Optimal Stopping in a Simple Model of Unemployment Insurance

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1. Model of Unemployment Insurance

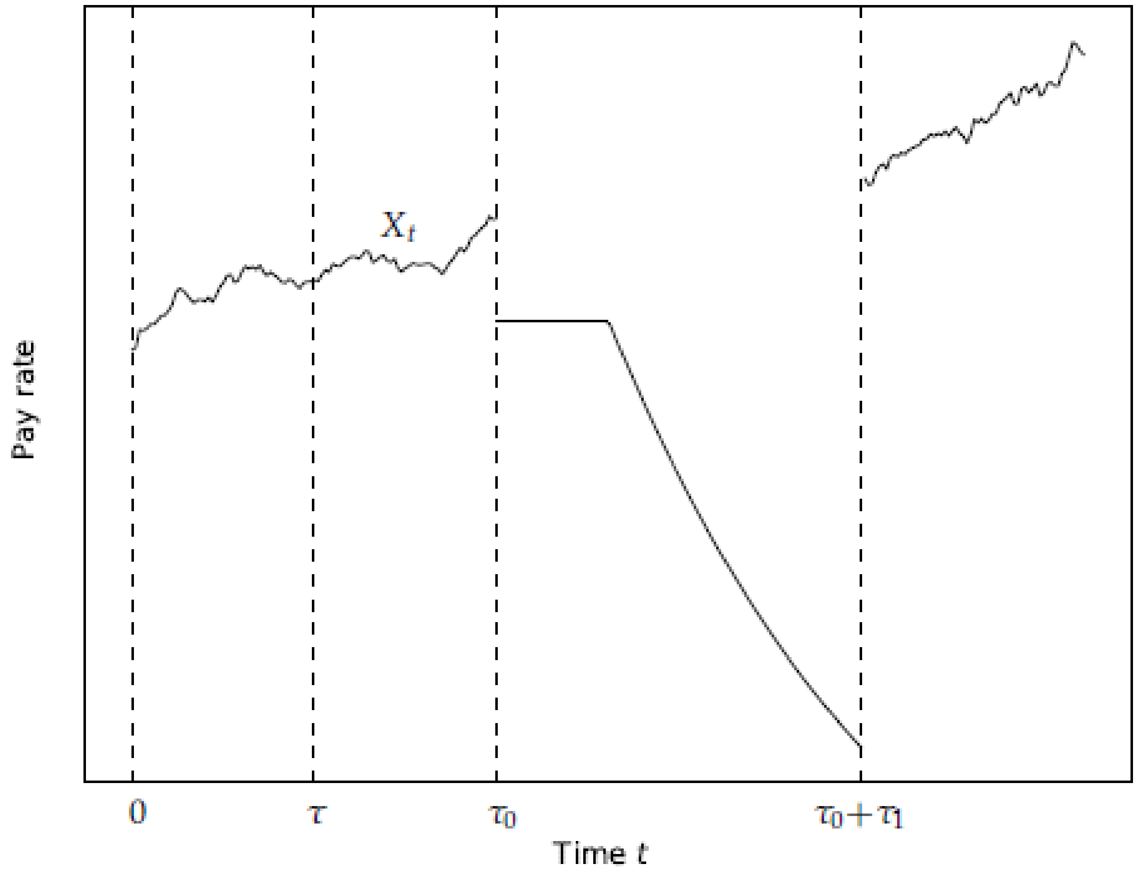
Unemployment Insurance (UI): an employed individual is covered against the risk of involuntary unemployment (e.g., redundancy).

Stylized UI model:

John is employed (courier, postman, waiter, shopping assistant) but he is concerned about losing the job.

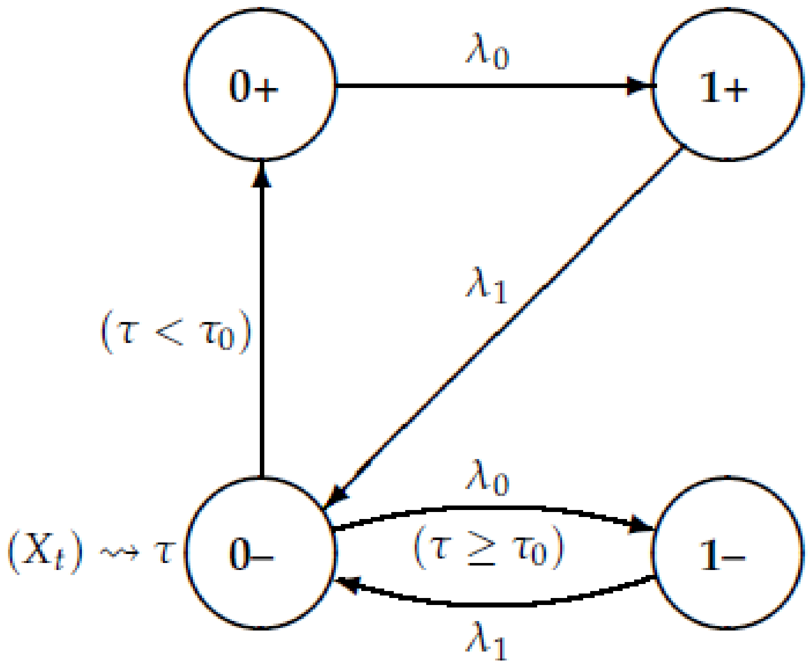
The employer / social services may have a UI scheme in place, e.g.

- one-off entry premium
- a benefit payment prop. to final wage, until a new job is found.



Notation and assumptions:

- $t = 0$: start of employment
- X_t: wage process
- τ: entry to UI (premium P)
- τ_0: loss of job
- $h(s)$: benefit schedule
- τ_1: time until new job



Specifically:

$$\tau_0 \sim \mathrm{Exp}(\lambda_0),\ \ \tau_1 \sim \mathrm{Exp}(\lambda_1) \quad \text{(independent)}$$

$$X_t\text{: GBM}(\mu, \sigma^2)\text{:} \quad \frac{\mathrm{d}X_t}{X_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \quad X_0 = x$$

$$X_t = x\exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma B_t\right\} \qquad (t \geq 0)$$

Legend:
0 (employed)
1 (unemployed)
+ (with insurance)
– (without insurance)

2. Optimal Stopping Problem

The question for John is **when** (rather than **if**) to join the scheme, i.e. how to choose an **optimal** entry time τ. His considerations:

- Delaying the entry makes sense due to inflation (because P is fixed).
- The wage is likely to grow with time (inflation + reward for experience).
- Some savings may be needed before the entry premium is affordable.
- On the other hand, delaying the decision is risky, as John may lose the job before entering the UI scheme ($\tau_0 < \tau$).

So there is a scope for optimizing the entry time — not too early but also not too late.

The decision should be based on observations of the wage process (X_t).

The target functional to maximize is **the expected future benefit.** Conditional on the final wage X_{τ_0} this is given by

$$X_{\tau_0}\,\mathrm{E}\left(\int_0^{\tau_1} \mathrm{e}^{-rs}\,h(s)\,\mathrm{d}s\right) = \beta X_{\tau_0}, \qquad r = \text{inflation rate},$$

$$\beta := \int_0^{\infty} \lambda_1 \mathrm{e}^{-\lambda_1 t} H(t)\,\mathrm{d}t, \qquad H(t) := \int_0^{t} \mathrm{e}^{-rs}\,h(s)\,\mathrm{d}s.$$

if the contract is entered immediately then the **net expected benefit** discounted to the entry time $t = 0$ is given by the **gain function**

$$g(x) := \mathrm{E}_x\left(\mathrm{e}^{-r\tau_0}\beta X_{\tau_0}\right) - P, \qquad x = X_0$$

Expectation is easy to compute using that $\mathrm{E}_x(X_t) = x\,\mathrm{e}^{\mu t}$:

$$\mathrm{E}_x\left(\mathrm{e}^{-r\tau_0} X_{\tau_0}\right) = \mathrm{E}_x\left(\mathrm{e}^{-r\tau_0}\,\mathrm{E}_x(X_{\tau_0} \mid \tau_0)\right)$$
$$= \mathrm{E}_x\left(\mathrm{e}^{-r\tau_0}(x\mathrm{e}^{\mu\tau_0})\right) = x\int_0^\infty \mathrm{e}^{(\mu-r)t}\,\lambda_0\,\mathrm{e}^{-\lambda_0 t}\,\mathrm{d}t = \frac{\lambda_0 x}{r+\lambda_0-\mu}.$$

Denoting $\tilde{r} := r + \lambda_0,\ \beta_1 := \dfrac{\beta\lambda_0}{\tilde{r}-\mu},$ the gain function is expressed as

$$g(x) = \beta_1 x - P. \tag{1}$$

Of course, this computation is meaningful as long as $\mu < r + \lambda_0 = \tilde{r}$.

This condition is assumed throughout (it is quite natural and realistic).

Now consider a delayed entry time $\tau > 0$ (tacitly assuming that $\tau < \infty$). Discounting first to τ (when the premium P is payable) and then to $t = 0$ yields the **expected net present value** of the total gain,

$$\mathrm{eNPV}(x;\tau) := \mathrm{E}_x\left[\mathrm{e}^{-r\tau}\left(\mathrm{e}^{-r(\tau_0-\tau)}\beta X_{\tau_0} - P\right)\mathbb{1}_{\{\tau<\tau_0\}}\right] \qquad (2)$$

Indicator in (2): entry time τ must occur prior to τ_0 (otherwise no gain).

Formally, τ in (2) has a finite (random) expiry date τ_0, but expectation involves averaging w.r.t. τ_0, so (2) can be rewritten as a **perpetual option**:

Lemma 1. $$\mathrm{eNPV}(x;\tau) = \mathrm{E}_x\left[\mathrm{e}^{-\tilde{r}\tau} g(X_\tau)\,\mathbb{1}_{\{\tau<\infty\}}\right] \qquad (3)$$

where $g(x) = \beta_1 x - P$ *(see (1)) and* $\tilde{r} = r + \lambda_0$.

Proof. Conditioning on τ (restricted to $\tau < \tau_0$), use $\tau_0 - \tau \sim \mathrm{Exp}(\lambda_0)$ and *strong Markov property* of (X_t). Note: $\tau = \infty$ does not contribute to (3).

To summarize, the optimal entry time $\tau = \tau^*$should maximize eNPV(*x;*τ), that is, it solves the following optimal stopping problem,

$$v(x) = \sup_{\tau} \mathrm{E}_x \left(\mathrm{e}^{-\tilde{r}\tau} g(X_\tau) \right), \tag{4}$$

where sup is taken over all stopping times adapted to $\mathcal{F}_t = \sigma(X_s, s \leq t)$.

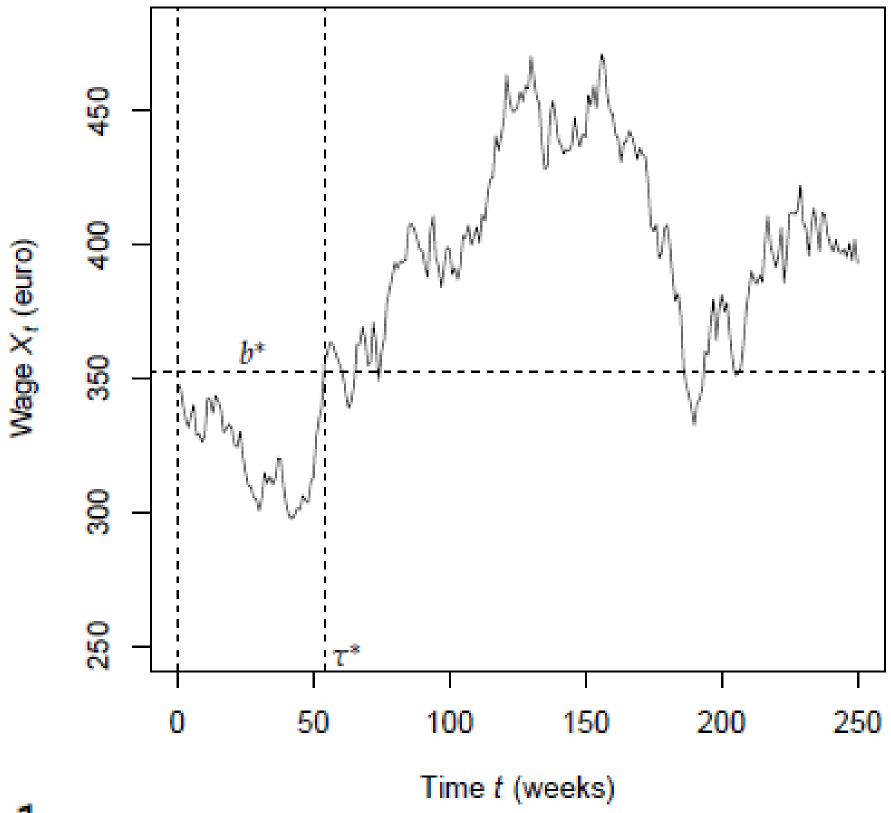
$v(x)$ in (4) is called the **value function** of the optimal stopping problem.

3. Optimal Stopping Rule

For the wage process (X_t), consider the hitting time τ_b of threshold $b \in \mathbb{R}$:

$$\tau_b := \inf\{t \geq 0 \colon X_t \geq b\} \in [0, \infty]. \qquad \inf \varnothing = \infty.$$

Since (X_t) is continuous, $X_{\tau_b} = b$ (P_x-a.s.).



Stopping Rule:

$$\tau^* = \begin{cases} \tau_{b^*} & \text{if } x \in [0, b^*], \\ 0 & \text{if } x \in [b^*, \infty). \end{cases}$$

$$b^* = \frac{Pq_*}{\beta_1(q_* - 1)},$$

$$q_* = \frac{1}{\sigma^2}\left(-(\mu - \tfrac{1}{2}\sigma^2) + \sqrt{(\mu - \tfrac{1}{2}\sigma^2)^2 + 2\tilde{r}\sigma^2}\right) > 1$$

The corresponding value function:

$$v(x) = \begin{cases} (\beta_1 b^* - P)\left(\frac{x}{b^*}\right)^{q_*}, & x \in [0, b^*], \\ \beta_1 x - P, & x \in [b^*, \infty). \end{cases}$$

When is $\tau_{b^*} < \infty$? For if $\tau_{b^*} = \infty$, then John will never get a UI policy!

$$\mathrm{P}_x(\tau_b < \infty) = \begin{cases} 1, & \mu - \frac{1}{2}\sigma^2 \geq 0, \\ \left(\frac{x}{b}\right)^{1-2\mu/\sigma^2}, & \mu - \frac{1}{2}\sigma^2 < 0. \end{cases}$$

$$\mathrm{E}_x(\tau_b) = \begin{cases} \infty, & \mu = \frac{1}{2}\sigma^2, \\ \frac{\ln(b/x)}{\mu - \frac{1}{2}\sigma^2}, & \mu > \frac{1}{2}\sigma^2. \end{cases}$$

4. Sketch of Proof

Need to find the value function $v(x)$ and identify the maximizer $\tau = \tau^*$.

Standard approach: **guess** the solution and then **verify** it is correct.

4.1. Guessing the Solution

Recall from (4) and (1): $v(x) = \sup_{\tau} \mathrm{E}_x \left(\mathrm{e}^{-r\tau} g(X_\tau) \right), \;\; g(x) = \beta_1 x - P.$

Picking $\tau = 0$ yields the lower estimate $v(x) \geq g(x)$.

Clearly, if $v(x) > g(x)$ then keep waiting; if $v(x) = g(x)$ then stop.

This motivates defining two regions, “continuation” and “stopping”:

$$\mathrm{C} := \{x \geq 0 \colon v(x) > g(x)\}, \qquad S := \{x \geq 0 \colon v(x) \leq g(x)\}.$$

By the Markov property, the same argument can be propagated to any time $t > 0$, provided that stopping has not yet occurred (i.e. $\tau \leq t$).

Thus, a plausible strategy: continue as long as $X_t \in C$ (i.e. $v(X_t) > g(X_t)$), but stop as soon as $X_t \in S$ (i.e. $v(X_t) \leq g(X_t)$). That is,

$$\tau^* = \inf\{t \geq 0 \colon X_t \in S\} = \inf\{t \geq 0 \colon v(X_t) \leq g(X_t)\} \in [0, \infty].$$

Furthermore, it is natural to hypothesize that

$$C = [0, b^*), \qquad S = [b^*, \infty).$$

This leads to a reduced optimal stopping problem over **hitting times**,

$$u(x) = \sup_{b \geq 0} \mathrm{E}_x\left(e^{-r\tau_b} g(X_{\tau_b})\right), \qquad \tau^* = \tau_{b^*} = \inf\{t \geq 0 \colon X_t \geq b^*\}$$

4.2. Free Boundary Problem

According to general theory, $u(x)$ is **harmonic** with respect to the process $\tilde{X}_t$ obtained from X_t by **killing** (discounting) with rate $\tilde{r} = r + \lambda_0$:

$$\mathrm{E}_x\left[\mathrm{e}^{-\tilde{r}(\tau_b \wedge t)} u(X_{\tau_b \wedge t})\right] = u(x) \quad (t \geq 0).$$

GBM X_t is a diffusion process with infinitesimal generator

$$L := \mu x \frac{\mathrm{d}}{\mathrm{d}x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} \quad (x > 0).$$

The generator of the killed process $\tilde{X}_t$ is then given by

$$\tilde{L} = L - \tilde{r} I, \qquad I = \text{identity operator.}$$

Thus, we arrive at the following free-boundary problem:

$$\begin{cases} Lu(x) - \tilde{r}u(x) = 0, \quad x \in (0,b), & \text{[harmonicity in } C\text{]} \\ u(b) = g(b), & \text{[stopping rule]} \\ u'(b) = g'(b), & \text{[“smooth fit principle”]} \\ u(0+) = 0, & [v(0) = 0] \end{cases}$$

(both $b > 0$ and $u(x)$ are unknown). Explicitly:

$$\begin{cases} \mu x u'(x) + \frac{1}{2}\sigma^2 x^2 u''(x) - \tilde{r}u(x) = 0, \quad x \in (0,b), \\ u(b) = \beta_1 b - P, \\ u'(b) = \beta_1, \\ u(0+) = 0. \end{cases}$$

Look for a solution in the form $u(x) = x^q \ (x > 0)$, hence

$$\tfrac{1}{2}\sigma^2 q(q-1) + \mu q - \tilde{r} = 0.$$

Solving and using the boundary / initial conditions yields

$$u(x) = \begin{cases} (\beta_1 b - P)\left(\dfrac{x}{b}\right)^{q_*}, & x \in [0, b], \\ \beta_1 x - P, & x \in [b, \infty), \end{cases}$$

where (in accord with Section 3)

$$q_* = \frac{1}{\sigma^2}\left(-(\mu - \tfrac{1}{2}\sigma^2) + \sqrt{(\mu - \tfrac{1}{2}\sigma^2)^2 + 2\tilde{r}\sigma^2}\right) > 1,$$

$$b = \frac{P q_*}{\beta_1 (q_* - 1)}$$

4.3. Verification

(i) $u(x) \geq v(x)$

By the *Itô formula* applied to $e^{-\tilde{r}t}u(X_t)$ we obtain

$$e^{-\tilde{r}t}u(X_t) = u(x) + \int_0^t e^{-\tilde{r}s}\big(Lu(X_s) - \tilde{r}u(X_s)\big)\,ds + M_t \quad (P_x\text{-a.s.}),$$

where

$$M_t := \int_0^t e^{-\tilde{r}s}u'(X_s)\,\sigma X_s\,dB_s$$

Strictly speaking, this requires that the function $u(x)$ is C^2, which breaks down at $x = b$, where it is only C^1. But $u(x)$ is strictly convex on $(0, b)$ and linear on (b, ∞), and in this situation the *Itô–Meyer formula* ensures that this representation is valid.

Now, by construction

$$Lu(x) - \tilde{r}u(x) = 0, \quad x \in (0, b).$$

Moreover, one can check that this equality extends to $x = b$.

On the other hand, for $x > b$ we have $u(x) = \beta_1 x - P$, so

$$\begin{aligned} Lu(x) - \tilde{r}u(x) &= \mu\beta_1 x - \tilde{r}(\beta_1 x - P) \\ &= \beta_1 x(\mu - \tilde{r}) + \tilde{r}P \\ &< \beta_1 b(\mu - \tilde{r}) + \tilde{r}P \\ &= \frac{P(\mu q_* - \tilde{r})}{q_* - 1} < 0, \end{aligned}$$

recalling the equation $\frac{1}{2}\sigma^2 q(q-1) + \mu q - \tilde{r} = 0$ and using that $q_* > 1$.

Thus, $Lu(x) - \tilde{r}u(x) \leq 0 \quad (x > 0)$.

Substituting into the Itô–Meyer formula:

$$u(x) + M_t \geq \mathrm{e}^{-\tilde{r}t} u(X_t) \quad (\mathrm{P}_x\text{-a.s.}).$$

Using this at a stopping time τ (so that $u(X_\tau) = g(X_\tau)$) and applying *Doob's optional sampling theorem* yields

$$u(x) \geq \mathrm{E}_x\left(\mathrm{e}^{-\tilde{r}\tau} g(X_\tau)\right)$$

and therefore

$$u(x) \geq \sup_\tau \mathrm{E}_x\left(\mathrm{e}^{-\tilde{r}\tau} g(X_\tau)\right) = v(x) \quad (x > 0)$$

[More precisely, (M_t) is a (continuous) *local martingale*, so we should use a localizing sequence of bounded stopping times $\tau_n \uparrow \infty$ such that the stopped process $(M_{\tau_n \wedge t})$ is a martingale.]

(ii) $u(x) \le v(x)$

For $x \in [b, +\infty)$ we already have $u(x) = g(x) \le v(x)$..

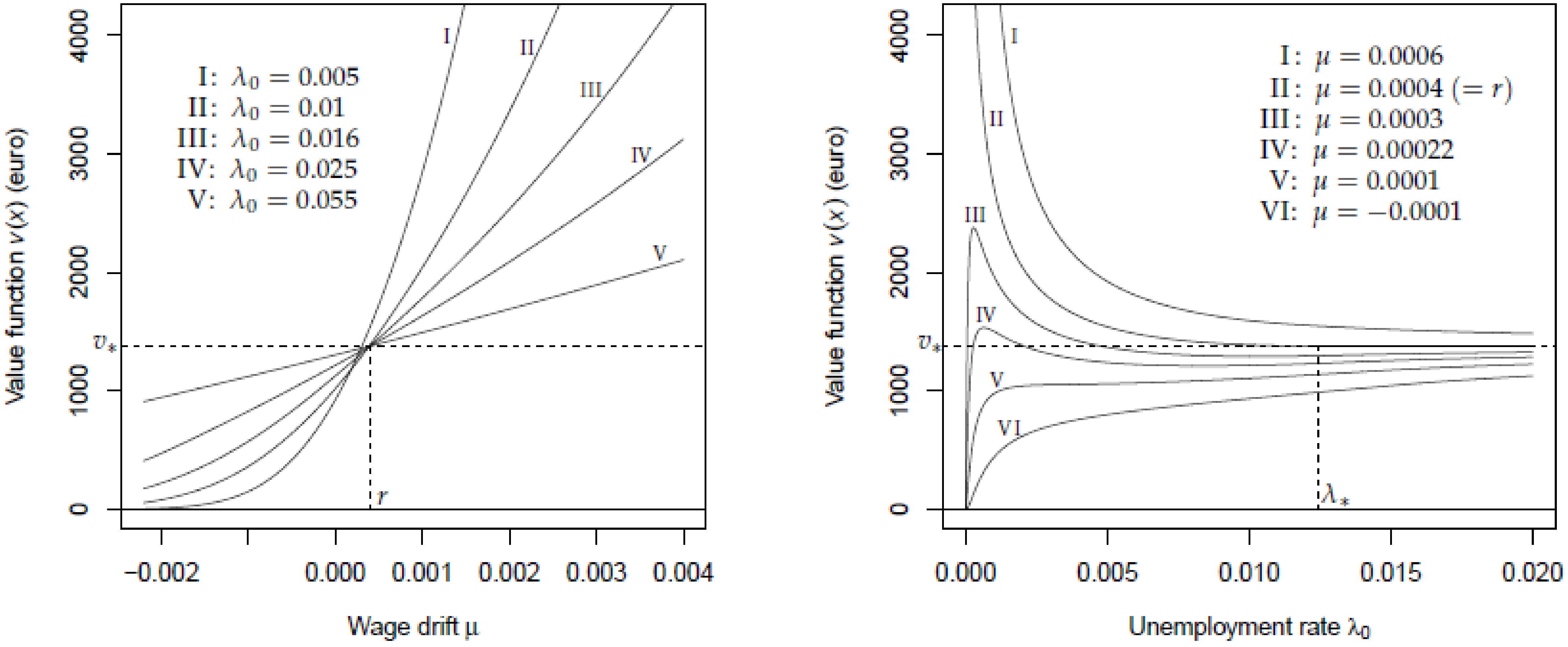
Fix $x \in (0, b)$ and consider the Itô–Meyer formula with t replaced by $\tau_n \wedge \tau_b$, where (τ_n) is the localizing sequence as before. Then

$$u(x) + M_{\tau_n \wedge \tau} = \mathrm{e}^{-\tilde{r}(\tau_n \wedge \tau_b)} u(X_{\tau_n \wedge \tau_b}) \quad (\mathrm{P}_x\text{-a.s.}).$$

Taking expectation: $u(x) = \mathrm{E}_x\left[\mathrm{e}^{-\tilde{r}(\tau_n \wedge \tau_b)} u(X_{\tau_n \wedge \tau_b})\right]$. Sending $n \to \infty$:

$$\begin{aligned} u(x) &= \mathrm{E}_x\left(\mathrm{e}^{-\tilde{r}\tau_b} u(b) \mathbb{1}_{\{\tau_b < \infty\}}\right) \\ &= \mathrm{E}_x\left(\mathrm{e}^{-\tilde{r}\tau_b} g(b) \mathbb{1}_{\{\tau_b < \infty\}}\right) \\ &= \mathrm{E}_x\left(\mathrm{e}^{-\tilde{r}\tau_b} g(X_{\tau_b}) \mathbb{1}_{\{\tau_b < \infty\}}\right) \le v(x), \quad \text{as required.} \end{aligned}$$

5. Parametric Dependencies of the Value Function



(a) $\mu \mapsto v(x)$ (b) $\lambda_0 \mapsto v(x)$

Economic interpretation:

- Monotonicity of $\mu \mapsto v(x)$ is meaningful (faster wage growth – better value)
- Behavior of $\lambda_0 \mapsto v(x)$ is more interesting:

  $\mu < r$: initially, better value with increasing unemployment rate λ_0 , but then things get worse (for larger μ)

  $\mu \geq r$: a counter-intuitive increase of the value as λ_0 drops to zero.

In particular, if $\mu = r$ and $\lambda_0 \downarrow 0$, then $b^* \to \infty$, (so stopping never occurs), but $v(x) \to \beta x$ – a money tree?

Explanation of the paradox: for small λ_0 , threshold b^* is high and the hitting probability is small, but the payoff is rather big, so the value stays positive. Thus, such artefacts caused by no constraint on the waiting times.

This can be rectified e.g. by introducing *mortality* (which is done in the paper).

6. American Call Option

Our model resembles the optimal stopping problem for the *(perpetual) American call option:* the holder of a call option may (but doesn't have to) buy an asset for a fixed price K, based on observations over the stock price (S_t), modelled as GBM. This leads to the optimal stopping problem

$$V(x) = \sup_{\tau} \mathrm{E}_x \left(\mathrm{e}^{-r\tau} (S_\tau - K)^+ \right)$$

Our optimal stopping problem can be rewritten in a similar form:

$$v(x) = \beta_1 \sup_{\tau} \mathrm{E}_x \left(\mathrm{e}^{-\tilde{r}\tau} (X_\tau - \tilde{K}) \right), \qquad \tilde{K} := P/\beta_1,$$

Differences:

- No truncation (financial gain is not the sole priority)
- The hitting time τ_{b^*} may be infinite – this is acceptable for ACO but not quite in the insurance context (John wants to get insured!)

7. Optimal Stopping with Utility

A more realistic formulation of the UI model should involve a certain **utility** to express John's preferences for satisfaction (e.g. impatience).

A simple example (akin to penalized regression):

$$\begin{aligned} v^{\dagger}(x) &= \sup_{\tau} \left[\kappa \, \mathrm{P}_x(\tau < \infty) + \mathrm{eNPV}(x;\tau) \right] \\ &= \sup_{\tau} \mathrm{E}_x \left(\kappa \, \mathbb{1}_{\{\tau<\infty\}} + \mathrm{e}^{-\tilde{r}\tau} g(X_\tau) \right) \end{aligned}$$

or, in a more standard form,

$$v^{\dagger}(x) = \sup_{\tau} \mathrm{E}_x \left(\mathrm{e}^{-\tilde{r}\tau} G(\tau, X_\tau) \right), \quad G(t,x) := \kappa \, \mathrm{e}^{\tilde{r}t} + g(x).$$

But this problem is time dependent, so not amenable to an exact solution.

Suboptimal solution (in the class of hitting times):

$$u^{\dagger}(x) = \sup_{b \geq 0} \left[\kappa \, \mathrm{P}_x(\tau_b < \infty) + \mathrm{eNPV}(x; \tau_b) \right].$$

Explicitly (may assume that $b \geq x$):

$$u^{\dagger}(x) = \sup_{b \geq x} \left[\kappa \left(\frac{x}{b} \right)^{1-2\mu/\sigma^2} + (\beta_1 b - P) \left(\frac{x}{b} \right)^{q_*} \right].$$

Solution:

$$b^{\dagger} = \frac{(P-\kappa) q_*}{\beta_1 (q_* - 1)} = b^* - \frac{\kappa q_*}{\beta_1 (q_* - 1)} \leq b^*.$$

$$u^{\dagger}(x) = \begin{cases} (\beta_1 b^{\dagger} + \kappa - P) \left(\frac{x}{b^{\dagger}} \right)^{q_*}, & x \in [0, b^{\dagger}], \\ \beta_1 x + \kappa - P, & x \in [b^{\dagger}, \infty) \end{cases}$$

For more details, including links with Expected Utility Theory and inclusion of consumption, see the paper in *Risks*.

Thank you for listening! 

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