Viability properties of mean field type control systems

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Viability property

Let X be a set.
Dynamic system: $X \ni x_0 \multimap \Phi^t(x_0) \subset X$ satisfying

$$\Phi^s \circ \Phi^r = \Phi^{s+r}.$$

A function $[0, T] \ni t \mapsto x(t) \in X$ is a motion if $x(t) \in \Phi^t(x(0))$.

A set $K \subset X$ is called viable if, for any $x_0 \in K$, there exists a motion $x(t)$, $t \in [0, T]$ such that

$$x(t) \in K, \quad x(0) = x_0.$$

Viability theory

Viability theory characterizes the viability property in the terms of nonsmooth analysis.

The viability theory provides

- sustainability conditions in the different areas of science;
- forms of dynamical systems with the desired properties;
- link between control theory and PDEs;

Nonsmooth analysis

Objects:

- directional derivatives;
- sub- and superdifferentials;
- tangent cone;
- normal cone.

Nonsmooth analysis is used in

- optimization;
- control theory;
- theory of PDEs;

Viability in $\mathbb{R}^d$

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U.$$

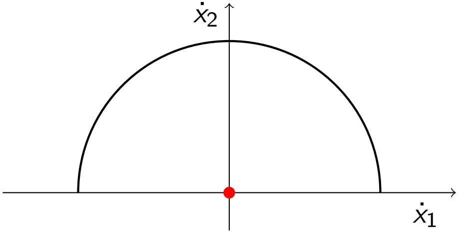
Here u is a control parameter. U is a metric compact.
Choice of u:

- u is constant;
- $u(t)$ is a measurable function;
- relaxed control: $\xi(t, du)$.

Example

$$\begin{cases} \dot{x}_1 = & \cos u \\ \dot{x}_2 = & \sin u \end{cases}, \quad u \in [0, \pi].$$

Is it possible to stay at $x_1 = x_2 = 0$, i.e. to minimize $\int_0^T \|x(t)\|^2 dt$ using only measurable controls?



Example

$$\begin{cases} \dot{x}_1 = & \cos u \\ \dot{x}_2 = & \sin u \end{cases}, \quad u \in [0, \pi].$$

Is it possible to stay at $x_1 = x_2 = 0$ using measurable controls?
No, but the control

$$u_\varepsilon(t) = \begin{cases} 0, & t \in [2k\varepsilon, (2k+1)\varepsilon) \\ \pi, & t \in [2(k+1)\varepsilon, 2(k+1)\varepsilon) \end{cases}$$

provides that the corresponding motion is

$$\|x_1(t)\| \leq \varepsilon, \quad \|x_2(t)\| = 0.$$

Relaxed controls

A weakly measurable function $[0, T] \ni t \mapsto \xi(t, du) \in \mathcal{P}(U)$ is called a relaxed control.

If $x_0 \in \mathbb{R}^d$, then the corresponding motion is a solution of

$$\dot{x}(t) = \int_U f(x(t), u)\xi(t, du), \quad x(0) = x_0.$$

Any relaxed control can be approximate by the measurable controls.

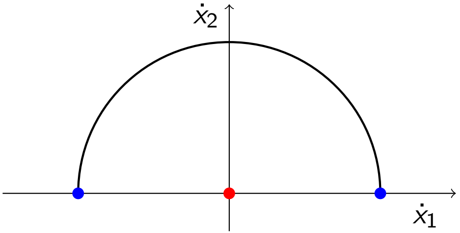
Example. Relaxed controls

If we put

$$u(t) \triangleq \frac{1}{2}\delta_0 + \frac{1}{2}\delta_\pi,$$

then the corresponding motion is

$$x_1(t) = x_2(t) \equiv 0.$$



Differential inclusion

Since

$$\left\{ \int_U f(x,u)\xi(du) : \xi \in \mathcal{P}(U) \right\} = \text{co}\{f(x,u) : u \in U\} =: F(x),$$

we can replace the control system with the differential inclusion

$$\dot{x} \in F(x).$$

Viability in $\mathbb{R}^d$

A set K is called viable under the differential inclusion $\dot{x} \in F(x)$ if, for any $x_0 \in K$, there exists a $x(\cdot)$ such that

- $x(0) = x_0$;
- $\dot{x}(t) \in F(x(t))$, $t \in [0, T]$;
- $x(t) \in K$, $t \in [0, T]$.

Viability theorem

Tangent cone:

$$T_K(x) \triangleq \left\{ v \in \mathbb{R}^d : \lim_{\tau \downarrow 0} \frac{1}{\tau} \operatorname{dist}(x + \tau v, K) = 0 \right\}.$$

Theorem (Nagumo)

The closed set $K \subset \mathbb{R}^d$ is viable if and only if

$$T_K(x) \cap F(x) \neq \varnothing, \quad x \in K.$$

Mean field type control problem

$$\frac{d}{dt}m(t) = \langle f(\cdot, m(t), u(t,\cdot)), \nabla\rangle m(t),$$

or (equivalently)

$$\partial_t m(t) + \mathrm{div}(f(x, m(t), u(t,x))m(t)) = 0.$$

Here $m(t)$ is a probability on the phase space.

Space of probability measures

- (X, ρ_X) is a metric space.
- $\mathcal{B}(X)$ is a Borel σ-algebra on X.
- $\mathcal{P}(X)$ stands for the set of probability measures.
- $\mathcal{P}^p(X)$ is the set of probabilities $m \in \mathcal{P}(X)$ such that, for some $x_0 \in X$, $\int_X (\rho_X(x, x_0))^p m(dx) < \infty$.
- p-th Kantorovich (Wasserstein) distance between $m', m'' \in \mathcal{P}^p(X)$:

$$W_p(m', m'') \triangleq \left[\inf_{\pi \in \Pi(m', m'')} \int_{X \times X} \rho_X^p(x', x'') \pi(d(x', x'')) \right]^{1/p}.$$

Here $\Pi(m', m'')$ denotes the set of probabilities on $X \times X$ with marginals equal to m' and m'' respectively.

Control of each agent

The mean field type control system corresponds to the control system of similar agents with the dynamics

$$\dot{x}(t) = f(x(t), m(t), u).$$

Notation and assumptions

- phase space: $\mathbb{T}^d \triangleq \mathbb{R}^d / \mathsf{Z}^d$;
- curves on $[s, r]$: $x(\cdot) \in C([s, r]; \mathbb{T}^d)$;
- if $x(\cdot) \in C([s, r]; \mathbb{T}^d)$, then

$$e_t(x(\cdot)) \triangleq x(t);$$

- $\mathrm{p}^i(x_1, x_2) = x_i$;
- f is Lipschitz in $\mathbb{T}^d \times \mathcal{P}^p(\mathbb{T}^d)$.

Mean field type differential inclusion

$$\frac{d}{dt}m(t) \in \langle F(x(t), m(t)), \nabla \rangle m(t),$$

where $F(x, m) \triangleq \text{co}\{f(x, m, u) : u \in U\}$.

Solutions of MFDI

We say that $[0, t] \ni t \mapsto m(t) \in \mathcal{P}^p(\mathbb{T}^d)$ solves MFDI if there exists a probability $\chi \in \mathcal{P}^p(C([0, T]; \mathbb{T}^d))$ such that

- χ-a.e. $x(\cdot)$ satisfy $\dot{x}(t) \in F(x(t), m(t))$;
- $m(t) = e_t \sharp \chi$.

Equivalent condition

Theorem (Marigonda, Quincampoix)

The flow of probabilities $m(t)$ solves MFDI iff there exists a function $v(t,x)$ such that

- $v(t,x) \in F(x, m(t))$ $m(t)$*-a.s.;*
- $\partial_t m(t) + \mathrm{div}(v(t,x)m(t)) = 0$ *in the sense of distributions.*

Viability in the Wasserstein space

We say that $K \subset \mathcal{P}^p(\mathbb{T}^d)$ is viable under MFDI if, given $m_0 \in K$, there exists $m(t)$, $t \in [0, T]$ such that

- $m(0) = m_0$;
- $m(t)$ solves MFDI;
- $m(t) \in K$, $t \in [0, T]$.

Distributions on tangent bundle

Let $m \in \mathcal{P}^p(\mathbb{T}^d)$.

- A probability $\beta \in \mathcal{P}^p(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\mathrm{p}^1 \sharp \beta = m$ is a tangent distribution to m;
- if $a > 0$, then

$$\mathcal{L}^a(m) \triangleq \{\beta \in \mathcal{P}^p(\mathbb{T}^d \times \mathbb{B}_a) : \mathrm{p}^1 \sharp \beta = m\};$$

- $\Theta^\tau(x, v) \triangleq x + \tau v$;
- $\Theta^\tau \sharp \beta$ is a shift of m along β.

Tangent cone in $\mathcal{P}^p(\mathbb{T}^d)$

Let $K \subset \mathcal{P}^p(\mathbb{T}^d)$.
We say that $\beta \in \mathcal{L}^a(m)$ is a tangent distributions to K at $m \in \mathcal{P}^p(\mathbb{T}^d)$ with the radius a, if there exist sequences $\{\tau_n\}_{n=1}^\infty \subset (0, +\infty)$, $\{\beta_n\}_{n=1}^\infty \subset \mathcal{L}^a(m)$ such that

$$\frac{1}{\tau_n}\mathrm{dist}(\Theta^{\tau_n}\sharp\beta_n, K) \to 0, \quad W_p(\beta_n, \beta) \to 0, \quad \tau_n \to 0 \text{ as } n \to \infty.$$

Denote the set of tangent distributions of the radius a to K at m by $\mathcal{T}_K^a(m)$.

Nagumo type viability theorem

Theorem (A.)

The closed set $K \subset \mathcal{P}^p(\mathbb{T}^d)$ is viable under MFDI if and only if there exists $a > 0$ such that

$$\mathcal{T}_K^a(m) \cap \mathcal{F}(m) \neq \varnothing, \quad m \in K.$$

Here

$$\mathcal{F}(m) \triangleq \{\beta \in \mathcal{P}^p(\mathbb{T}^d \times \mathbb{R}^d) : \mathrm{p}^1 \sharp \beta = m, \quad \mathrm{supp}(\beta) \subset \mathrm{gr}(F(\cdot, m)\}.$$

Proof is by compactness arguments.

Feedback control

Aim: design the way providing viability or approximate viability.

Feedback control in $\mathbb{R}^d$

$$\dot{x} = f(x, u(x)).$$

- any function $\mathfrak{u} : \mathbb{R}^d \to U$ is called a feedback strategy;
- if $\Delta = \{t_i\}_{i=0}^n$, then $x[t, x_0, \mathfrak{u}, \Delta] = x(t)$ is step-by-step motion satisfying

$$\dot{x}(t) = f(x(t), \mathfrak{u}[x(t_i)]), \quad t \in [t_i, t_{i+1}], \quad x(0) = x_0.$$

Extremal shift

Let $K \subset \mathbb{R}^d$.

- if $x \in K$, then put $\hat{\mathfrak{u}}[x]$ an arbitrary control;
- if $x \notin K$, then pick y to be a nearest to x point of K;

$$\langle x - y, f(x, \hat{\mathfrak{u}}[x])\rangle = \min_{u \in U} \langle x - y, f(x, u)\rangle.$$

Normal cones

- Proximal normal cone:
  $N_K^P(x) \triangleq \{w \in \mathbb{R}^d : \mathrm{dist}(x + tw, K) = t\|w\|\};$
- Normal cone:
  $N_K(x) = T_K^*(x) = \{w \in \mathbb{R}^d : \langle w, v\rangle \leq 0 \;\; \forall v \in T_K(x)\};$
- $$N_K(x) = \mathrm{co}\,\mathrm{Lim}_{y \to x : y \in K}\, N_K^P(y).$$

Clarke-Ledyaev viability theorem

Let

$$H(x, w) \triangleq \min_{u \in U} \langle w, f(x, u) \rangle.$$

Theorem (Clarke, Ledyaev)

Assume that

$$H(x, w) \leq 0 \text{ for every } x \in K, \quad w \in N_K^P(x).$$

Then, for any $x_0 \in K$

$$\lim_{\delta \downarrow 0} \sup\{\text{dist}(x[t, x_0, \hat{\mathfrak{u}}, \Delta], K) : \quad t \in [0, T],$$
$$\Delta \text{ is a partition of } [0, T], \quad d(\Delta) \leq \delta\} = 0.$$

Clarke-Ledyaev viability theorem

The set K is viable if and only if $H(x, w) \leq 0$ for every $x \in K, \quad w \in N_K^P(x)$.

Feedback control in $\mathcal{P}^2(\mathbb{T}^d)$

$$\frac{d}{dt}m(t) = \langle f(\cdot, m(t), u(\cdot, m(t))), \nabla\rangle m(t).$$

Feedback strategy: $\mathfrak{u}[m]$ is a probability on $\mathcal{P}^2(\mathbb{T}^d \times U)$ such that $\mathrm{p}^1 \sharp \mathfrak{u}[m] = m$.

Motion produced by feedback strategy

Given $[s, r] \ni t \mapsto m(t) \in \mathcal{P}^2(\mathbb{T}^d)$, let $\mathrm{traj}^{s,r}_{m(\cdot)}$ assign to $(y, u) \in \mathbb{T}^d \times U$ the solution of the initial value problem

$$\dot{x}(t) = f(x(t), m(t), u), \quad x(s) = y.$$

Motion produced by feedback strategy

Let $m_0 \in \mathcal{P}^2(\mathbb{T}^d)$, $\Delta = \{t_i\}_{i=0}^n$, we say that $m(t) = m[t, m_0, \mathfrak{u}, \Delta]$ is a step-by-step motion if there exist $\chi_0, \ldots, \chi_{n-1}$ such that

- $m(0) = m_0$;
- $\chi_i \in \mathcal{P}^2(C([t_i, t_{i+1}]; \mathbb{T}^d))$;
- $m(t) = e_t \sharp \chi_i$, $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, n-1$;
- $\chi_i = \mathrm{traj}_{m(\cdot)}^{t_i, t_{i+1}} \sharp \mathfrak{u}[m(t_i)]$.

Extremal shift in $\mathcal{P}^2(\mathbb{T}^d)$

- If $x, y \in \mathbb{T}^d$, then $\ell(x, y)$ is a vector $w \in x - y$ of the minimal norm;
- If, additionally, $m \in \mathcal{P}^2(\mathbb{T}^d)$, then let $\hat{u}(x, y, m)$ minimize

$$\langle \ell(x, y), f(x, m, u) \rangle.$$

- Let $\nu \in \mathcal{P}^2(\mathbb{T}^d)$, be a nearest to m element of K;
- π optimal plan between m and ν.

$$\hat{\mathrm{u}}[m] \triangleq (\mathrm{p}^1, \hat{u}(\cdot, \cdot, m)) \sharp \pi.$$

Proximal normals in $\mathcal{P}^2(\mathbb{T}^d)$

Let

- $K \subset \mathcal{P}^2(\mathbb{T}^d)$,
- $m \in K$,
- $\mu \in \mathcal{P}^2(\mathbb{T}^d)$ be such that $\mathrm{dist}(\mu, K) = W_2(\mu, m)$;
- π be a optimal plan between μ and m.

Then,

$$\gamma \triangleq (\mathrm{p}^2, \ell)\sharp\pi$$

is called a proximal normal.

The set of proximal normals is denoted by $\mathcal{N}_K^P(m)$.

Clarke-Ledyaev type viability theorem

Let $m \in \mathcal{P}^2(\mathbb{T}^d)$, $\gamma \in \mathcal{P}^2(\mathbb{T}^d \times \mathbb{R}^d)$, $\mathrm{p}^1 \sharp \gamma = m$. Put

$$\mathcal{H}(m,\gamma) \triangleq \int_{\mathbb{T}^d \times \mathbb{R}^d} \min_{u \in U} \langle w, f(x,m,u) \rangle \gamma(d(x,w)).$$

Theorem (A., Marigonda, Quincampoix)

Assume that $\mathcal{H}(m,\gamma) \leq 0$, *for any* $m \in K$, $\gamma \in \mathcal{N}_K^P(m)$. *Then,* $m_0 \in K$

$$\lim_{\delta \downarrow 0} \sup \{ \mathrm{dist}(m[t, m_0, \hat{\mathfrak{u}}, \Delta], K) : \quad t \in [0, T],$$
$$\Delta \textit{ is a partition of } [0,T], \quad d(\Delta) \leq \delta \} = 0.$$

Clarke-Ledyaev type viability theorem

The set K is viable under MFDI if and only if $\mathcal{H}(m, \gamma) \leq 0$, for any $m \in K$, $\gamma \in \mathcal{N}_K^P(m)$.

Example

Let $G \subset \mathbb{T}^d$ be closed; $K \triangleq \mathcal{P}^2(G)$.

Property. K is viable under MFDI if and only if

$$\min_{u \in U} \langle w, f(x, m, u) \rangle \leq 0$$

for every $x \in G$, $w \in N_K^P(x)$.

Conclusion

- We define tangent and proximal normal sets to a set in the space of probabilities.
- The definitions are reasonable due to the viability theorems.
- Future works include normal set and the relation between normal, proximal normal and tangent elements.

Thank you for your attention!