On the value of a non-Markovian Dynkin games with partial and asymmetric information

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Joint work with Tiziano De Angelis and Nikita Merkulov

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Outline

1. Motivation
2. Randomised stopping times
3. Dynkin games with asymmetric information
4. Main result
5. Sketch of the proof

Optimal stopping problem

Optimal stopping

$$x \mapsto \sup_{\sigma \leq T} \mathbb{E}_x \left[g(X_\sigma) \right]$$

Horizon: $T > 0$

Probability space: $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$

Markov process: $(X_t)_{t \in [0,T]}$

Control: σ - an $\mathcal{F}_t$-stopping time

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Introduce an opponent:

Dynkin, E.B. (1969) *Game variant of a problem of optimal stopping* Soviet Math. Dokl.

Zero-sum Markovian stopping game (Dynkin game)

Minimiser

chooses stopping time τ

Maximiser

chooses stopping time σ

$$N(x,\tau,\sigma) = \mathbb{E}_x\Big[\mathbf{1}_{\{\tau\le\sigma\}\cap\{\tau<T\}} f(X_\tau) + \mathbf{1}_{\{\sigma<\tau\}\cap\{\sigma<T\}} g(X_\sigma) + \mathbf{1}_{\sigma=\tau=T} h(X_T)\Big]$$

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Value of the game

$$V_*(x) = V^*(x)$$

$$V_*(x) = \sup_\sigma \inf_\tau N(x,\tau,\sigma)$$

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Nash equilibrium (τ^*,σ^*)

$$N(x,\tau,\sigma^*) \ge N(x,\tau^*,\sigma^*) \quad \forall\tau$$

$$N(x,\tau^*,\sigma) \le N(x,\tau^*,\sigma^*) \quad \forall\sigma$$

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E. Ekström, G. Peskir (2008) *Optimal Stopping Games for Markov Processes*, SICON

Assumptions:

$$f(x) \ge h(x) \ge g(x)$$

Theorem. If (X_t) is strong Markov and càdlàg, and f, g, h continuous, then the game has a value:

$$\sup_\sigma \inf_\tau N(x,\tau,\sigma) = \inf_\tau \sup_\sigma N(x,\tau,\sigma).$$

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Theorem. If, additionally, (X_t) is quasi left-continous, then there is a Nash equilibrium.

Zero-sum non-Markovian stopping game

$$N(\tau,\sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau\le\sigma\}\cap\{\tau<T\}} f_\tau + \mathbf{1}_{\{\sigma<\tau\}\cap\{\sigma<T\}} g_\sigma + \mathbf{1}_{\sigma=\tau=T} h\Big]$$

J.P. Lepeltier, M.A. Maingueneau (1984) *Le Jeu de Dynkin en Theorie Generale Sans L'Hypothese de Mokobodski*, Stochastics

(f_t), (g_t) càdlàg bounded processes

$$f_t \ge g_t, \qquad f_{T-} \ge h \ge g_{T-}$$

Theorem. The value exists:

$$\sup_\sigma \inf_\tau N(\tau,\sigma) = \inf_\tau \sup_\sigma N(\tau,\sigma).$$

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Proof. Snell envelope type approach + ε-optimal strategies

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Touzi, Vieille (2002) *Continuous-time Dynkin games with mixed strategies*, SICON

Game options

Y. Kifer (2000) *Game options*, Finance and Stochastics

Buyer gets a payoff $(S_t - K)^+$ when exercises at t

Seller incurs a penalty $(S_t - K)^+ + L$ when recalls at t

Theorem. Price = value of the game

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Theorem. Price = value of the game

What if players have access to different information, for example, one is an insider?

Partial/asymmetric information

$$N(\tau,\sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau \le \sigma\} \cap \{\tau < T\}} f_\tau + \mathbf{1}_{\{\sigma < \tau\} \cap \{\sigma < T\}} g_\sigma + \mathbf{1}_{\sigma = \tau = T} h\Big]$$

- $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ filtered probability space satisfying usual conditions
- f, g are $(\mathcal{F}_t)$-adapted processes; h is an $(\mathcal{F}_T)$-measurable random variable

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Minimiser

Observation $(\mathcal{F}^1_t) \subset (\mathcal{F}_t)$

Chooses $(\mathcal{F}^1_t)$-stopping time τ

Maximiser

Observation $(\mathcal{F}^2_t) \subset (\mathcal{F}_t)$

Chooses $(\mathcal{F}^2_t)$-stopping time σ

- Does the game has a value

$$\sup_\sigma \inf_\tau N(\tau,\sigma) = \inf_\tau \sup_\sigma N(\tau,\sigma)?$$

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- Does the game has a value

$$\sup_\sigma \inf_\tau N(\tau,\sigma) = \inf_\tau \sup_\sigma N(\tau,\sigma)?$$

- It is well known that the value does not exist in general.

Randomised stopping times

Definition

For $(\mathcal{G}_t) \subset (\mathcal{F}_t)$, a r.v. τ_R is a $(\mathcal{G}_t)$-randomised stopping time if there are

- $U \sim U(0,1)$ independent from $\mathcal{F}_T$, and
- $(\mathcal{G}_t)$-adapted non-decreasing càdlàg (ξ_t) with $\xi_{0-} = 0$ and $\xi_T = 1$

such that

$$\tau_R = \inf\{t \geq 0 : \xi_t > U\}.$$

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Examples.

- pure stopping time τ corresponds to $\xi_t = \mathbf{1}_{\tau \leq t}$
- stopping with intensity $\lambda(t)$ corresponds to absolutely continuous (ξ_t)
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Remark. This definition is equivalent to a more classical definition of mixed strategy when $\tau_R : \Omega \times (0,1) \to [0, T]$ is such that

- it is $\mathcal{G}_1 \otimes \mathcal{B}(0,1)$-measurable,
- $\omega \mapsto \tau_R(\omega, u)$ is a $(\mathcal{G}_t)$-stopping time for each fixed $u \in (0,1)$.

Existing literature

Grün (2013) *On Dynkin games with infomplete information*, SICON

- Markovian setting with a diffusion (X_t)
- finite number of regimes θ selecting payoff functions $f^\theta(x)$, $g^\theta(x)$ and $h^\theta(x)$
- only one player knows θ
- existence of value for randomised strategies for both players
- convex and differential techniques adapted from Cardaliaguet (2007) *Differential games with asymmetric information* SICON

Gensbittel, Grün (2019) *Zero-sum stopping games with asymmetric information*, Mathematics of Operations Research

- each player observes own stochastic process (finite state space Markov process)
- payoff depends on both processes
- existence of value in randomised strategies
- convex and differential techniques as in the other paper

Existing literature

De Angelis, Ekstrom, Glover (2018) *Dynkin games with incomplete and asymmetric information* arxiv:1810.07674

- univariate dynamics $dX_t = \mu^\theta(X_t)dt + \sigma(X_t)dW_t$
- verification result proved
- explicit solution for GBM and linear payoffs: ξ_t is absolutely continuous wrt local time at some $B \in \mathbb{R}$

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Touzi, Vieille (2002) *Continuous-time Dynkin games with mixed strategies*, SICON

- distinct from the above approaches
- uses Sion's min-max theorem
- payoffs (f_t) and (g_t) semimartingales with integrable sup-norm

Main theorem

$$N(\tau,\sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau\le\sigma\}\cap\{\tau<T\}} f_\tau + \mathbf{1}_{\{\sigma<\tau\}\cap\{\sigma<T\}} g_\sigma + \mathbf{1}_{\sigma=\tau=T} h\Big]$$

Minimiser

Observation $(\mathcal{F}^1_t) \subset (\mathcal{F}_t)$

Chooses $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$

Maximiser

Observation $(\mathcal{F}^2_t) \subset (\mathcal{F}_t)$

Chooses $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$

Theorem

Under the assumptions on the next slide, the value exists in randomised strategies, i.e.

$$\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} N(\tau,\sigma) = \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma).$$

Assumptions

$$N(\tau,\sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau\le\sigma\}\cap\{\tau<T\}} f_\tau + \mathbf{1}_{\{\sigma<\tau\}\cap\{\sigma<T\}} g_\sigma + \mathbf{1}_{\sigma=\tau=T} h\Big]$$

- All filtrations satisfy usual conditions
- $\mathbb{E}\big[\sup_{t\in[0,T]}\big(|f_t| + |g_t|\big)\big] < \infty$
- $f_t \ge g_t$ and $f_T \ge h \ge g_T$
- $f_t = f_t^1 + f_t^2$, $g = g_t^1 + g_t^2$, where
  - $(f_t^1), (g_t^1)$ are $(\mathcal{F}_t)$-adapted regular processes

    P.A. Meyer (1978) *Convergence faible et compacité des temps d'arrêt, d'après Baxter et Chacón*, Sèminaire de probabilités

  - $(f_t^2), (g_t^2)$ are $(\mathcal{F}_t)$-adapted càdlàg piecewise constant processes of integrable variation with no jumps at 0 and T
  - either (f_t^2) is non-increasing or (g_t^2) is non-decreasing

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Our framework encompasses virtually all (known to us) examples of zero-sum Dynkin games (in continuous time) with partial/asymmetric information.

Program

1. Reformulate as a game between singular controllers
2. Show existence of value when one player uses absolutely continuous controls
3. Extend to general singular controls

Reformulation

Lemma

Let $\tau \in \mathcal{T}^R(\mathcal{F}^1_t)$, $\sigma \in \mathcal{T}^R(\mathcal{F}^2_t)$ *with generating processes* ξ_t, ζ_t *and independent randomisation devices.*

$$\mathbb{E}\left[f_\tau \mathbf{1}_{\{\tau \le \sigma\}\cap\{\tau < T\}}\right] = \mathbb{E}\left[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t\right]$$

$$\mathbb{E}\left[g_\sigma \mathbf{1}_{\{\sigma < \tau\}\cap\{\sigma < T\}}\right] = \mathbb{E}\left[\int_{[0,T)} g_t(1-\xi_t)d\zeta_t\right]$$

Reformulation

Lemma

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$$\mathbb{E}\left[g_\sigma \mathbf{1}_{\{\sigma < \tau\} \cap \{\sigma < T\}}\right] = \mathbb{E}\left[\int_{[0,T)} g_t (1 - \xi_t) d\zeta_t\right]$$

$$N(\tau, \sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau \le \sigma\} \cap \{\tau < T\}} f_\tau + \mathbf{1}_{\{\sigma < \tau\} \cap \{\sigma < T\}} g_\sigma + \mathbf{1}_{\sigma = \tau = T} h\Big]$$

$$\Downarrow$$

$$N(\xi, \zeta) = \mathbb{E}\left[\int_{[0,T)} f_t (1 - \zeta_{t-}) d\xi_t + \int_{[0,T)} g_t (1 - \xi_t) d\zeta_t + h \Delta\xi_T \Delta\zeta_T\right]$$

Sion's min-max theorem

Theorem

Let X be a convex subset of a linear topological space and Y a compact convex subset of a linear topological space. Let N be a real-valued function on $X \times Y$ such that

1. *$N(x, \cdot)$ is upper semi continuous and quasi-concave on Y for each $x \in X$,*
2. *$N(\cdot, y)$ is lower semi continuous and quasi-convex on X for each $y \in Y$,*

Then

$$\inf_{x \in X} \sup_{y \in Y} N(x, y) = \sup_{y \in Y} \inf_{x \in X} N(x, y).$$

M. Sion (1958) *On general minmax theorems*, Pacific J. Math.

H. Komiya (1988) *Elementary proof for Sion's minmax theorem*, Kodai Math. J.

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$$N(\tau, \sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau \le \sigma\} \cap \{\tau < T\}} f_\tau + \mathbf{1}_{\{\sigma < \tau\} \cap \{\sigma < T\}} g_\sigma + \mathbf{1}_{\sigma = \tau = T} h\Big]$$

Value with continuous controls

$$\mathcal{A}(\mathcal{G}_t) := \{\rho \,:\, \rho \text{ is } (\mathcal{G}_t)\text{-adapted with } t \mapsto \rho_t(\omega) \text{ càdlàg, non-decreasing, } \rho_{0-}(\omega) = 0 \text{ and } \rho_T(\omega) = 1 \text{ for all } \omega \in \Omega\}.$$

$$\mathcal{A}_{ac}(\mathcal{G}_t) := \{\rho \in \mathcal{A}(\mathcal{G}_t) :\, t \mapsto \rho_t(\omega) \text{ is absolutely continuous on } [0, T)\}.$$

Theorem

Assume (g^2_t) is non-decreasing. Then

$$\inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}^1_t)} \sup_{\zeta \in \mathcal{A}(\mathcal{F}^2_t)} N(\xi, \zeta) = \sup_{\zeta \in \mathcal{A}(\mathcal{F}^2_t)} \inf_{\xi \in \mathcal{A}_{ac}(\mathcal{F}^1_t)} N(\xi, \zeta).$$

Proof

Embed sets $\mathcal{A}_{ac}(\mathcal{F}_t^1)$ and $\mathcal{A}(\mathcal{F}_t^2)$ in

$$L^2\big([0,T]\times\Omega, \mathcal{B}([0,T])\times\mathcal{F}, \lambda\times\mathbb{P}\big)$$

with the weak topology.

Weak compactness of $\mathcal{A}(\mathcal{F}_t^2)$:

- Banach-Alaoglu $\implies$ unit ball is weakly compact
- closedness in L^2
- convexity + strong closedness $\implies$ weak closedness

Convexity of $N(\xi,\cdot)$: trivial, see

$$N(\xi,\zeta)=\mathbb{E}\Big[\int_{[0,T)} f_t(1-\zeta_{t-})d\xi_t+\int_{[0,T)} g_t(1-\xi_t)d\zeta_t+h\Delta\xi_T\Delta\zeta_T\Big]$$

Upper semicontinuity for regular f, g

Fix $\xi \in \mathcal{A}_{ac}(\mathcal{F}^1_t)$. We need to prove the upper semicontinuity of

$$\zeta \mapsto N(\xi, \zeta).$$

Consider a sequence $(\zeta^n)_{n\geq 1} \subset \mathcal{A}(\mathcal{F}^2_t)$ converging to $\zeta \in \mathcal{A}(\mathcal{F}^2_t)$ strongly in L^2. We have to show that

$$\limsup_{n\to\infty} N(\xi, \zeta^n) \leq N(\xi, \zeta).$$

Then, as level sets are convex, this implies their weak closedness, so upper semicontinuity in the weak topology.

Assume, by contradiction, that $\limsup_{n\to\infty} N(\xi, \zeta^n) > N(\xi, \zeta)$. There is a subsequence (n_k) over which we have $(\mathbb{P} \times \lambda)$-a.e. convergence of ζ^{n_k} to ζ and $\lim_{k\to\infty} N(\xi, \zeta^{n_k}) > N(\xi, \zeta)$. Denote this subsequence as (ζ^n).

Since ξ is absolutely continuous on $[0, T)$, by dominated convergence

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t(1 - \zeta^n_{t-}) d\xi_t\right] = \mathbb{E}\left[\int_{[0,T)} f_t(1 - \zeta_{t-}) d\xi_t\right].$$

So we have convergence for the first term of

$$N(\xi, \zeta^n) = \mathbb{E}\left[\int_{[0,T)} f_t(1 - \zeta^n_{t-}) d\xi_t + \int_{[0,T)} g_t(1 - \xi_t) d\zeta^n_t + h\Delta\xi_T \Delta\zeta^n_T\right].$$

$$
\begin{aligned}
&\mathbb{E}\Big[\int_{[0,T)} g_t(1-\xi_t)d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\Big] \\
&= \mathbb{E}\Big[\int_{[0,T)} g_t(1-\xi_{t-})d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\Big] \\
&= \mathbb{E}\Big[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t^n + (h-g_T)\Delta\xi_T\Delta\zeta_T^n\Big],
\end{aligned}
$$

$$\begin{aligned}
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&=\mathbb{E}\Big[\int_{[0,T)} g_t(1-\xi_{t-})d\zeta_t^n + h\Delta\xi_T\Delta\zeta_T^n\Big]\\
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\end{aligned}$$

We prove a lot of results of this kind

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t^n\Big] = \mathbb{E}\Big[\int_{[0,T]} g_t(1-\xi_{t-})d\zeta_t\Big],$$

and

$$\limsup_{n\to\infty}\mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T^n\big] \le \mathbb{E}\big[(h-g_T)\Delta\xi_T\Delta\zeta_T\big].$$

In conclusion

$$\limsup_{n\to\infty} N(\xi,\zeta^n) \le N(\xi,\zeta),$$

a contradiction.

Approximation with continuous controls

Theorem

Assume (g^2_t) is non-decreasing. Then for any $\xi \in \mathcal{A}(\mathcal{F}^1_t)$ there is a sequence $(\xi^n) \subset \mathcal{A}_{ac}(\mathcal{F}^1_t)$ such that

$$\limsup_{n\to\infty} N(\xi^n, \zeta) \le N(\xi, \zeta).$$

Proof that the value exists:

$$\inf_{\xi\in\mathcal{A}_{ac}(\mathcal{F}^1_t)} \sup_{\zeta\in\mathcal{A}(\mathcal{F}^2_t)} N(\xi,\zeta) \ge \inf_{\xi\in\mathcal{A}(\mathcal{F}^1_t)} \sup_{\zeta\in\mathcal{A}(\mathcal{F}^2_t)} N(\xi,\zeta)$$

Approximation with continuous controls

Theorem

Assume (g_t^2) is non-decreasing. Then for any $\xi \in \mathcal{A}(\mathcal{F}_t^1)$ there is a sequence $(\xi^n) \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ such that

$$\limsup_{n\to\infty} N(\xi^n, \zeta) \le N(\xi, \zeta).$$

Proof that the value exists:

$$\begin{aligned}\inf_{\xi\in\mathcal{A}_{ac}(\mathcal{F}_t^1)} \sup_{\zeta\in\mathcal{A}(\mathcal{F}_t^2)} N(\xi,\zeta) &\ge \inf_{\xi\in\mathcal{A}(\mathcal{F}_t^1)} \sup_{\zeta\in\mathcal{A}(\mathcal{F}_t^2)} N(\xi,\zeta) \\ &\ge \sup_{\zeta\in\mathcal{A}(\mathcal{F}_t^2)} \inf_{\xi\in\mathcal{A}(\mathcal{F}_t^1)} N(\xi,\zeta)\end{aligned}$$

Approximation with continuous controls

Theorem

Assume (g_t^2) is non-decreasing. Then for any $\xi \in \mathcal{A}(\mathcal{F}_t^1)$ there is a sequence $(\xi^n) \subset \mathcal{A}_{ac}(\mathcal{F}_t^1)$ such that

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&\ge \sup_{\zeta\in\mathcal{A}(\mathcal{F}_t^2)} \inf_{\xi\in\mathcal{A}_{ac}(\mathcal{F}_t^1)} N(\xi,\zeta),
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Approximation with continuous controls

Theorem

Assume (g^2_t) is non-decreasing. Then for any $\xi \in \mathcal{A}(\mathcal{F}^1_t)$ there is a sequence $(\xi^n) \subset \mathcal{A}_{ac}(\mathcal{F}^1_t)$ such that

$$\limsup_{n\to\infty} N(\xi^n, \zeta) \le N(\xi, \zeta).$$

Proof that the value exists:

$$\begin{aligned}\inf_{\xi\in\mathcal{A}_{ac}(\mathcal{F}^1_t)} \sup_{\zeta\in\mathcal{A}(\mathcal{F}^2_t)} N(\xi,\zeta) &\ge \inf_{\xi\in\mathcal{A}(\mathcal{F}^1_t)} \sup_{\zeta\in\mathcal{A}(\mathcal{F}^2_t)} N(\xi,\zeta) \\ &\ge \sup_{\zeta\in\mathcal{A}(\mathcal{F}^2_t)} \inf_{\xi\in\mathcal{A}(\mathcal{F}^1_t)} N(\xi,\zeta) \\ &\ge \sup_{\zeta\in\mathcal{A}(\mathcal{F}^2_t)} \inf_{\xi\in\mathcal{A}_{ac}(\mathcal{F}^1_t)} N(\xi,\zeta),\end{aligned}$$

so

$$\inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} N(\tau,\sigma) = \sup_{\sigma\in\mathcal{T}^R(\mathcal{F}^2_t)} \inf_{\tau\in\mathcal{T}^R(\mathcal{F}^1_t)} N(\tau,\sigma).$$

Summary

$$N(\tau,\sigma) = \mathbb{E}\Big[\mathbf{1}_{\{\tau\le\sigma\}\cap\{\tau<T\}} f_\tau + \mathbf{1}_{\{\sigma<\tau\}\cap\{\sigma<T\}} g_\sigma + \mathbf{1}_{\sigma=\tau=T} h\Big]$$

- Value of a stopping game with partial/asymmetric information in randomised strategies.
- Value may not exist in pure (non-randomised) stopping times.
- Necessity of assumptions shown with counterexamples.
- Framework encompasses most of such games from the literature.
- The proof goes through a reformulation as a game between singular controllers and application of Sion's theorem.

T. De Angelis, N. Merkulov, J. Palczewski (2020) *On the value of non-Markovian Dynkin games with partial and asymmetric information*, https://arxiv.org/abs/2007.10643

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Thank you