Sharp density estimates for a stochastic process introduced by Yor

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LSA Meeting

Introduction

Plan of my talk

- Stochastic process and its applications
- Kolmogorov equation: large deviation principle
- Yosida's parametrix and open problem

Joint work in collaboration with Stefano Pagliarani: *A Yosida's parametrix approach to Varadhan's estimates for a degenerate diffusion under the weak Hörmander condition* (2021 - Submitted)

The stochastic process

$$\begin{cases} \mathbf{X}_t = x_0 \exp\left(\mu t + \sigma \mathbf{W}_t\right), \\ \mathbf{Y}_t = y_0 + \int_0^t \mathbf{X}_s \, ds \end{cases}$$

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The density $p = p(x, y, t; x_0, y_0, 0)$

satisfies the Kolmogorov equation

$\mathscr{L} p(x, y, t; x_0, y_0, 0) = 0, \qquad (x, y, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$

$$\mathscr{L} := \frac{\sigma^2 x^2}{2} \partial_x^2 + x\left(\mu + \frac{\sigma^2}{2}\right)\partial_x + x\partial_y - \partial_t,$$

Applications

Arithmetic average Path-dependent Options in the Black & Scholes theory:

$$\begin{cases} S_t = S_0 \exp(\mu t + \sigma W_t) \\ B_t = B_0 \exp(rt) \end{cases}$$

Assume that $V = V(t, S_t, A_t)$ also depends on the average price

$$A_t = \int_0^t S_\tau \, d\tau,$$

then

$$\begin{cases} \partial_t V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V + S\partial_A V + r\,(S\partial_s V - V) = 0, \\ V(T, S_T, A_T) = \varphi(S_T, A_T) \end{cases}$$

What is known

[Yor](1992)(2001), [Geman, Yor](1993), [Matsumoto, Yor](2005)

$$\Gamma(x,y,t,x_0,y_0,t_0) = \frac{1}{2xx_0} p\left(\frac{1}{2}\log\left(\frac{x_0}{x}\right), \frac{y_0-y}{x}, \frac{t-t_0}{2}\right),$$

where

$$p(w,y,t) = \frac{e^{\frac{\pi^2}{2t}}}{\pi\sqrt{2\pi t}} \exp\left(-\frac{1+e^{2w}}{2y}\right) \frac{e^w}{y^2} q\left(\frac{e^w}{y}, t\right),$$

and

$$q(z,t) = \int_0^\infty e^{-\frac{\xi^2}{2t}} e^{-z\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi.$$

Other results

- Upper and lower bounds for the fundamental solution of $\mathscr{L}$

$$\frac{c^-}{t^2}\exp\left(-C^-\Psi(x,y,t)\right) \leq \Gamma(x,y,t) \leq \frac{C^+}{t^2}\exp\left(-c^+\Psi(x,y,t)\right)$$

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- Existence of a fundamental solution and estimates for

$$\mathscr{L}u = x\partial_x(ax\partial_x u) + bx\partial_x u + x\partial_y u - \partial_t u$$

with $a = a(x,y,t), b = b(x,y,t)$ smooth, bounded, $a \ge \lambda > 0$
[Cibelli, P., Rossi] Sharp estimates for Geman–Yor processes and applications to arithmetic average asian options (2019)

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- Same result with locally Hölder continuous coefficients a and b.
[Anceschi, Muzzioli, P.] Existence of a fundamental solution of partial differential equations associated to Asian options (2021)

Our main result

Large deviation principle [Varadhan](1967), [Molchanov](1975), [Azencott](1984)

$\mathscr{L} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \partial_{x_i} \partial_{x_j} - \partial_t$

uniformly parabolic with Hölder continuous coefficients. Then

$$t \log(\Gamma(x, t; y, 0)) \longrightarrow -\frac{d(x, y)^2}{2} \quad \text{as } t \to 0.$$

$$\frac{\log(\Gamma(x, t; y, 0))}{-\frac{d(x,y)^2}{2t}} \longrightarrow 1 \quad \text{as } t \to 0.$$

uniformly on compact sets.

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uniformly on compact sets. Compare with

$$\frac{c^-}{t^{n/2}}\exp\left(-C^-\frac{|x-y|^2}{t}\right) \leq \Gamma(x,t;y,0) \leq \frac{C^+}{t^{n/2}}\exp\left(-c^+\frac{|x-y|^2}{t}\right)$$

Our main result

Theorem [Pagliarani, P.] $\mathscr{L} = \frac{x^2}{2}\partial_x^2 + x\partial_y - \partial_t$

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Theorem [Cibelli, P., Rossi] Let Γ be the fundamental solution of $x^2\partial_{xx} + x\partial_y - \partial_t$. Then

$$\frac{c_\varepsilon^-}{t^2}\exp\left(-C^-\Psi(x, y+\varepsilon t, t-\varepsilon t)\right) \leq$$

$$\Gamma(x,y,t) \leq$$

$$\frac{C_\varepsilon^+}{t^2}\exp\left(-c^+\Psi(x, y-\varepsilon, t+\varepsilon)\right),$$

The function Ψ

An optimal control problem

$$\Psi(x_0, y_0, t_0) := \inf_{\omega \in L^2([0,t_0])} \int_0^{t_0} \omega^2(s) ds \quad \text{subject to constraint}$$

$$\begin{cases} x'(s) = \omega(s) x(s), & x(0) = x_0, \quad x(t_0) = 1, \\ y'(s) = x(s), & y(0) = y_0, \quad y(t_0) = 0. \end{cases}$$

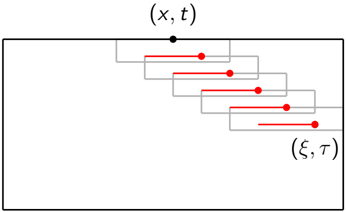
Recall the Stochastic Differential Equation

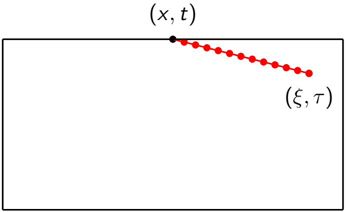
$$\begin{cases} dX_t = X_t dW_t, & X_0 = x_0 \\ dY_t = X_t \, dt, & Y_0 = y_0. \end{cases}$$

Lower bound

Method of the proof: Harnack chains:

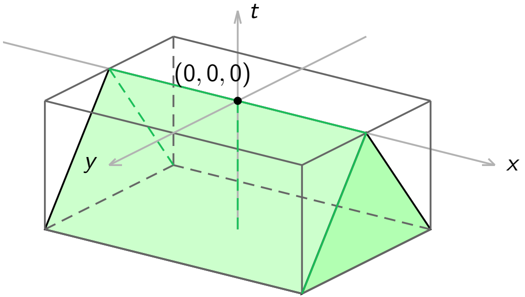
$u(\xi,\tau) \leq C^k u(x,t)$ (heat equation)





Harnack inequality

We borrow the Harnack inequality from the theory of Kolmogorov equations: $\partial_x\big(a(x,y,t)\partial_x u\big) + x\partial_y u = \partial_t u$



Invariant Harnack inequality

If $u_{xx}(x,y,t) + xu_y(x,y,t) = u_t(x,y,t)$, then:

- dilations: $v(x,y,t) = u(rx, r^3 y, r^2 t)$
  satisfies $v_{xx} + xv_y = v_t$

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- translations: $w(x,y,t) = u(x+\xi, y+\eta-\xi t, t+\tau)$
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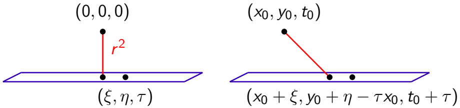

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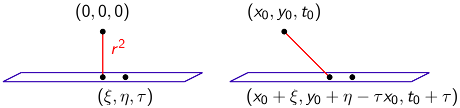

- $\mathscr{L}$-admissible paths (Yor operator):
  $\mathscr{L} := X^2 + Y, \quad X = x\partial_x, \quad Y = x\partial_y - \partial_t$

  $\gamma'(t) = \omega(t) X(\gamma(t)) + Y(\gamma(t))$

The optimal control problem

$$\Psi(x_0, y_0, t_0) := \inf_{\omega \in L^2([0,t_0])} \int_0^{t_0} \omega^2(s)ds \quad \text{subject to constraint}$$

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Upper and lower bounds

Let Γ be the fundamental solution of $x^2\partial_{xx} + x\partial_y - \partial_t$. Then

$$\Gamma(x, y, t) \geq \frac{c_\varepsilon^-}{t^2} \exp\left(-C^-\Psi(x, y + \varepsilon t, t - \varepsilon t)\right)$$

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Proof (of the upper bound). The Value Function Ψ is a solution of the Hamilton-Jacobi-Bellman equation:

$$Y\Psi(x,y,t) + \tfrac{1}{4}\big(X\Psi(x,y,t)\big)^2 = 0$$

We then compare $\Gamma(x,y,t)$ with $\exp\left(-c^+\Psi(x,y,t)\right)$.

The function Ψ

Let

$$g(r) = \begin{cases} \frac{\sinh(\sqrt{r})}{\sqrt{r}}, & r > 0, \\ \frac{\sin(\sqrt{-r})}{\sqrt{-r}}, & -\pi^2 < r < 0. \end{cases}$$

$$E = \frac{4}{(t-t_0)^2} g^{-1}\left(\frac{y_0 - y}{(t-t_0)\sqrt{xx_0}}\right).$$

Then

$$\Psi(x_1, y_1, t_1; x_0, y_0, t_0) = (t_1 - t_0)E + \frac{4(x_1 + x_0)}{y_0 - y_1} \pm 4\sqrt{E + \frac{4x_1x_0}{(y_0 - y_1)^2}},$$

("$-$" if $E \geq -\frac{\pi^2}{t_1 - t_0}$, "$+$" otherwise).

A novel representation of Ψ

Invariance property [Monti, Pascucci] (2009)

$x^2\partial_{xx}u(x,y,t) + x\partial_y u(x,y,t) = \partial_t u(x,y,t)$ if, and only if,

$$w(x,y,t) = u(x_0 x, y_0 + x_0 y, t_0 + t)$$

satisfies $x^2\partial_{xx}w(x,y,t) + x\partial_y w(x,y,t) = \partial_t w(x,y,t)$

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Define

$$(x,y,t) \circ (x_0,y_0,t_0) := (x_0 x, y_0 + x_0 y, t_0 + t)$$

$$(x,y,t)^{-1} \circ (x_0,y_0,t_0) := \left(\tfrac{x_0}{x}, \tfrac{y_0 - x_0}{x}, t_0 - t\right)$$

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$$h(x,y,t;x_0,y_0,t_0) := \frac{(t_0 - t)\sqrt{x\,x_0}}{y - y_0}$$

A novel representation of Ψ (cont)

$$\Psi(x,y,t;x_0,y_0,t_0) = \frac{4}{t_0-t}\left(G\left(\frac{(t_0-t)\sqrt{x\,x_0}}{y-y_0}\right) + \frac{(t_0-t)\sqrt{x\,x_0}}{y-y_0}\left(\sqrt{\frac{x}{x_0}} + \sqrt{\frac{x_0}{x}} - 2\right)\right)$$

with accurate esitmates of G.

Yosida's parametrix

Levi's Prametrix

$$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)\partial_{x_i}\partial_{x_j} - \partial_t$$

$$\mathscr{L}_{(\xi,\tau)} := \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(\xi,\tau)\partial_{x_i}\partial_{x_j} - \partial_t$$

Let $Z(x,t;\xi,\tau)$ denote the *fundamental solution* of $\mathscr{L}_{(\xi,\tau)}$.

Levi's Prametrix

$\mathscr{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)\partial_{x_i}\partial_{x_j} - \partial_t$

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Let $Z(x,t;\xi,\tau)$ denote the *fundamental solution* of $\mathscr{L}_{(\xi,\tau)}$.

$\Gamma(x,t;\xi,\tau) := Z(x,t;\xi,\tau) + R(x,t;\xi,\tau)$

Lemma: if the a_{ij}'s are α–Hölder continuous, then

$$|\mathscr{L} Z(x,t;\xi,\tau)| \leq \frac{C}{(t-\tau)^{1-\alpha/2}}\Gamma^{+}(x,t;\xi,\tau)$$

Levi's Prametrix (cont.)

$$\Gamma(x,t;\xi,\tau) := Z(x,t;\xi,\tau) + \int_\tau^t \left(\int_{\mathbb{R}^n} Z(x,t;y,s)\Phi(y,s;\xi,\tau)dy \right) ds$$

$$\Phi(x,t;\xi,\tau) = \sum_{k=1}^{\infty} \mathscr{L}_k Z(x,t;\xi,\tau)$$

$$\mathscr{L}_{k+1} Z(x,t;\xi,\tau) = \int_\tau^t \left(\int_{\mathbb{R}^n} \mathscr{L} Z(x,t;y,s)\mathscr{L}_k(y,s;\xi,\tau)dy \right) ds$$

Levi's Prametrix (cont.)

Kolmogorov-Chapman identity

$$
|\mathscr{L}_{k+1} Z(x,t;\xi,\tau)| \leq \int_\tau^t \frac{C}{(t-s)^{1-\alpha/2}} \frac{C^k}{(t-s)^{1-k\alpha/2}} \cdot \\
\cdot \left(\int_{\mathbb{R}^n} \Gamma^+(x,t;y,s)\Gamma^+(y,s;\xi,\tau)dy \right) ds \leq \\
\frac{C^{k+1}}{(t-\tau)^{1-(k+1)\alpha/2}} \Gamma^+(x,t;\xi,\tau)
$$

Yosida's Prametrix

$$Z_0(x, y, t; \xi, \eta, \tau) := \frac{C}{x\xi(t-\tau)^2} \exp\left(-\frac{\Psi(x, y, t; \xi, \eta, \tau)}{2}\right)$$

Yosida's Prametrix

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$$|\mathscr{L} Z_0(x,y,t;\xi,\eta,\tau)| \notin L^1(]\tau,t[\times\mathbb{R}^2)$$

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$$|\mathscr{L} Z_0(x,y,t;\xi,\eta,\tau)| \notin L^1(]\tau,t[\times\mathbb{R}^2)$$

Find $U = U(x,y,t;\xi,\eta,\tau)$ such that

$$Z_Y(x,y,t;\xi,\eta,\tau) := U(x,y,t;\xi,\eta,\tau)\cdot\frac{C}{(t-\tau)^2}\exp\left(-\frac{\Psi(x,y,t;\xi,\eta,\tau)}{2}\right)$$

and that

$$|\mathscr{L} Z_Y(x,y,t;\xi,\eta,\tau)| \in L^1(]\tau,t[\times\mathbb{R}^2)$$

Yosida's Prametrix (cont.)

The construction of U relies on the Hamilton-Jacobi-Bellman equation:

$$Y\Psi(x,y,t)+\frac{1}{4}\big(X\Psi(x,y,t)\big)^2=0$$

Yosida's Prametrix (cont.)

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Proof of the Chapman-Kolmogorov equation

$$\int_{\mathbb{R}^2} \Gamma_Y^+(x,y,t;\xi,\eta,\tau)\Gamma_Y^+(\xi,\eta,\tau;x_0,y_0,t_0)d\xi\, d\eta = \Gamma_Y^+(x,y,t;x_0,y_0,t_0)$$

based on *numerical evidence*.

Many thanks for your attention!

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And congratulations
to Valentin and Stanislav!