On the Ambrosio–Figalli–Trevisan superposition principle

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Let us consider the Cauchy problem for the Fokker–Planck–Kolmogorov equation

$$\partial_t \mu_t = \partial_{x_i}\partial_{x_j}\big(a^{ij}\mu_t\big) - \partial_{x_i}\big(b^i\mu_t\big), \quad \mu_0 = \nu. \tag{1}$$

Below we write this equation in the short form

$$\partial_t \mu_t = L^*\mu_t,$$

where L^* is the formal adjoint operator to the differential operator

$$Lu = a^{ij}\partial_{x_i}\partial_{x_j}u + b^i\partial_{x_i}u,$$

We assume throughout that the matrix

$$A(t,x) = (a^{ij}(t,x))_{i,j\leq d}$$

is symmetric and nonnegative definite and the functions

$$(t,x) \mapsto a^{ij}(t,x) \text{ and } (t,x) \mapsto b^{i}(t,x)$$

are Borel measurable on $[0,T] \times \mathbb{R}^d$.

By a solution we mean a mapping $t \mapsto \mu_t$ from $[0, T]$ to the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ that is continuous with respect to the weak topology and satisfies the integral equality

$$\int_{\mathbb{R}^d} \varphi \, d\mu_t = \int_{\mathbb{R}^d} \varphi \, d\nu + \int_0^t \int_{\mathbb{R}^d} L\varphi \, d\mu_s \, ds$$

for all $t \in [0, T]$ and all $\varphi \in C_0^\infty(\mathbb{R}^d)$, where it is assumed that a^{ij} and b^i are locally (i.e., on compact sets in $[0, T] \times \mathbb{R}^d$) integrable with respect to the measure $\mu_t \, dt$:

$$a^{ij}, b^i \in L^1_{loc}(\mu_t \, dt).$$

Let ν be a probability Borel measure on $\mathbb{R}^d$. Recall that a probability measure P_ν on the path space

$$\Omega_d := C([0, T], \mathbb{R}^d)$$

of continuous functions $\omega\colon [0, T] \to \mathbb{R}^d$ with its standard sup-norm $\|\omega\| = \sup_t |\omega(t)|$ is a solution of the martingale problem with the operator L and the initial condition ν if

(M1) $P_\nu\big(\omega\colon \omega(0) \in B\big) = \nu(B)$ for all Borel sets $B \subset \mathbb{R}^d$,

(M2) for every function $f \in C_0^\infty(\mathbb{R}^d)$, the function

$$(\omega, t) \mapsto f(\omega(t)) - f(\omega(0)) - \int_0^t Lf(s, \omega(s))\, ds$$

is a martingale with respect to the measure P_ν and the natural filtration $\mathcal{F}_t = \sigma(\omega(s), s \in [0, t])$,

MAIN QUESTION

Is it true that for every probability solution μ_t of the Cauchy problem

$$\partial_t \mu_t = L^* \mu_t, \quad \mu_0 = \nu,$$

there exists a solution P_ν of the corresponding martingale problem such that

(M3) for every function $f \in C_0^\infty(\mathbb{R}^d)$, there holds the equality

$$\int_{\mathbb{R}^d} f \, d\mu_t = \int_{\Omega_d} f(\omega(t)) \, P_\nu(d\omega) \quad \forall\, t \in [0, T]?$$

The latter means that μ_t is the law of $\omega(t)$ under P_ν.

FPK equations

We recall known existence and uniqueness results for Fokker–Planck–Kolmogorov equations.
Suppose that

- b is locally bounded, i.e., for every ball $U \subset \mathbb{R}^d$, there is a number $B = B(U) \geq 0$ such that

$$|b(x,t)| \leq B(U) \quad \forall\, x \in U,\ t \in [0,T],$$

- A is locally Lipschitzian in x and locally strictly positive, i.e., for every ball $U \subset \mathbb{R}^d$, there exist numbers $\lambda = \lambda(U) \geq 0$, $\alpha = \alpha(U) > 0$ and $m = m(U) > 0$ such that

$$|a^{ij}(x,t) - a^{ij}(y,t)| \leq \lambda|x-y|, \quad \alpha \cdot \mathrm{I} \leq A(x,t) \leq m \cdot \mathrm{I}$$

for every $x, y \in U,\ t \in [0,T]$.

Under this assumptions there hold

- **Existence:** for every probability measure ν there exists a subprobability solution $\mu = \mu_t(dx)\, dt$ ($\mu_t \geq 0$ and $\mu_t(\mathbb{R}^d) \leq 1$) of the Cauchy problem (1). Moreover if at least one of the following two conditions is fulfilled:
  (i) $(1+|x|)^{-2}|a^{ij}|,\ (1+|x|)^{-1}|b|, \in L^1(\mathbb{R}^d \times [0,T], \mu)$,
  (ii) there exist a nonnegative function $V \in C^2(\mathbb{R}^d)$ (Lyapunov function) and a number $M \geq 0$ such that $\lim_{|x|\to\infty} V(x) = +\infty$ and $LV \leq MV$,
  then μ_t are probability measures ($\mu_t \geq 0$ and $\mu_t(\mathbb{R}^d) = 1$).
- **Uniqueness:** if at least one of the conditions (i) or (ii) is fulfilled, then such solution is unique.

Example

Let $A = I$. Suppose that for some numbers $\gamma_1 > 0$ and $\gamma_2 > 0$ we have

$$\langle b(x,t), x\rangle \leq \gamma_1 + \gamma_2 |x|^2.$$

Then there exists a unique probability solution.

Let us consider examples when the Cauchy problem for the Fokker–Palnck–Kolmogorov equation has at least two distinct probability solutions.

Example

(I) There exists a smooth function B on $\mathbb{R}$ such that the probability solution ν of the equation $\nu'' - (B\nu)' = 0$ is not invariant measure for the corresponding semigroup T_t with the generator L but only subinvariant: $T_t^*\nu < \nu$ if $t > 0$.

(II) Let now $C(y) = (C^1(y), C^2(y))$ be a smooth vector field on $\mathbb{R}^2$ for which there are two different probability solutions σ_1 and σ_2 of the equation $\Delta\sigma - \mathrm{div}(C\sigma) = 0$.

Set

$$\mu_t^1 = \nu \otimes \sigma_1,$$

$$\mu_t^2 = (\nu - T_t^*\nu) \otimes (\sigma_2 - \sigma_1) + \nu \otimes \sigma_1 = (\nu - T_t^*\nu) \otimes \sigma_2 + T_t^*\nu \otimes \sigma_1$$

We construct two different probability solutions of the Cauchy problem $\partial_t\mu_t = \Delta\mu - \mathrm{div}(b\mu)$, $\mu_0 = \nu \otimes \sigma_1$, where $b = (B(x), C^1(y), C^2(y))$.

Note that one can take

$$\nu(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx, \quad \sigma_1(dy) = \frac{1}{2\pi} e^{-|y|^2/2}\, dy.$$

Set

$$\eta_t(dxdy) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{x^2+|y|^2}{4t}}\, dxdy$$

Let us consider

$$\widetilde{\mu_t}^i = \begin{cases} \mu^i_{t-1/2}, & t \geq 1/2 \\ \eta_t & t \in (0, 1/2) \end{cases}, \quad i = 1, 2.$$

Then $\widetilde{\mu_t}^1$ and $\widetilde{\mu_t}^2$ are distinct probability solutions of the Cauchy problem (1) with $A = I$, $\widetilde{b}(t,x) = b(x)$ if $t \geq 1/2$ and $\widetilde{b}(t,x) = 0$ if $t < 1/2$, and $\nu = \delta_0$.

It is well known that in the case where $A = 0$ and the field b just continuous, the Cauchy problem (1) have more than one solution. Let $d = 1$, $A = 0$ and $b(x) = \sqrt[3]{x^2}$. Then the Cauchy problem for the ordinary equation $\dot{x} = b(x)$, $x(0) = 0$, has two distinct solutions $x_1(t) = t^3/3$ and $x_2(t) = 0$. The measures $\delta_{x_1(t)}$ and $\delta_{x_2(t)}$ are two distinct solutions to the Cauchy problem

$$\partial_t \mu_t = -\partial_x (b\mu_t), \quad \mu_0 = \delta_0.$$

The book
V.I.Bogachev, N.V. Krylov, M.Rockner, S.V.Shaposhnikov
Fokker–Planck–Kolmogorov equations, AMS, 2015,
gives a survey of recent results about FPK equations.

Martingale problem

Let us recall two well known results for martingale problems. The following statement is a part of Theorem 10.1.3 (D.W.Sroock, S.R.S.Varadhan).

Theorem

Assume that A and b are locally bounded and A is locally Lipschitzian in x and locally strictly positive. Then for each $(s, z) \in [0, \infty) \times \mathbb{R}^d$ *there is at most one solution to the martingale problem with L and the initial condition* $\nu = \delta_z$ *at* $t = s$*.*

It is interesting to compare this statement and the above examples where the Cauchy problem for the Fokker–Planck–Kolmogorov equation has several probability solutions.

Next assertion is Theorem 10.2.2.
(D.W.Sroock, S.R.S.Varadhan).

Theorem

Assume that A and b are locally bounded and A is locally Lipschitzian in x and locally strictly positive. Suppose that

$$\|A(x,t)\| \le \gamma_1(1+|x|^2), \quad \langle b(x,t), x\rangle \le \gamma_2 + \gamma_3|x|^2.$$

then the martingale problem for L is well posed.

In the case $A = 0$ (the continuity equation) the following superposition principle of L. Ambrosio (2005) is known. If $\mu = \mu_t\, dt$ with probability measures μ_t on $\mathbb{R}^d$ satisfies the continuity equation

$$\partial_t \mu_t + \operatorname{div}(b\mu_t) = 0$$

and $|b(x,t)|/(1+|x|)$ is μ-integrable, then there exists a nonnegative Borel measure η on the space

$$\mathbb{R}^d \times C([0,T], \mathbb{R}^d)$$

concentrated on the set of pairs (x, ω) such that ω is an absolutely continuous solution of the integral equation

$$\omega(t) = x + \int_0^t b(\omega(s), s)\, ds$$

and, for each function $\varphi \in C_b(\mathbb{R}^d)$ and each $t \in [0, T]$, one has

$$\int \varphi(x)\, \mu_t(dx) = \int \varphi(\omega(t))\, \eta(dx d\omega).$$

In other words, the measure μ_t coincides with the image of η under the evaluation mapping $(x, \omega) \mapsto \omega(t)$. Of course, the integral on the right coincides with the integral against the projection of η on $C([0, T], \mathbb{R}^d)$.

The case of possibly nonzero A and bounded A and b was first considered by A. Figalli (2008), who proved that every probability solution to the Cauchy problem for the Fokker–Planck–Kolmgorov equation is represented by a martingale measure on the path space. Generalizing this seminal achievement, D. Trevisan (2016) obtained the following important and very general result.

Suppose that a mapping $t \mapsto \mu_t$ from $[0, T]$ to the space of probability measures $\mathcal{P}(\mathbb{R}^d)$ is continuous with respect to the weak topology and satisfies the Cauchy problem (1). Suppose also that it satisfies the condition

$$\int_0^T \int_{\mathbb{R}^d} \big[\|A(t,x)\| + |b(t,x)|\big]\, \mu_t(dx)\, dt < \infty. \qquad (2)$$

Then there exists a solution P_ν of the martingale problem with L and ν such that for every function $f \in C_0^\infty(\mathbb{R}^d)$, there holds the equality

$$\int_{\mathbb{R}^d} f\, d\mu_t = \int_{\Omega_d} f(\omega(t))\, P_\nu(d\omega) \quad \forall\, t \in [0, T].$$

Question

Let $V \in C^2(\mathbb{R}^d)$, $V \geq 0$ and $V(x) \to +\infty$ if $|x| \to \infty$.
Is it true, that

$$LV \leq C + CV \quad \Longrightarrow \quad \text{Superposition Principle?}$$

Theorem

Suppose that $\{\mu_t\}$ *is a solution of the Cauchy problem* $\partial_t\mu_t = L^*\mu_t$ *with* $\mu_0 = \nu$ *and there exists a nonnegative function* $V \in C^2(\mathbb{R}^d)$ *such that* $V \in L^1(\nu)$ *and for some numbers* $C \geq 0$ *one has*

$$\lim_{|x|\to+\infty} V(x) = +\infty, \quad LV(t,x) \leq C + CV(x).$$

Then

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} V\, d\mu_t < \infty \quad \text{and} \quad \int_0^T \int_{\mathbb{R}^d} |LV|\, d\mu_t\, dt < \infty.$$

«Proof»
We have

$$\int V\,d\mu_t = \int V\,d\nu + \int_0^t \int LV\,d\mu_s\,ds.$$

Thus

$$\int V\,d\mu_t \le \int V\,d\nu + Ct + C\int_0^t \int V\,d\mu_s\,ds$$

and the announced bound on the integral of V against μ_t is obtained with the aid of Gronwall's inequality.
Let $(LV)_+ = \max\{LV, 0\}$ and $(LV)_- = \max\{-LV, 0\}$. Recall that $C \ge 0$ and $V \ge 0$. Then $(LV)_+ \le C + CV$ and

$$\int_0^T \int (LV)_-\,d\mu_s\,ds \le \int V\,d\nu + \int_0^t \int (LV)_+\,d\mu_s\,ds.$$

Note that $|LV| = (LV)_+ + (LV)_-$.

Question

Let $V \in C^2(\mathbb{R}^d)$, $V \geq 0$ and $V(x) \to +\infty$ if $|x| \to \infty$.
Is it true, that

$$\int_0^T \int |LV| \, d\mu_s \, ds < \infty \quad \Longrightarrow \quad \text{Superposition Principle?}$$

Throughout we assume that the coefficients are Borel measurable on $[0, T] \times \mathbb{R}^d$,

$$a^{ij}, b^i \in L^1([0, T] \times U, \mu_t\, dt)$$

for every ball U in $\mathbb{R}^d$, and the following condition is fulfilled:

$$\int_0^T \int_{\mathbb{R}^d} \frac{\|A(t,x)\| + |\langle b(t,x), x\rangle|}{(1+|x|)^2}\, \mu_t(dx)\, dt < \infty. \tag{3}$$

Theorem

Suppose that $\{\mu_t\}$ *is a solution to the Cauchy problem* $\partial_t \mu_t = L^* \mu_t$ *on* $[0, T]$ *with* $\mu_0 = \nu$ *and* (3) *is fulfilled. Then there exists a Borel probability measure* P_ν *on* $\Omega_d = C([0, T], \mathbb{R}^d)$ *for which all assertions* (M1), (M2) *and* (M3) *are true.*

Note that the integrability of $(1+|x|)^{-2}|\langle b(t,x),x\rangle|$ can hold even in the case where the function $(1+|x|)^{-1}|b(t,x)|$ is not integrable with respect to the solution.

Example

We observe that the standard Gaussian density γ is a stationary solution to the equation with $A = I$ and $b(x) = -x$, so it remains a solution for the equation with a perturbed drift $-x + v(x)$, where a smooth vector field v is chosen such that $\operatorname{div}(\gamma v) = 0$ and

$$(x, v(x)) = 0.$$

For example, we can take v of the form

$$v(x) = \gamma(x)^{-1} h(|x|^2) Ux$$

with an orthogonal operator U such that $(Ux, x) = 0$. Of course, h can be rapidly increasing, so that $|v|\gamma$ will not be integrable.

Not only is the assumption of integrability of $(1+|x|)^{-2}|\langle b(t,x), x\rangle|$ weaker than the assumption of integrability of $(1+|x|)^{-1}|b(t,x)|$, but it is also simpler to verify.

Corollary

Let $\log(1+|x|^2) \in L^1(\nu)$ *and*

$$\|A(t,x)\| \leq C + C|x|^2 \log(1+|x|^2),$$

$$\langle b(t,x), x\rangle \leq C + C|x|^2 \log(1+|x|^2).$$

Then the hypotheses of the main theorem are fulfilled, hence its conclusion holds. In particular, it holds without any assumptions about ν *if*

$$\|A(t,x)\| + |\langle b(t,x), x\rangle| \leq C + C|x|^2.$$

The proof consists of three steps:

I. Smoothing of the coefficients and verification of the conditions with the Lyapunov function.

We obtain new operator L_ε and new solution μ_t^ε.

II. The representation of μ_t^ε by a solution P_ν^ε to the martingale problem with L_ε.

III. Verification of the compactness of the family of measures P_ν^ε and the proof of the fact that the limit is the required measure.

The main difficulty is the approximation procedure. When reducing the general case to that of smooth coefficients, we encounter two problems:
1) it is necessary to control that the solutions with smoothed coefficients converge to the considered solution, which is not automatic due to the lack of uniqueness,
2) for a priori estimates it is necessary to keep condition (3) uniformly.

«Proof»
Let
$$\mu_t^\varepsilon = \int\int \omega_\varepsilon(x-y, t-s)\mu_s(dy)\, ds.$$
Then $\mu_t^\varepsilon \to \mu_t$ and
$$\partial_t \mu_t^\varepsilon = \partial_{x_i}\partial_{x_j}\big(q_\varepsilon^{ij}\mu_t^\varepsilon\big) - \partial_{x_i}\big(h_\varepsilon^i \mu_t^\varepsilon\big),$$
where
$$q_\varepsilon^{ij} = \frac{\big(a^{ij}\mu_t\big)_\varepsilon}{\mu_t^\varepsilon}, \quad h_\varepsilon^i = \frac{\big(b^i\mu_t\big)_\varepsilon}{\mu_t^\varepsilon}.$$

We have

$$\langle h_\varepsilon(x,t), x\rangle =$$

$$\frac{1}{\mu_t^\varepsilon}\int\int\langle b(y,s), y\rangle\omega_\varepsilon(x-y, t-s)\mu_s(dy)\,ds +$$

$$+\frac{1}{\mu_t^\varepsilon}\int\int\langle b(y,s), x-y\rangle\omega_\varepsilon(x-y, t-s)\mu_s(dy)\,ds.$$

In order to estimate the second term we observe that for some function ω

$$(x_i - y_i)\omega_\varepsilon(x - y, t - s) = \varepsilon^2 \partial_{y_i} \eta_\varepsilon(x - y, t - s).$$

Then we can apply the Fokker–Planck–Kolmogorov equation and estimate the second term through the diffusion coefficient, Indeed

$$\frac{1}{\mu_t^\varepsilon}\int\int\langle b(y,s),x-y\rangle\omega_\varepsilon(x-y,t-s)\mu_s(dy)\,ds=$$

$$\frac{\varepsilon^2}{\mu_t^\varepsilon}\int\int\langle b(y,s),\nabla_y\eta_\varepsilon(x-y,t-s)\rangle\mu_s(dy)\,ds=$$

$$-\frac{\varepsilon^2}{\mu_t^\varepsilon}\int\int a^{ij}(y,s)\partial_{y_iy_j}\eta_\varepsilon(x-y,t-s)\mu_s(dy)\,ds-$$

$$\frac{\varepsilon^2}{\mu_t^\varepsilon}\int\int\partial_s\eta_\varepsilon(x-y,t-s)\mu_s(dy)\,ds.$$

Nonlinear FPK equations

The superposition principle applies not only to linear Fokker–Planck–Kolmogorov equations, but also to nonlinear equations. Let $\{\mu_t\}$ be a solution to the Cauchy problem

$$\partial_t \mu_t = \partial_{x_i}\partial_{x_j}(a^{ij}(t,x,\mu)\mu_t) - \partial_{x_i}(b^i(t,x,\mu)\mu_t), \quad \mu_0 = \nu.$$

Typical global assumptions are expressed in terms of a Lyapunov function V:

$$L_\mu V \leq C(\mu) + C(\mu)V, \quad V \in L^1(\nu).$$

If $V(x) = \log(1 + |x|^2)$, then the solution $\{\mu_t\}$ satisfies condition (3). Given a solution $\{\mu_t\}$, we can regard it as a solution to the linear operator L_μ. Therefore, there exists the corresponding solution P_ν to the martingale problem such that μ_t is the one-dimensional distribution of the measure P_ν on $C([0, T], \mathbb{R}^d)$.

Hence we can assume that the measure P_ν solves the martingale problem with the operator L_μ that depends on P_ν through μ, i.e., solves the martingale problem corresponding to the stochastic McKean–Vlasov equation. Thus, using the superposition principle and solutions to the Fokker–Planck–Kolmogorov equation one can construct solutions to the martingale problem for nonlinear stochastic equations.

The presented talk is based on the joint paper with
V.I.Bogachev and M.Rockner
On the Ambrosio–Figalli–Trevisan Superposition Principle for Probability Solutions to Fokker–Planck–Kolmogorov Equations.
Journal of Dynamics and Differential Equations (2020).

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THANK YOU!