The Normal distribution on Hyperbolic space

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Plan

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\Delta v \\ v\big|_{t=0} = v_0(x) \end{cases}$$

The Normal distribution on $\mathbb{R}^n$ is the kernel of the fundamental solution:

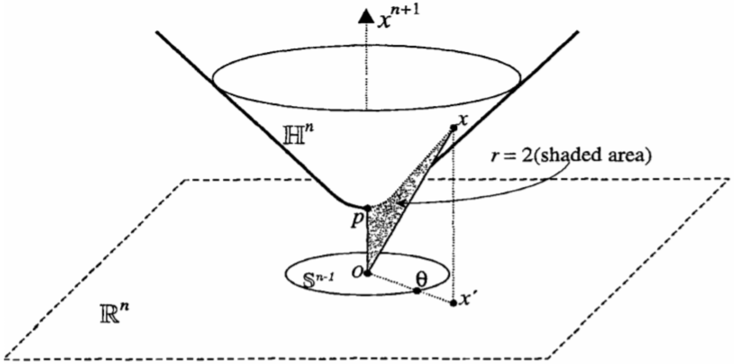
$$v(x,t) = \int_{\mathbb{R}^n} \frac{1}{(2\pi t)^{n/2}} \exp(-\frac{|x-y|^2}{2t}) v_0(y) dy,$$

$$K(x,y) = \frac{1}{(2\pi t)^{n/2}} \exp(-\frac{|x-y|^2}{2t})$$

1. Recall basic statements about Hyperbolic geometry.
2. The Laplace-Beltrami operator in $\mathbb{H}^n$.
3. Fourier transform on Hyperbolic Space in $\mathbb{H}^n$.
4. Fundamental solution of the heat equation in $\mathbb{H}^n$.

n-dimensional Hyperbolic Plane

$\mathbb{H}^n = \{(x_1, .., x_{n+1}) : x_i \in \mathbb{R}, x_{n+1} > 0, x_{n+1}^2 - x_1^2 - ... - x_n^2 = 1\}$
The polar coordinates on $\mathbb{H}^n$: $(\theta, r), \theta \in \mathbb{S}^{n-1}, r > 0$.
$x_{n+1} = \cosh r, \quad x_i = \theta_i \sinh r, i = 1, ..., n.$



The canonical Hyperbolic metric in Cartesian coordinates:

$$g_{\mathbb{H}^n} = \frac{1}{(1 - |y|^2)^2} g_{\mathbb{R}^n}, \quad y = \frac{x'}{x^{n+1} + 1}, |y| = \sum (y^i)^2$$

The metrics $g_{\mathbb{H}^n}$ has the following representation in the polar coordinates

$$g_{\mathbb{H}^n} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}}.$$

The Laplace-Beltrami operator on $\mathbb{H}^n$

The Laplace-Beltrami operator in coordinates, g - metric tensor, g^{ij} - components of the inverse of the metric tensor:

$$\Delta = \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x_i} \left(\sqrt{det(g)} g^{ij} \frac{\partial}{\partial x_j} \right)$$

The Laplace-Beltrami operator in polar coordinate:

$$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\cosh r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{n-1}} =$$

$$= \frac{1}{\sinh^{n-1} r} \frac{\partial}{\partial r} (\sinh^{n-1} r \frac{\partial}{\partial r}) + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^{n-1}},$$

$\Delta_{\mathbb{S}^{n-1}}$ - the Laplace-Beltrami operator on sphere $\mathbb{S}^{n-1}$.
$\Delta_{\mathbb{S}^{n-1}} = \frac{\partial}{\partial r^2} + (n-2)\cos r \frac{\partial}{\partial r} + \frac{1}{\sin^2 r} \Delta_{\mathbb{S}^{n-2}}$

The D'Alambertian operator

Consider the bilinear form on $\mathbb{R}^{n+1}$

$$[x, y] = x_{n+1}y_{n+1} - (x_1 y_1 + ... + x_n y_n).$$

$\Gamma = \{x \in \mathbb{R}^{n+1} | [x, x] > 0, x_{n+1} > 0\}$
The D'Alambertian operator in cartesian coordinate on Γ:

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} ... + \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial x_{n+1}^2}$$

Lemma 1

The D'Alambertian operator in polar coordinates $x = (tw \sinh r, t \cosh r), t > 0, r \geq 0, w \in \mathbb{S}^{n-1}$*:*

$$\Box = -(\frac{\partial^2}{\partial t^2} + \frac{n}{t}\frac{\partial}{\partial t}) + \frac{1}{t^2}\Delta_{\mathbb{H}^n},$$

where $\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1) \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}}.$

The Laplace-Beltrami operator on H^n

$$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\cosh r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}} =$$

$$= \frac{1}{\sinh^{n-1} r}\frac{\partial}{\partial}(\sinh^{n-1} r \frac{\partial}{\partial}) + \frac{1}{\sinh^2 r}\Delta_{\mathbb{S}^{n-1}},$$

Claim 1

Let's $w \in \mathbb{S}^{n-1}$, $b(w) = (w,1)$ *and* $[x, b(w)] = x_{n+1} - x_1 w_1 - x_2 w_2 - \dots - x_n w_n$. *Then* $[x, b(w)]^\mu$ *is an eigenfunction of* $\Delta_{\mathbb{H}^n}$ *with the eigenvalue* $\mu(\mu-1) + n\mu$.

Corollary 1

If $\mu = i\lambda - \rho, \rho = \frac{n-1}{2}$, *then* $\Delta_{\mathbb{H}^n}[x, b(w)]^{i\lambda-\rho} = -(\lambda^2 + \rho^2)[x, b(w)]^{i\lambda-\rho}$

Proof of the claim 1

$$\Box[x,b(w)]^\mu = -\frac{\partial^2[x,b(w)]^\mu}{\partial x_{n+1}^2} + \sum_{i=1}^{n}\frac{\partial^2[x,b(w)]^\mu}{\partial x_i^2} =$$
$$= \mu(\mu-1)[x,b(w)]^{\mu-2}(-1+w_1^2+...+w_n^2) = 0.$$

Using polar coordinates on Γ, $x = (t\sinh r w', t\cosh r) \in \Gamma$
$[x,b(w)]^\mu = t^\mu(\cosh r - \sinh r\langle w',w\rangle)^\mu.$

$$\Box[x,b(w)]^\mu = -(\frac{\partial^2}{\partial t^2} + \frac{n}{t}\frac{\partial}{\partial t})t^\mu(\cosh r - \sinh r\langle w',w\rangle)^\mu +$$
$$\frac{1}{t^2}\Delta_{\mathbb{H}^n} t^\mu(\cosh r - \sinh r\langle w',w\rangle)^\mu = 0$$

$$\Delta_{\mathbb{H}^n} t^\mu(\cosh r - \sinh r\langle w',w\rangle)^\mu = (\frac{\partial^2}{\partial t^2} + \frac{n}{t}\frac{\partial}{\partial t})t^\mu(\cosh r - \sinh r\langle w',w\rangle)^\mu$$

$$= (\mu(\mu-1) + n\mu)t^\mu(\cosh r - \sinh r\langle w',w\rangle)^\mu$$

The spherical function on $\mathbb{H}^n$

Definition 1

The spherical function on $\mathbb{H}^n$ is given by

$$\Phi_\lambda(x) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} [x, b(w)]^{i\lambda-\rho} dw,$$

σ_{n-1} *is the surface area of the sphere* $\mathbb{S}^{n-1}$

The spherical function $\Phi_\lambda(x)$ is an eigenfunction of Δ with the eigenvalue $-(\lambda^2 + \rho^2)$. Moreover, $\Phi_\lambda(0) = 1$.

$$\Delta_{\mathbb{H}^n} \Phi_\lambda(x) = \Delta_{\mathbb{H}^n} \frac{1}{\sigma_{n-1}} \int\limits_{\mathbb{S}^{n-1}} [x, b(w)]^{i\lambda-\rho} dw =$$

$$= \frac{1}{\sigma_{n-1}} \int\limits_{\mathbb{S}^{n-1}} \Delta_{\mathbb{H}^n} [x, b(w)]^{i\lambda-\rho} dw = \frac{-(\lambda^2 + \rho^2)}{\sigma_{n-1}} \int\limits_{\mathbb{S}^{n-1}} [x, b(w)]^{i\lambda-\rho} dw$$

The spherical function in the polar coordinates

In the polar coordinates on $\mathbb{H}^n$, $x = (\sinh rw', \cosh r)$, $r > 0$, $w' \in \mathbb{S}^{n-1}$.
$[x, b(w)]^\mu = (\cosh r - |w'|\langle \frac{w'}{|w'|}, w\rangle)^\mu = (\cosh r - \sinh r \cos(\widehat{f, w}))$,
$f = (f_1(w'_1, ..., w'_n), ..., f_n(w'_1, ..., w'_n))$

$$\Phi_\lambda(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{S}^{n-1}} [x, b(w)]^{i\lambda-\rho} dw =$$

$$= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_0^\pi (\cosh r - \sinh r \cos\theta)^{i\lambda-\rho} (sin\theta)^{2\rho-1} d\theta$$

Legendre polynomials:

$$P_\nu^\mu(\cosh\alpha) = \frac{2^\mu}{\sqrt{\pi}(\sinh\alpha)^\mu \Gamma(\frac{1}{2} - \mu)} \int_0^\pi (\cosh\alpha - \sinh\alpha\cos t)^{\nu+\mu} (\sin t)^{-2\mu} dt$$

$$\alpha = -r, \quad \mu = \frac{1}{2} - \rho, \quad \nu = i\lambda - \frac{1}{2}$$

$$\Phi_\lambda(x) = 2^{\rho-1/2} \Gamma(\rho + \frac{1}{2}) (\sinh r)^{1/2-\rho} P_{i\lambda-\frac{1}{2}}^{\frac{1}{2}-\rho}(\cosh r)$$

The spherical function for H^3

$n = 3$, then $\rho = 1, \mu = -\frac{1}{2}, \Gamma(\rho + \frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

$$\Phi_\lambda(r) = \sqrt{2}\Gamma(\frac{3}{2})(\sinh r)^{-\frac{1}{2}} P^{-\frac{1}{2}}_{i\lambda - \frac{1}{2}}(\cosh r).$$

$$P_\nu^{-\frac{1}{2}}(z) = (\frac{2}{\pi})^{\frac{1}{2}} \frac{(z^2-1)^{-\frac{1}{4}}}{2\nu+1} \left(\left[z + (z^2-1)^{\frac{1}{2}} \right]^{\nu+\frac{1}{2}} - \left[z + (z^2-1)^{\frac{1}{2}} \right]^{-\nu-\frac{1}{2}} \right).$$

$z = \cosh r$, then

$$\Phi_\lambda(r) = \sqrt{\frac{\pi}{2}}(\sinh r)^{-\frac{1}{2}} \sqrt{\frac{2}{\pi}} \frac{(\sinh r)^{-\frac{1}{2}}}{2i\lambda} [(\cosh r + \sinh r)^{i\lambda} -$$

$$-(\cosh r + \sinh r)^{-i\lambda}] = \frac{1}{2i\lambda \sinh r} 2i \sin(\lambda r) = \frac{\sinh(\lambda r)}{\lambda \sinh r}$$

The Fourier transform in $\mathbb{H}^n$

Definition 2

Let $f \in C_c^\infty(\mathbb{H}^n), w \in \mathbb{S}^{n-1}$. The Fourier transform of f in $\mathbb{H}^n$ is given by

$$\hat{f}(\lambda, w) = \int\limits_{\mathbb{H}^n} f(x)[x, b(w)]^{i\lambda-\rho} dx$$

When f is rotation invariant and radial, i.e. $f(k \circ x) = f(x)$ for all rotation k of $\mathbb{H}^n$, the Fourier transform of f is independent of $w \in \mathbb{S}^{n-1}$ and can be written

$$\hat{f}(x) = \int\limits_{\mathbb{H}^n} f(x)\Phi_\lambda(x)dx = \sigma_{n-1} \int\limits_0^\infty f_0(r)\Phi_\lambda(r) \sinh^{n-1} r dr,$$

where σ_{n-1} - the area surface of the sphere $\mathbb{S}^{n-1}$.

Properties of the Fourier transform in $L^1(\mathbb{R})$

The Fourier transform of a function f in $L^1(R)$ is defined by

$$\hat{f}(x) = \int_{\mathbb{R}} f(y) \exp(-2\pi i y x) dy$$

Theorem 1 (Properties of the Fourier Transform)

(I) Fourier Inversion. If $f \in L^1(\mathbb{R})$ and f is everywhere differentiable, then $\hat{\hat{f}}(-x) = f(x)$, for all x.
(2) Convolution. Suppose that f and g are in $L^1(\mathbb{R})$ and define the convolution $f * g$ by

$$f * g(x) = \int_{\mathbb{R}} f(x-y) g(y) dy.$$

Then $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$.

Properties of the Fourier transform in $\mathbb{H}^n$

Theorem 2 (Properties of the Fourier Transform)

(I) Fourier Inversion. Let $f \in C_c^\infty$ and radial. Then

$$f(x) = \frac{2^{n-1}}{2\pi\sigma_{n-1}} \int_0^\infty \hat{f}(\lambda)\Phi_\lambda(x)|c(\lambda)|^{-2} d\lambda,$$

where $c(\lambda) = \frac{2^{2\rho-1}\Gamma(\rho+1)\Gamma(i\lambda)}{\pi^{1/2}\Gamma(\rho+i\lambda)}$.

(2) Convolution. Suppose that f and g are rotation-invariant and radial function on $\mathbb{H}^n$ and

$$(a) \int_{\mathbb{H}^n} |f(x)|\Phi_0(x)dx < \infty \quad (b) \int_{\mathbb{H}^n} |g(x_{n+1})|\Phi_0(x)dx < \infty$$

and define the convolution $(f * g)$ by $(f * g)(x) = \int_{\mathbb{H}^n} f(y)g([x, y])dy$. Then $\widehat{f * g} = \hat{f} \cdot \hat{g}$.

Fundamental solution of the heat equation

$$\frac{1}{2}\Delta_{\mathbb{H}^n}\psi(t,r) = \frac{\partial}{\partial t}\psi(t,r), \tag{1}$$

$\Delta_{\mathbb{H}^n} = \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r}$,
$\psi(t,r)$ - fundamental solution. $s(r) = \sinh^{n-1} r, \Phi(r) = \Phi_0(\cosh r)$.
Applying the Fourier Transform to (1):

$$\underbrace{\frac{1}{2}(n-1)\int_0^\infty \Phi(r)\coth r \frac{\partial \psi(t,r)}{\partial r}\sinh^{n-1}(r)dr}_{(1)} +$$

$$+\underbrace{\frac{1}{2}\int_0^\infty \Phi(r)\frac{\partial^2\psi(t,r)}{\partial r^2}\sinh^{n-1}(r)dr}_{(2)} = \int_0^\infty \Phi(r)\frac{\partial\psi(t,r)}{\partial t}\sinh^{n-1}(r)dr$$

$$(1) \quad (n-1)\int_0^\infty \Phi(r)\coth r\frac{\partial\psi}{\partial r}\sinh^{n-1} r dr = \int_0^\infty \Phi(r)\frac{\partial\psi}{\partial r}\frac{\partial s}{\partial r}dr = \Phi(r)\frac{\partial s}{\partial r}\psi|_0^\infty -$$

$$-\int_0^\infty \psi\left(\frac{\partial\Phi}{\partial r}\frac{\partial s}{\partial r} + \Phi(r)\frac{\partial^2 s}{\partial r^2}\right)dr = -\int_0^\infty \psi\left(\frac{\partial\Phi}{\partial r}\frac{\partial s}{\partial r} + \Phi(r)\frac{\partial^2 s}{\partial r^2}\right)dr.$$

$$(2) \quad \int_0^\infty \Phi(r)\frac{\partial^2\psi}{\partial r^2}\sinh^{n-1} r dr = \int_0^\infty \psi(r,t)\frac{\partial^2}{\partial r^2}\left(\Phi(r)s(r)\right)dr =$$

$$= \int_0^\infty \psi(r,t)\left(s(r)\frac{\partial^2\Phi(r)}{\partial r^2} + 2\frac{\partial\Phi(r)}{\partial r}\frac{\partial s(r)}{\partial r} + \Phi(r)\frac{\partial^2 s(r)}{\partial r^2}\right)dr$$

$$\text{Left side: } \sigma_{n-1} \int_0^\infty \Phi(r) \Delta_{\mathbb{H}^n} \psi(t,r) \sinh^{n-1} r dr = \sigma_{n-1} \int_0^\infty \psi(t,r)(s(r) \frac{\partial^2 \Phi(r)}{\partial r^2} +$$

$$+ \frac{\partial \Phi(r)}{\partial r} \frac{\partial s(r)}{\partial r}) \sinh^{n-1} r dr = \sigma_{n-1} \int_0^\infty \psi(t,r) \Delta_{\mathbb{H}^n} \Phi(r) \sinh^{n-1} r dr =$$

$$= -(\lambda^2 + \rho^2) \sigma_{n-1} \int_0^\infty \psi(t,r) \Phi(r) \sinh^{n-1} r dr = -(\lambda^2 + \rho^2) \hat{\psi}(\lambda, t).$$

$$\text{Right side: } \qquad \sigma_{n-1} \int_0^\infty \Phi(r) \frac{\partial \psi(t,r)}{\partial t} \sinh^{n-1} r dr =$$

$$\sigma_{n-1} \frac{\partial}{\partial t} \int_0^\infty \Phi(r) \psi(t,r) \sinh^{n-1} r dr = \frac{\partial}{\partial t} \hat{\psi}(\lambda, t)$$

Then we have

$$\frac{-(\lambda^2+\rho^2)}{2}\hat{\psi}(\lambda,t)=\frac{\partial}{\partial t}\hat{\psi}(\lambda,t)$$

Therefore,

$$\hat{\psi}(\lambda,t)=\exp(\frac{-(\lambda^2+\rho^2)}{2}t),\quad \rho=\frac{n-1}{2}$$

Using Fourier Inversion formula we get

$$\psi(t,r)=\frac{2^{n-1}}{2\pi\sigma_{n-1}}\int\limits_0^\infty \exp(\frac{-(\lambda^2+\rho^2)}{2}t)\Phi_\lambda(x)|c(\lambda)|^{-2}d\lambda,$$

where $c(\lambda)=\frac{2^{2\rho-1}\Gamma(\rho+1)\Gamma(i\lambda)}{\pi^{1/2}\Gamma(\rho+i\lambda)}=\frac{2^{n-2}\Gamma(\frac{n}{2})\Gamma(i\lambda)}{\sqrt{\pi}\Gamma(\rho+i\lambda)}$ - Harish-Chandra c-function.

$$|c(\lambda)|^{-2}=\frac{\pi}{2^{2n-4}\Gamma(\frac{n}{2})^2}\frac{\Gamma(\rho+i\lambda)^2}{\Gamma(i\lambda)^2}$$

Fundamental solution in $\mathbb{H}^3$

$$|c(\lambda)|^{-2} = \frac{\pi}{2^{2n-4}\Gamma(\frac{n}{2})^2}\frac{\Gamma(\rho+i\lambda)^2}{\Gamma(i\lambda)^2} = \frac{\pi}{4\Gamma(\frac{3}{2})^2}\frac{\Gamma(1+i\lambda)^2}{\Gamma(i\lambda)^2} = \pi^2\lambda^2 \quad \Phi_\lambda(x) = \frac{\sinh(\lambda r)}{\lambda \sinh r}$$

$$\psi(t,r) = \frac{2^{n-1}}{2\pi\sigma_{n-1}}\int_0^\infty \exp(\frac{-(\lambda^2+\rho^2)}{2}t)\Phi_\lambda(x)|c(\lambda)|^{-2}d\lambda =$$

$$= \frac{\exp(-\frac{t}{2})}{2\pi^2}\int_0^\infty \exp(-\frac{\lambda^2}{2}t)\frac{\sinh(\lambda r)}{\lambda\sinh r}\pi^2\lambda^2 d\lambda =$$

$$= \frac{\exp(-\frac{t}{2})}{2\sinh r}\int_0^\infty \lambda\exp(-\frac{\lambda^2}{2}t)\sinh(\lambda r)d\lambda =$$

$$= \frac{\exp(-\frac{t}{2})}{2\sinh r}\frac{r\sqrt{\pi}}{\sqrt{2}t^{3/2}}\exp(-\frac{r^2}{2t}) = \frac{r}{(2\pi t)^{3/2}\sinh r}\exp(-\frac{t^2+r^2}{2t})$$

Fundamental solution in $\mathbb{H}^3$

Definition 3

The normal distributed rotation invariant random variable with parameter t on $\mathbb{H}^3$ has distribution function

$$\psi_t(r) = \frac{r}{(2\pi t)^{3/2}\sinh r}\exp\left(-\frac{t^2 + r^2}{2t}\right).$$

- MG Sur,F. I. Karpelevicii and V. N. Tutubalin. Limit Theorems for the Compositions of Distributions in the Lobachevsky Plane and Space. *Theory Probab. Appl., 4(4), 399–402. (4 pages)*.
- Boris Rubin. Radon,Cosine and SineTransforms on Real Hyperbolic Space *Advances in Mathematics 170, 206–223 (2002)*.
- Audrey Terras. Harmonic Analysis on Symmetric Spaces—Euclidean Space, the Sphere, and the Poincaré Upper Half-Plane.
- Audrey Terras. Noneuclidean Harmonic Analysis, the Central Limit Theorem, and Long Transmission Lines with Random Inhomogeneities. *Journal of multivariate analysis 15, 261-276 (1984)*

Thank you for attention