_+ + 1) + \epsilon)\mathbb{E}_{x,0} 1(\tau > T_{2k+1})(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

So, summing up from zero to m, we estimate (on the next slide)

$$\mathbb{E}_{x,0} X^2_{T_{2m+2}\wedge\tau} - x^2$$

$$\leq ((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)$$

$$-((2r_- - d) - \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau)$$

$$= \sum_{k=0}^{m} \Big(-((2r_- - d) - \epsilon)\mathbb{E}_{x,0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau)$$

$$+((2r_+ + d) + \epsilon)\mathbb{E}_{x,0}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)\Big).$$

$$\mathbb{E}_{x,0} X^2_{T_{2m+2}\wedge\tau} - x^2$$

$$\leq \sum_{k=0}^{m} \left(-((2r_- - d) - \epsilon)\mathbb{E}_{x,0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau)\right.$$

$$\left. +((2r_+ + d) + \epsilon)\mathbb{E}_{x,0}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau)\right),$$

or, changing the lhs and rhs and dropping $-\mathbb{E}_{x,0} X^2_{T_{2m+2}\wedge\tau}$,

$$x^2 \geq ((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau) \tag{13}$$

$$-((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau).$$

Ctd

Now we show that the term $+\sum$ here dominates the negative term $-\sum$

$$1(\tau > T_{2k+1}) \le 1(\tau > T_{2k}) \Longrightarrow \mathbb{P}_{x,0}(\tau > T_{2k+1}) \le \mathbb{P}_{x,0}(\tau > T_{2k});$$

$$\overline{\lambda}_0 \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) - \underline{\lambda}_1 \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$= \overline{\lambda}_0 \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) 1(\tau \ge T_{2k})$$

$$-\underline{\lambda}_1 \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) 1(\tau \ge T_{2k+1})$$

$$= \overline{\lambda}_0 \mathbb{E}_{x,0} 1(\tau > T_{2k}) \mathbb{E}_{X_{T_{2k}}}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$-\underline{\lambda}_1 \mathbb{E}_{x,0} 1(\tau > T_{2k+1}) \mathbb{E}_{X_{T_{2k+1}}}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$\ge \overline{\lambda}_0 \mathbb{E}_{x,0} 1(\tau > T_{2k}) \overline{\lambda}_0^{-1} - \underline{\lambda}_1 \mathbb{E}_{x,0} 1(\tau > T_{2k+1}) \underline{\lambda}_1^{-1} \ge 0$$

Therefore, we estimate

$$((2r_+ + d) + \epsilon)\sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$\leq ((2r_+ + d) + \epsilon)\frac{\overline{\lambda}_0}{\underline{\lambda}_1}\sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$$

$$= q((2r_- - d) - \epsilon)\sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau).$$

By Corollary 3

$$\mathbb{E}_{x,z}(T_{2k+2}\wedge\tau - T_{2k+1}\wedge\tau) \leq \frac{\overline{\lambda}_0}{\underline{\lambda}_1}\mathbb{E}_{x,z}(T_{2k+1}\wedge\tau - T_{2k}\wedge\tau) \quad (14)$$

Return to (13):

$x^2 \geq ((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau)$
$-((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$

Due to (8) $[\overline{\lambda}_0(2r_+ + d + \epsilon) = q\underline{\lambda}_1(2r_- - d - \epsilon)$ with $q < 1]$ we estimate the absolute value of the negative term in (13) from above:

$$((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau)$$

$$\stackrel{(14)}{\leq} ((2r_+ + d) + \epsilon) \frac{\overline{\lambda}_0}{\underline{\lambda}_1} \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \qquad (15)$$

$$\stackrel{(8)}{=} q((2r_- - d) - \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau).$$

So, due to the balance condition

(recall (8): $\overline{\lambda}_0(2r_+ + d + \epsilon) = q\underline{\lambda}_1(2r_- - d - \epsilon)$)

$$
\begin{aligned}
x^2 &\overset{(13)}{\geq} ((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\quad -((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau) \\
&\overset{(15)}{\geq} (1-q)((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\overset{(8)}{\geq} \frac{1-q}{2}((2r_- - d) - \epsilon) \sum_{k=0}^{m} (\mathbb{E}_{x,0}(T_{2k+1} \wedge \tau - T_{2k} \wedge \tau) \\
&\quad + \frac{1-q}{2q}((2r_+ + d) + \epsilon) \sum_{k=0}^{m} \mathbb{E}_{x,0}(T_{2k+2} \wedge \tau - T_{2k+1} \wedge \tau).
\end{aligned}
$$

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Denoting $c := \min\left(\frac{1-q}{2q}((2r_+ + d)+\epsilon), \frac{1-q}{2}((2r_- - d)-\epsilon)\right)$, we conclude that

$$x^2 \geq c\sum_{k=0}^{2m}\mathbb{E}_{x,0}(T_{k+1}\wedge\tau - T_k\wedge\tau).$$

So, as $m\uparrow\infty$, we get

$$\sum_{k=0}^{\infty}\mathbb{E}_{x,0}(T_{k+1}\wedge\tau - T_k\wedge\tau) \leq c^{-1}x^2.$$

Due to (12) (with $\mathbb{E}_{x,0}T_0 = 0$), this implies that

$$\mathbb{E}_{x,0}\tau \leq c^{-1}x^2, \tag{16}$$

as required. For $z = 1$ the proof is likewise, cf. (12). QED

Further research

The next step would be to allow a multiplicative Wiener process in the SDequation:

$$dX_t = b(X_t, Z_t)dt + \sigma(X_t, Z_t)dW_t,$$

with similar intensities for the jumps of the component Z,

$$\lambda_z(x), \; z = 1, 2.$$

There should be no big new difficulties, at least, for uniformly nondegenerate bounded $\sigma\sigma^*$; however, it should be carefully written to make sure.

Thanks

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Thanks

80 & 75

Again congratulations to Stanislav Molchanov and Valenetin Konakov