Perturbations of parabolic equations and diffusion processes with degeneration: boundary problems and metastability

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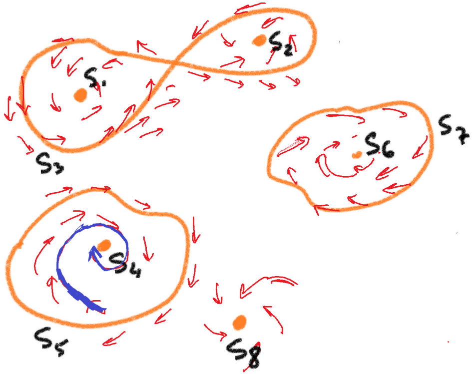
Based on joint work with Mark Freidlin



Concept for reference: Small perturbations of dynamical systems with multiple invariant sets (Freidlin-Wentzell theory).

Freidlin-Wentzell Theory

The process is governed by

$$dX_t^{x,\varepsilon} = v(X_t^{x,\varepsilon})dt + \varepsilon dW_t, \quad X_0^{x,\varepsilon} = x.$$

For times $t = t(\varepsilon)$ such that $1 \ll t(\varepsilon) \ll e^{\lambda/\varepsilon^2}$ for each $\lambda > 0$, $X_{t(\varepsilon)}^{x,\varepsilon}$ is very close to the “nearest" stable attractor. It takes exponential, in ε^2, time to go from one attractor to another.

The order of transitions and the transsition times are determined by constants V_{ij} (values of the quasi-potential).

Quasi-potential and the action functional

$$V_{ij} = \inf(\mathbf{S}(\varphi) : \varphi(0) \in S_i,\ \varphi(T) \in S_j),$$

where

$$\mathbf{S}(\varphi) = \frac{1}{2}\int_0^T ||\dot{\varphi}(t) - v(\varphi(t))||^2 dt$$

(“difficulty" of following the curve φ for time T).

For example, if x is close to S_i, it takes time of order $\exp(\min_j(V_{ij})/\varepsilon^2)$ for $X_t^{x,\varepsilon}$ to go to the “next" attractor. The process of transitions between the attractors resembles a Markov process with very small (ε-dependent) transition rates.

Metastability

Generically, there exist $0 = \lambda_0 < \lambda_1 < ... < \lambda_n = \infty$ such that for almost every x and each time scale $t(\varepsilon)$ satisfying

$$\lambda_k/\varepsilon^2 \ll \ln(t(\varepsilon)) \ll \lambda_{k+1}/\varepsilon^2,$$

$X^{x,\varepsilon}_{t(\varepsilon)}$ is found in the vicinity of S_i with i determined by k and x.

S_i is the metastable state of the process starting at x at the time scale $t(\varepsilon)$.

Random perturbations of degenerate diffusions

Degenerate process:

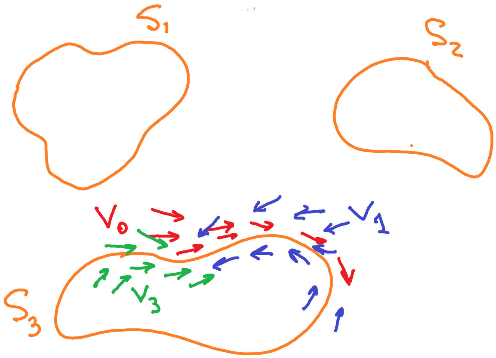
$$dX_t^x = v_0(X_t^x)dt + \sum_{i=1}^{d} v_i(X_t^x) \circ dW_t^i, \qquad X_0^x = x \in \mathbb{R}^d,$$

The Stratonovich form is convenient here since it allows one to provide a coordinate-independent description of the process. The generator:

$$Lu = L_0 + \frac{1}{2}\sum_{i=1}^{d} L_i^2,$$

where L_i is the operator of differentiation along v_i.

We assume that $S_1, ..., S_m \subset \mathbb{R}^d$ are smooth non-intersecting surfaces (or curves), the process is non-degenerate outside the surfaces, each of the surfaces is invariant for the process, and the diffusion restricted to a single surface is an ergodic process.



Perturbed (non-degenerate) process:

$$dX_t^{x,\varepsilon} = (v_0 + \varepsilon^2 \tilde{v}_0)(X_t^{x,\varepsilon})dt + \sum_{i=1}^{d} v_i(X_t^{x,\varepsilon}) \circ dW_t^i +$$

$$+\varepsilon \sum_{i=1}^{d} \tilde{v}_i(X_t^{x,\varepsilon}) \circ d\tilde{W}_t^i, \quad X_t^{x,\varepsilon} = x.$$

non-degeneracy: $\mathrm{span}(\tilde{v}_1(x), ..., \tilde{v}_d(x)) = \mathbb{R}^d$ for each x.

Generator: $L^\varepsilon = L + \varepsilon^2 \tilde{L}$, with

$$\tilde{L}u = \tilde{L}_0 + \frac{1}{2} \sum_{i=1}^{d} \tilde{L}_i^2,$$

where $\tilde{L}_i$ is differentiation along $\tilde{v}_i$.

Goal: Understand the behavior of $X^{x,\varepsilon}_{t(\varepsilon)}$ at different time scales $t(\varepsilon)$.

Now, the characteristic time scales are not going to be exponential in ε^2. Instead, we'll have

$$t_0(\varepsilon) \ll t_1(\varepsilon) \ll ... \ll t_n(\varepsilon)$$

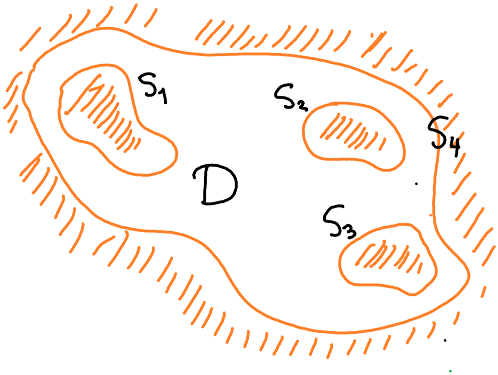
with $t_0 \equiv 1$, $t_1 = |\ln(\varepsilon)|$, $t_2, ..., t_{n-1}$ -powers of ε, $t_n \equiv \infty$, such that $X^{x,\varepsilon}_{t(\varepsilon)}$ has a limit, provided that

$$t_k(\varepsilon) \ll t(\varepsilon) \ll t_{k+1}(\varepsilon).$$

The limiting distribution will not necessarily be supported on one of the surfaces ...

Setup suitable for PDE results

Let D be bounded domain with $\partial D = S_1 \bigcup S_2 \bigcup ... \bigcup S_m$. Here, the surfaces are $(d-1) - \textit{dimensional}$. The process $X_t^{x,\varepsilon}$ is considered on $\overline{D}$, reflected on the boundary.



First initial-boundary value problem:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = L^\varepsilon u^\varepsilon(t,x), \quad t>0, x\in D;$$
$$u^\varepsilon(0,x) = g(x), \quad x\in D; \qquad u^\varepsilon(t,x) = \psi(x), \quad t>0, x\in\partial D,$$

$g\in C(\overline{D})$, $\psi\in C(\partial D)$.

Second initial-boundary value problem:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = L^\varepsilon u^\varepsilon(t,x), \quad t>0, x\in D;$$
$$u^\varepsilon(0,x) = g(x), \quad x\in D; \qquad \frac{\partial u^\varepsilon(t,x)}{\partial n^\varepsilon(x)} = 0, \quad t>0, x\in\partial D,$$

where $n^\varepsilon(x)$ is the co-normal to ∂D at x.

Theorem: For each of the problems, there is a finite sequence of characteristic time scales $t_0(\varepsilon) \ll t_1(\varepsilon) \ll ... \ll t_n(\varepsilon)$ such that

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t(\varepsilon), x) = c_k$$

uniformly on any compact subset of D, provided that $t_k(\varepsilon) \ll t(\varepsilon) \ll t_{k+1}(\varepsilon)$. The constants c_k are determined by integrating the initial and/or boundary data with respect to the measures $\pi_1, ..., \pi_m, \nu_1, ..., \nu_m$, μ. These measures need to be explained.

The "special" measures

$\pi_1, ..., \pi_m$ - just the invariant measures for the unperturbed process restricted to $S_1,...,S_m$.

Each ν_k is defined as the limit of the exit measures. That is, we consider the process $X_t^{x,\varepsilon}$ conditioned on reaching ∂D at S_k.
Theorem: The measures induced by such process, stopped at S_k, converge, as $\varepsilon \downarrow 0$, for each $x \in D$, to the same measure, which will be called ν_k.

μ is the invariant measure for the unperturbed process in D. Such a measure exists if all the boundary components are repelling (to be discussed next).

Near-boundary behavior

Simple 1-d example: Consider the process on $[0, \infty)$:

$$dX_t^x = \beta X_t^x dt + \sqrt{2\alpha} X_t^x dW_t, \quad X_0^x = x > 0, \quad (\alpha > 0)$$

Generator:

$$Lu = \alpha x^2 u'' + \beta x u'.$$

We have the **attracting** and **repelling** cases:

$$\lim_{t\to\infty} X_t^x = 0 \ \text{ with positive probability if } \ \alpha > \beta,$$

$$\lim_{t\to\infty} X_t^x \neq 0 \ \text{ with probability one if } \ \alpha < \beta.$$

Near-boundary behavior, classifying boundary components

Fix $S = S_k$. Use (y, z)- local coordinates; y - along S, z - in the orthogonal direction. The generator of the process $X_t^{x,\varepsilon}$ in (y, z) coordinates can be written as:

$$L^\varepsilon u = L_y u + z^2\alpha(y)\frac{\partial^2 u}{\partial z^2} + z\beta(y)\frac{\partial u}{\partial z} + z\mathcal{D}_y\frac{\partial u}{\partial z} + Ru + \varepsilon^2\tilde{L}u.$$

L_y - restriction of L to S;
$\mathcal{D}_y$ differential operator with first-order derivatives in y;
β and α are the leading terms for the drift and diffusion coefficients in the direction normal to the boundary;
R - perturbation that contains higher-order terms in z, and can be viewed as a perturbation when z is small.
Define

$$\bar{\alpha} = \int_S \alpha(y)d\pi(y), \;\; \bar{\beta} = \int_S \beta(y)d\pi(y),$$

where π is the invariant measure on S.

$$\bar{\alpha} = \int_S \alpha(y) d\pi(y), \;\; \bar{\beta} = \int_S \beta(y) d\pi(y),$$

Attracting surface: $\bar{\alpha} > \bar{\beta}$
Repelling surface: $\bar{\alpha} < \bar{\beta}$

However, understanding further properties of the process near the boundary requires more delicate analysis.

Time to approach the a boundary component or to leave its neighborhood

Lemma. If $\bar{\alpha} > \bar{\beta}$ ($\bar{\alpha} < \bar{\beta}$), then there exist $\gamma > 0$ ($\gamma < 0$) and a positive-valued function $\varphi \in C^1(S)$ satisfying $\int_S \varphi d\pi = 1$ such that

$$L_y\varphi + \alpha\gamma(\gamma - 1)\varphi + \beta\gamma\varphi + \gamma\mathcal{D}_y\varphi = 0.$$

Such γ are φ are determined uniquely.

This lemma associates one number, γ, to each component of the boundary. For example, the time it takes the process $X_t^{x,\varepsilon}$, starting at $x \in S$, to leave a $\varkappa$-wide neighborhood of S scales as $(\varkappa/\varepsilon)^\gamma$ if ε and $\varkappa$ are small, $\varepsilon \ll \varkappa$, and the boundary is attracting.

We used:

$$L^{\varepsilon}u \approx L_y u + z^2\alpha(y)\frac{\partial^2 u}{\partial z^2} + z\beta(y)\frac{\partial u}{\partial z} + z\mathcal{D}_y\frac{\partial u}{\partial z}.$$

This works for $\varepsilon \ll z \ll 1$. For $z \sim \varepsilon$, we need $(y, z/\varepsilon)$ coordinates. The operator there looks like:

$$L^{\varepsilon}u \approx L_y u + (z^2\alpha(y) + \rho(y))\frac{\partial^2 u}{\partial z^2} + z\beta(y)\frac{\partial u}{\partial z} + z\mathcal{D}_y\frac{\partial u}{\partial z}.$$

The exit distribution can be understood from this operator. $\rho(y)$ is the coefficient, on the boundary, in the perturbation $\tilde{L}$ at the diffusion term orthogonal to the boundary.

Understanding meta-stable distributions for $X_t^{x,\varepsilon}$.

Need to:

(a) Undestand the times to approach S_k and to leave a neighborhood of S_k - discussed above (based on the spectral lemma).

(b) Understand the transition probabilities between different S_k - these are nearly ε-independent. (Understood by conditioning the non-perturbed process not to return to S_k.)

(c) Once we understood the transition probabilities and times, in which order are S_k visited? (Similar to hierarchy of cycles.)

Parameter-dependent Markov Renewal Processes

Let $x \in \partial D$, let $\sigma_0^{x,\varepsilon} = 0$, and, assuming that $X^{x,\varepsilon}_{\sigma_n^{x,\varepsilon}} \in S_k$, let

$$\sigma_{n+1}^{x,\varepsilon} = \inf\{t \geq \sigma_n^{x,\varepsilon} : X_t^{x,\varepsilon} \in \partial D \setminus S_k\}.$$

Markov renewal process $(\mathbf{X}_n^{x,\varepsilon}, \mathbf{T}_n^{x,\varepsilon})$, $n \in \mathbb{Z}_+$, is defined as:

$$\mathbf{X}_n^{x,\varepsilon} = X^{x,\varepsilon}_{\sigma_n^{x,\varepsilon}}, \quad \mathbf{T}_n^{x,\varepsilon} = \sigma_n^{x,\varepsilon} - \sigma_{n-1}^{x,\varepsilon}, \quad n \geq 1.$$

The corresponding semi-Markov process on $S_1 \bigcup ... \bigcup S_m$ is just $\mathcal{X}_t^{x,\varepsilon} = X^{x,\varepsilon}_{\sigma_n^{x,\varepsilon}}$ for $\sigma_n^{x,\varepsilon} \leq t < \sigma_{n+1}^{x,\varepsilon}$, $n \geq 0$.

Abstract Formulation

Let $(\mathbf{X}_n^{x,\varepsilon}, \mathbf{T}_n^{x,\varepsilon})$ be a Markov renewal process on the state space $M = S_1 \bigcup ... \bigcup S_m$.

Now, $S_1, ..., S_m$ need not be smooth surfaces, but are just disjoint measurable sets in a metric space M.

$Q^\varepsilon(x, S_k)$ - transition probability from $x \in M$ to S_k. We assume that $Q^\varepsilon(x, S_k) = 0$ for $x \in S_k$.

$\mathbf{T}_n^{x,\varepsilon}$, conditioned on $\mathbf{X}_n^{x,\varepsilon} \in S_k$, is assummed to be the same as that of a random variable $\xi_k^{x,\varepsilon}$ (there is no dependence on n since the process is assumed to be time-homogeneous).

Assumptions on the Markov Renewal Process

(a) There are quantities $q_{ij}(\varepsilon)$ and $\tau_{ij}(\varepsilon)$ such that

$$\lim_{\varepsilon\downarrow 0} \frac{Q^\varepsilon(x,S_j)}{q_{ij}(\varepsilon)} = 1, \quad \lim_{\varepsilon\downarrow 0} \frac{\mathrm{E}\xi_j^{x,\varepsilon}}{\tau_{ij}(\varepsilon)} = 1, \quad \text{uniformly in } x \in S_i, \quad i \neq j,$$

provided that $Q^\varepsilon(x,S_j)$ is not identically zero.

(b) $\xi_j^{x,\varepsilon}/\tau_{ij}(\varepsilon)$ are uniformly integrable in $x \in S_i$, $\varepsilon > 0$ and that $\mathrm{P}(\xi_j^{x,\varepsilon}/\tau_{ij}(\varepsilon) < c) \to 0$ as $c \downarrow 0$, uniformly in $x \in S_i$, $\varepsilon > 0$.

(c) **Complete Asymptotic Regularity.**

$$\lim_{\varepsilon\downarrow 0} \frac{q_{a_1 b_1}(\varepsilon)}{q_{c_1 d_1}(\varepsilon)} \cdot \frac{q_{a_2 b_2}(\varepsilon)}{q_{c_2 d_2}(\varepsilon)} \cdots \frac{q_{a_r b_r}(\varepsilon)}{q_{c_r d_r}(\varepsilon)} \cdot \frac{\tau_{ab}(\varepsilon)}{\tau_{cd}(\varepsilon)} \in [0,\infty]$$

exist for every $r \in \mathbb{N}$ and every $a, a_i, b, b_i, c, c_i, c, d_i$ with $a_i \neq b_i$, $c_i \neq d_i$, $a \neq b$, and $c \neq d$, for which the ratios appearing in the limits are defined.

We are interested in the behavior of the semi-Markov process $\mathcal{X}_{t(\varepsilon)}^{x,\varepsilon}$, where $\mathcal{X}_t^{x,\varepsilon} = \mathbf{X}_n^{x,\varepsilon}$ for $\mathbf{T}_0^{x,\varepsilon} + ...\mathbf{T}_n^{x,\varepsilon} \leq t < \mathbf{T}_0^{x,\varepsilon} + ...\mathbf{T}_{n+1}^{x,\varepsilon}$.

Under the above assumptions, there is a finite sequence of characteristic time scales $t_0(\varepsilon) \ll t_1(\varepsilon) \ll ... \ll t_n(\varepsilon)$, with $t_0 \equiv 1$ and $t_n \equiv \infty$, such that the limiting distribution of $\mathcal{X}_{t(\varepsilon)}^{x,\varepsilon}$ can be identified, as long as

$$t_i(\varepsilon) \ll t(\varepsilon) \ll t_{i+1}(\varepsilon)$$

for some i.