On a mixed singular/switching control problem with multiple regimes

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Let $W = \{W_t : t \geq 0\}$ be a k-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The stochastic controlled process $(X^{\xi,\varsigma}, I^{\varsigma})$ evolves as:

$$X_t^{\xi,\varsigma} = X_{\tilde{\tau}_i}^{\xi,\varsigma} - \int_{\tilde{\tau}_i}^{t} b(X_s^{\xi,\varsigma}, \ell_i)\mathrm{d}s + \int_{\tilde{\tau}_i}^{t} \sigma(X_s^{\xi,\varsigma}, \ell_i)\mathrm{d}W_s - \int_{[\tilde{\tau}_i, t)} \mathbb{n}_s \mathrm{d}\zeta_s,$$

$$I_t^{\varsigma} = \ell_i \quad \text{for } t \in [\tilde{\tau}_i, \tilde{\tau}_{i+1}) \text{ and } i \geq 0,$$

where $X_0^{\xi,\varsigma} = \tilde{x} \in \overline{\mathcal{O}} \subset \mathbb{R}^d$, $I_{0-}^{\varsigma} = \tilde{\ell} \in \mathbb{I} := \{1, 2, \ldots, m\}$, $\tilde{\tau}_i := \tau_i \wedge \tau$, τ represents the first exit time of the process $X^{\xi,\varsigma}$ from the set $\mathcal{O}$, and

$$b_\ell := b(\cdot, \ell) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^d \text{ and } \sigma_\ell := \sigma(\cdot, \ell) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^d \times \mathbb{R}^k, \text{ with } \ell \in \mathbb{I}.$$

Control processes

The control process (ξ, ς) is in $\mathcal{U} \times \mathcal{S}$ where the singular control $\xi = (\mathbb{n}, \zeta)$ belongs to the class $\mathcal{U}$ of admissible controls that satisfy

$$\begin{cases} (\mathbb{n}_t, \zeta_t) \in \mathbb{R}^d \times \mathbb{R}_+, \ t \geq 0, \text{ such that } X_t^{\xi,\varsigma} \in \mathcal{O} \ \ t \in [0, \tau), \\ (\mathbb{n}, \zeta) \text{ is adapted to the filtration } \mathbb{F}, \\ \zeta_{0-} = 0 \text{ and } \zeta_t \text{ is non-decreasing and is right continuous} \\ \text{with left hand limits, } t \geq 0, \text{ and } \ |\mathbb{n}_t| = 1 \ \mathrm{d}\zeta_t\text{-a.s., } \ t \geq 0, \end{cases}$$

and the switching control process $\varsigma := (\tau_i, \ell_i)_{i \geq 0}$ belongs to the class $\mathcal{S}$ of switching regime sequences that satisfy

$$\begin{cases} \varsigma \text{ is a sequence of } \mathbb{F}\text{-stopping times and regimes in } \mathbb{I}, \text{ i.e,} \\ \varsigma = (\tau_i, \ell_i)_{i \geq 0} \text{ is such that} \\ 0 = \tau_0 \leq \tau_1 < \tau_2 < \cdots, \ \tau_i \uparrow \infty \text{ as } i \uparrow \infty \ \mathbb{P}\text{-a.s.,} \\ \ \text{and for each } i \geq 0, \ \ell_i \in \mathbb{I} = \{1, 2, \ldots, m\}. \end{cases}$$

The process (ξ, ς) is chosen in such a way that it will minimize the cost criterion

$$V_{\xi,\varsigma}(\tilde{x},\tilde{\ell}) := \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\big[\, \mathrm{e}^{-r(\tau_{i+1})}\, \vartheta_{\ell_i,\ell_{i+1}} \mathbb{1}_{\{\tau_{i+1}<\tau\}}\big] + \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{[\tilde{\tau}_i,\tilde{\tau}_{i+1})} \mathrm{e}^{-r(s)}[h_{\ell_i}(X^{\xi,\varsigma}_s)\mathrm{d}s + g_{\ell_i}(X^{\xi,\varsigma}_{s-})\circ \mathrm{d}\zeta_s]\bigg],$$

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$$
\begin{aligned}
V_{\xi,\varsigma}(\tilde{x}, \tilde{\ell}) := & \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\big[\mathrm{e}^{-r(\tau_{i+1})} \vartheta_{\ell_i,\ell_{i+1}} \mathbb{1}_{\{\tau_{i+1} < \tau\}} \big] \\
& + \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{[\tilde{\tau}_i, \tilde{\tau}_{i+1})} \mathrm{e}^{-r(s)} [h_{\ell_i}(X^{\xi,\varsigma}_s)\mathrm{d}s + g_{\ell_i}(X^{\xi,\varsigma}_{s-}) \circ \mathrm{d}\zeta_s] \bigg],
\end{aligned}
$$

where $r(t) := \int_0^t c(X^{\xi,\varsigma}_s, I^{\varsigma}_s)\mathrm{d}s = \sum_{i \geq 0} \int_{t \wedge \tilde{\tau}_i}^{t \wedge \tilde{\tau}_{i+1}} c_{\ell_i}(X^{\xi,\varsigma}_s)\mathrm{d}s$ and

$$
\begin{aligned}
\int_{[\tilde{\tau}_i, t \wedge \tilde{\tau}_{i+1})} \mathrm{e}^{-r(s)} g_{\ell_i}(X^{\xi,\varsigma}_{s-}) \circ \mathrm{d}\zeta_s := & \int_{\tilde{\tau}_i}^{t \wedge \tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)} g_{\ell_i}(X^{\xi,\varsigma}_s)\mathrm{d}\zeta^{\mathrm{c}}_s \\
& + \sum_{\tilde{\tau}_i \leq s < t \wedge \tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)} \Delta\zeta_s \int_0^1 g_{\ell_i}(X^{\xi,\varsigma}_{s-} - \lambda \mathbb{n}_s \Delta\zeta_s)\mathrm{d}\lambda.
\end{aligned}
$$

One of the main goals of this paper is to verify that the value function

$$V_{\tilde{\ell}}(\tilde{x}) := \inf_{\xi,\varsigma \in \mathcal{U}\times\mathcal{S}} V_{\xi,\varsigma}(\tilde{x},\tilde{\ell}), \text{ for } (\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}, \tag{1}$$

is in $\mathrm{C}^{0,1} \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}$.

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The cost for switching from the regime ℓ to κ is given by a constant $\vartheta_{\ell,\kappa} \geq 0$ and we assume that

$$\vartheta_{\ell_1,\ell_3} \leq \vartheta_{\ell_1,\ell_2} + \vartheta_{\ell_2,\ell_3}, \text{ for } \ell_3 \neq \ell_1, \ell_2, \tag{2}$$

which means that it is cheaper to switch directly from regime ℓ_1 to ℓ_3 than using the intermediate regime ℓ_2.

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which means that it is cheaper to switch directly from regime ℓ_1 to ℓ_3 than using the intermediate regime ℓ_2.

Additionally, we assume that there is no *loop of zero cost*, i.e.,

$$\begin{aligned} &\text{no family of regimes } \{\ell_0, \ell_1, \dots, \ell_n, \ell_0\} \\ &\text{such that } \vartheta_{\ell_0,\ell_1} = \vartheta_{\ell_1,\ell_2} = \cdots = \vartheta_{\ell_n,\ell_0} = 0. \end{aligned} \tag{3}$$

From (3) and by dynamic programming principle, we identify heuristically that the value function V is associated with the following HJB equation

$$\max\{[c_\ell - \mathcal{L}_\ell]u_\ell - h_\ell, |\,\mathrm{D}^1 u_\ell| - g_\ell, u_\ell - \mathcal{M}_\ell u\} = 0 \text{ in } \mathcal{O},$$
$$\text{s.t. } u_\ell = 0 \text{ on } \partial\mathcal{O},$$

where $u = (u_1, \dots, u_m) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^m$ and, for $\ell \in \mathbb{I}$, the operators $\mathcal{L}_\ell$ and $\mathcal{M}_\ell$ are defined by

$$\mathcal{L}_\ell u_\ell = \mathrm{tr}[a_\ell \,\mathrm{D}^2 u_\ell] - \langle b_\ell, \mathrm{D}^1 u_\ell \rangle,$$
$$\mathcal{M}_\ell u = \min_{\kappa \in \mathbb{I} \setminus \{\ell\}} \{u_\kappa + \vartheta_{\ell,\kappa}\}.$$

If the value function V_ℓ satisfies the previous HJB equation, we have that for each $\ell \in \mathbb{I}$ fixed, the domain set $\mathcal{O}$ is divided into three parts. Consider $\mathcal{C}_\ell := \mathcal{O} \setminus (\mathcal{N}_\ell \cup \mathcal{S}_\ell)$, where

$$\mathcal{N}_\ell := \{x \in \mathcal{O} : |\,\mathrm{D}^1 V_\ell| = g_\ell\} \quad \text{and} \quad \mathcal{S}_\ell = \{x \in \mathcal{O} : V_\ell = \mathcal{M}_\ell V\}.$$

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The open set $\mathcal{C}_\ell$ is where V_ℓ satisfies the elliptic partial differential equation $[c_\ell - \mathcal{L}_\ell] V_\ell = h_\ell$, which suggests that the 'optimal control' (ξ^*, ς^*) corresponding to this problem will not be exercised when the process X^{ξ^*, ς^*} is in $\mathcal{C}_\ell$. Otherwise, either ξ^* or ς^* will be exercised on X^{ξ^*, ς^*} in the following way:

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(i) if $X^{\xi^*,\varsigma^*} \in \mathcal{N}_\ell \setminus \mathcal{S}_\ell$, the singular control ξ^* will act on X^{ξ^*,ς^*} in such a way that X^{ξ^*,ς^*} will be pushed back to some point $y \in \partial\mathcal{C}_\ell$;

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(ii) if $X^{\xi^*,\varsigma^*} \in \mathcal{S}_\ell$, the switching control ς^* will be executed in such a way that the process X^{ξ^*,ς^*} will switch to some regime $\kappa \neq \ell$ at time $\tau_\kappa \leq \tau$.

Assumptions

(H1) The switching costs sequence $\{\vartheta_{\ell,\kappa}\}_{\ell,\kappa\in\mathbb{I}}$ is such that $\vartheta_{\ell,\kappa} \geq 0$ and (2) and (3) hold.

(H2) The domain set $\mathcal{O}$ is an open and bounded set such that its boundary $\partial\mathcal{O}$ is of class $\mathrm{C}^{4,\alpha'}$, with $\alpha' \in (0,1)$ fixed.

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Let ℓ be in $\mathbb{I}$. Then:

(H3) The functions $h_\ell, g_\ell \in \mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})$ are non-negative and $||h_\ell||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}$, $||g_\ell||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}$ are bounded by some finite positive constant Λ.

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(H4) Let $\mathcal{S}(d)$ be the set of $d\times d$ symmetric matrices. The coefficients of the differential part of $\mathcal{L}_\ell$, $a_\ell = (a_{\ell\,ij})_{d\times d} : \overline{\mathcal{O}} \longrightarrow \mathcal{S}(d)$, $b_\ell = (b_{\ell\,1},\dots,b_{\ell\,d}) : \overline{\mathcal{O}} \longrightarrow \mathbb{R}^d$ and $c_\ell : \overline{\mathcal{O}} \longrightarrow \mathbb{R}$, are such that $a_{\ell\,ij}, b_{\ell\,i}, c_\ell \in \mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})$, $c_\ell > 0$ on $\mathcal{O}$ and $||a_{\ell\,ij}||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}, ||b_{\ell\,i}||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}, ||c_\ell||_{\mathrm{C}^{2,\alpha'}(\overline{\mathcal{O}})}$ are bounded by some finite positive constant Λ. We assume that there exists a real number $\theta>0$ such that

$$\langle a_\ell(x)\zeta,\zeta\rangle \geq \theta|\zeta|^2,\ \text{for all } x\in\overline{\mathcal{O}},\ \zeta\in\mathbb{R}^d. \tag{4}$$

Under assumptions (H1)–(H4), the first main goal obtained in this document is as follows.

Theorem 1

The HJB equation

$$\max\{[c_\ell - \mathcal{L}_\ell]u_\ell - h_\ell, |\,\mathrm{D}^1 u_\ell| - g_\ell, u_\ell - \mathcal{M}_\ell u\} = 0 \text{ in } \mathcal{O},$$
$$\text{s.t. } u_\ell = 0 \text{ on } \partial\mathcal{O},$$

has a unique non-negative strong solution (in the almost everywhere sense) $u = (u_1, \dots, u_m)$ where $u_\ell \in \mathrm{C}^{0,1}(\overline{\mathcal{O}}) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ for each $\ell \in \mathbb{I}$.

Recall that

$$\mathcal{L}_\ell u_\ell = \mathrm{tr}[a_\ell \,\mathrm{D}^2 u_\ell] - \langle b_\ell, \mathrm{D}^1 u_\ell \rangle,$$
$$\mathcal{M}_\ell u = \min_{\kappa \in \mathbb{I} \setminus \{\ell\}} \{u_\kappa + \vartheta_{\ell,\kappa}\}.$$

In addition to the statement in (H2), we need to assume:

(H5) The domain set $\mathcal{O}$ is an open, convex and bounded set such that its boundary $\partial\mathcal{O}$ is of class $\mathrm{C}^{4,\alpha'}$, with $\alpha' \in (0,1)$ fixed.

Under assumptions (H1) and (H3)–(H5), the second main goal obtained in this document is as follows.

Theorem 2

Let V be the value function given by (1). Then $V_{\tilde{\ell}}(\tilde{x}) = u_{\tilde{\ell}}(\tilde{x})$ for $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$.

Penalized HJB equation

$$\max\left\{[c_\ell - \mathcal{L}_\ell]u_\ell^\varepsilon + \psi_\varepsilon(|\operatorname{D}^1 u_\ell^\varepsilon|^2 - g_\ell^2) - h_\ell, u_\ell^\varepsilon - \mathcal{M}_\ell u^\varepsilon\right\} = 0, \text{ in } \mathcal{O},$$
$$\text{s.t. } u_\ell^\varepsilon = 0, \text{ on } \partial\mathcal{O},$$

where the *penalized function* $\psi_\varepsilon(t) = \varphi(t/\varepsilon)$, with $\varepsilon \in (0,1)$ and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ that is in $\mathrm{C}^\infty(\mathbb{R})$ and satisfies

$$\varphi(t) = 0,\ t \leq 0, \quad \varphi(t) > 0,\ t > 0,$$
$$\varphi(t) = t - 1,\ t \geq 2, \quad \varphi'(t) \geq 0,\ \varphi''(t) \geq 0.$$

Under assumptions (H1)–(H4), we have the following result.

Proposition 3

For each $\varepsilon \in (0,1)$ fixed, there exists a unique non-negative strong solution $u^\varepsilon = (u_1^\varepsilon, \dots, u_m^\varepsilon)$ to the HJB equation

$$\max\left\{[c_\ell - \mathcal{L}_\ell]u_\ell^\varepsilon + \psi_\varepsilon(|\mathrm{D}^1 u_\ell^\varepsilon|^2 - g_\ell^2) - h_\ell, u_\ell^\varepsilon - \mathcal{M}_\ell u^\varepsilon\right\} = 0, \text{ in } \mathcal{O}, \quad \text{s.t. } u_\ell^\varepsilon = 0, \text{ on } \partial\mathcal{O}, \tag{5}$$

where $u_\ell^\varepsilon \in \mathrm{C}^{0,1}(\overline{\mathcal{O}}) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ for each $\ell \in \mathbb{I}$,

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where $u^{\varepsilon}_\ell \in \operatorname{C}^{0,1}(\overline{\mathcal{O}}) \cap \operatorname{W}^{2,\infty}_{\text{loc}}(\mathcal{O})$ for each $\ell \in \mathbb{I}$, satisfying

$$0 \leq u^{\varepsilon}_\ell \leq C_1 \quad \text{and} \quad |\operatorname{D}^1 u^{\varepsilon}_\ell| \leq C_2, \quad \text{in } \overline{\mathcal{O}}, \tag{6}$$

for some positive constants C_1, C_2 independent of ε.

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where $u_\ell^{\varepsilon} \in \mathrm{C}^{0,1}(\overline{\mathcal{O}}) \cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ for each $\ell \in \mathbb{I}$, satisfying

$$0 \leq u_\ell^{\varepsilon} \leq C_1 \quad \text{and} \quad |\,\mathrm{D}^1 u_\ell^{\varepsilon}| \leq C_2, \quad \text{in } \overline{\mathcal{O}}, \tag{6}$$

for some positive constants C_1, C_2 independent of ε. Besides, for each $B_{\beta r} \subset \mathcal{O}$, there exists a positive constant C_3 independent of ε such that

$$||\,\mathrm{D}^2 u_\ell^{\varepsilon}||_{\mathrm{L}^p(B_{\beta r})} \leq C_5, \quad \text{for each } p \in (1,\infty). \tag{7}$$

Then, from (6)–(7) and using again Arzelà-Ascoli compactness criterion and that $(\mathrm{L}^p(B_{\beta r}), ||\cdot||_{\mathrm{L}^p(B_{\beta r})})$ is a reflexive space, we have that there exist a sub-sequence $\{u_\ell^{\varepsilon_n}\}_{n\geq 1}$ of $\{u_\ell^{\varepsilon}\}_{\varepsilon\in(0,1)}$ and a u_ℓ in $\mathrm{C}^{0,1}(\overline{\mathcal{O}})\cap \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{O})$ such that

$$\begin{aligned} &u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} u_\ell \text{ in } \mathrm{C}(\overline{\mathcal{O}}),\ \partial_i u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_i u_\ell \text{ in } \mathrm{C}_{\mathrm{loc}}(\mathcal{O}),\\ &\partial_{ij} u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_{ij} u_\ell, \text{ weakly } \mathrm{L}^p_{\mathrm{loc}}(\mathcal{O}), \text{ for each } p\in(1,\infty). \end{aligned} \tag{8}$$

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$$\begin{aligned}
&u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} u_\ell \text{ in } \mathrm{C}(\overline{\mathcal{O}}),\ \partial_i u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_i u_\ell \text{ in } \mathrm{C}_{\mathrm{loc}}(\mathcal{O}),\\
&\partial_{ij} u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_{ij} u_\ell, \text{ weakly } \mathrm{L}^p_{\mathrm{loc}}(\mathcal{O}), \text{ for each } p\in(1,\infty).
\end{aligned} \tag{8}$$

It can be proven that $u=(u_1,\dots,u_m)$, where $u_\ell=\lim_{\varepsilon_n\to 0} u_\ell^{\varepsilon_n}$ for each $\ell\in\mathbb{I}$, is the unique solution to the HJB equation

$$\max\{[c_\ell-\mathcal{L}_\ell]u_\ell-h_\ell, |\,\mathrm{D}^1 u_\ell|-g_\ell, u_\ell-\mathcal{M}_\ell u\}=0 \text{ in } \mathcal{O},$$
$$\text{s.t. } u_\ell=0 \text{ on } \partial\mathcal{O},$$

ε-penalized absolutely continuous/switching control problem

Define the penalized controls set $\mathcal{U}^{\varepsilon}$ in the following way

$$\mathcal{U}^{\varepsilon} := \{\xi = (\mathbb{n}, \zeta) \in \mathcal{U} : \zeta_t \text{ is absolutely continuous, } 0 \leq \dot{\zeta}_t \leq 2C/\varepsilon\}, \quad (9)$$

with $\varepsilon \in (0,1)$ fixed, where C is some fixed positive constant independent of ε.

For each $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and $(\xi, \varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$, the process $X_t^{\xi,\varsigma} = \{X_t^{\xi,\varsigma} : t \geq 0\}$ evolves as

$$X_t^{\xi,\varsigma} = X_{\tilde{\tau}_i}^{\xi,\varsigma} - \int_{\tilde{\tau}_i}^{t} [b(X_s^{\xi,\varsigma}, \ell_i) + \mathbb{n}_t \dot{\zeta}_t] \mathrm{d}s + \int_{\tilde{\tau}_i}^{t} \sigma(X_s^{\xi,\varsigma}, \ell_i) \mathrm{d}W_s, \quad (10)$$

$$I_t = \ell_i \quad \text{for } t \in [\tilde{\tau}_i, \tilde{\tau}_{i+1}) \text{ and } i \geq 0, \quad (11)$$

where $\tilde{\tau}_i = \tau_i \wedge \tau$ and $\tau = \inf\{t > 0 : X_t^{\xi,\varsigma} \notin \mathcal{O}\}$.

Penalized cost

The penalized cost of $(\xi, \varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$ is defined by

$$\mathcal{V}_{\xi,\varsigma}(\tilde{x}, \tilde{\ell}) = \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\left[\mathrm{e}^{-r(\tau_{i+1})}\, \vartheta_{\ell_i,\ell_{i+1}} \mathbb{1}_{\{\tau_{i+1} < \tau\}} \right] + \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\left[\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)} [h_{\ell_i}(X_s^{\xi,\varsigma}) + l_{\ell_i}^{\varepsilon}(\dot{\zeta}_s \mathbb{n}_s, X_s^{\xi,\varsigma})] \mathrm{d}s \right],$$

where

$$l_{\ell}^{\varepsilon}(y, x) := l^{\varepsilon}(y, x, \ell) := \sup_{\gamma \in \mathbb{R}^d} \{\langle \gamma, y \rangle - H_{\ell}^{\varepsilon}(\gamma, x)\}, \quad \text{for } y \in \mathbb{R}^d.$$

is the Legendre transform of

$$H_{\ell}^{\varepsilon}(\gamma, x) := H^{\varepsilon}(\gamma, x, \ell) := \psi_{\varepsilon}(|\gamma|^2 - g_{\ell}(x)^2).$$

Penalized cost

The penalized cost of $(\xi, \varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$ is defined by

$$\mathcal{V}_{\xi,\varsigma}(\tilde{x}, \tilde{\ell}) = \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}} \Big[\mathrm{e}^{-r(\tau_{i+1})} \vartheta_{\ell_i,\ell_{i+1}} \mathbb{1}_{\{\tau_{i+1}<\tau\}} \Big] + \sum_{i \geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}} \Big[\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)} [h_{\ell_i}(X_s^{\xi,\varsigma}) + l_{\ell_i}^{\varepsilon}(\dot{\zeta}_s \mathbb{n}_s, X_s^{\xi,\varsigma})] \mathrm{d}s \Big],$$

where

$$l_{\ell}^{\varepsilon}(y, x) := l^{\varepsilon}(y, x, \ell) := \sup_{\gamma \in \mathbb{R}^d} \{\langle \gamma, y \rangle - H_{\ell}^{\varepsilon}(\gamma, x)\}, \quad \text{for } y \in \mathbb{R}^d.$$

is the Legendre transform of

$$H_{\ell}^{\varepsilon}(\gamma, x) := H^{\varepsilon}(\gamma, x, \ell) := \psi_{\varepsilon}(|\gamma|^2 - g_{\ell}(x)^2).$$

For each $(x, \ell) \in \overline{\mathcal{O}} \times \mathbb{I}$ fixed, $H_{\ell}^{\varepsilon}(\gamma, x)$ is a C^2 and convex function with respect to the variable $\gamma \in \mathbb{R}^d$, since $\psi_{\varepsilon} \in \mathrm{C}^{\infty}(\mathbb{R})$ is convex.

The value function for this problem is given by

$$V_{\tilde{\ell}}^{\varepsilon}(\tilde{x}) := \inf_{(\xi,\varsigma)\in\mathcal{U}^{\varepsilon}} \mathcal{V}_{\xi,\varsigma}(\tilde{x},\tilde{\ell}), \tag{12}$$

whose corresponding HJB equation is

$$\max\Bigg\{[c_\ell - \mathcal{L}_\ell]u_\ell^{\varepsilon} + \sup_{y\in\mathbb{R}^d}\{\langle \mathrm{D}^1 u_\ell^{\varepsilon}, y\rangle - l_\ell^{\varepsilon}(y,\cdot)\} - h_\ell, u_\ell^{\varepsilon} - \mathcal{M}_\ell u^{\varepsilon}\Bigg\} = 0, \text{ in } \mathcal{O},$$

$$\text{s.t. } u_\ell^{\varepsilon} = 0, \text{ on } \partial\mathcal{O}.$$

The value function for this problem is given by

$$V^{\varepsilon}_{\tilde{\ell}}(\tilde{x}) := \inf_{(\xi,\varsigma)\in\mathcal{U}^{\varepsilon}} \mathcal{V}_{\xi,\varsigma}(\tilde{x},\tilde{\ell}), \tag{12}$$

whose corresponding HJB equation is

$$\max\Big\{[c_\ell - \mathcal{L}_\ell]u^{\varepsilon}_{\ell} + \sup_{y\in\mathbb{R}^d}\{\langle \mathrm{D}^1 u^{\varepsilon}_{\ell}, y\rangle - l^{\varepsilon}_{\ell}(y,\cdot)\} - h_\ell, u^{\varepsilon}_{\ell} - \mathcal{M}_\ell u^{\varepsilon}\Big\} = 0, \text{ in } \mathcal{O},$$
$$\text{s.t. } u^{\varepsilon}_{\ell} = 0, \text{ on } \partial\mathcal{O}.$$

Observe that the previous equation can be rewritten as

$$\max\{[c_\ell - \mathcal{L}_\ell]u^{\varepsilon}_{\ell} + \psi_\varepsilon(|\mathrm{D}^1 u^{\varepsilon}_{\ell}|^2 - g^2_\ell) - h_\ell, u^{\varepsilon}_{\ell} - \mathcal{M}_\ell u^{\varepsilon}\} = 0, \text{ in } \mathcal{O},$$
$$\text{s.t. } u^{\varepsilon}_{\ell} = 0, \text{ on } \partial\mathcal{O},$$

because of $H^{\varepsilon}_{\ell}(\gamma, x) = \sup_{y\in\mathbb{R}^d}\{\langle\gamma, y\rangle - l^{\varepsilon}_{\ell}(y,x)\}$.

Lemma 4 (Verification Lemma. First part)

Let $\varepsilon \in (0,1)$ be fixed. Then, $u^{\varepsilon}_{\tilde{\ell}}(\tilde{x}) \leq \mathcal{V}_{\zeta,\varsigma}(\tilde{x},\tilde{\ell})$ for each $(\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and $(\xi,\varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$. Therefore, $u^{\varepsilon}_{\tilde{\ell}}(\tilde{x}) \leq V^{\varepsilon}_{\tilde{\ell}}(\tilde{x})$ for each $(\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and $(\xi,\varsigma) \in \mathcal{U}^{\varepsilon} \times \mathcal{S}$.

Lemma 4 (Verification Lemma. First part)

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To prove the lemma above, we need to use the characterization of u^{ε}_{ℓ} as a limit (when $\delta_{\hat{n}} \downarrow 0$) of a sub-sequence $\{u^{\varepsilon,\delta_{\hat{n}}}_{\ell}\}_{\delta_{\hat{n}}\in(0,1)}$ of $\{u^{\varepsilon,\delta}_{\ell}\}_{\delta\in(0,1)} \subset \mathrm{C}^{4,\alpha'}(\overline{\mathcal{O}})$ whose elements satisfy

$$[c_{\ell} - \mathcal{L}_{\ell}]u^{\varepsilon,\delta}_{\ell} + \psi_{\varepsilon}(|\,\mathrm{D}^1 u^{\varepsilon,\delta}_{\ell}|^2 - g^2_{\ell}) + \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi_{\delta}(u^{\varepsilon,\delta}_{\ell} - u^{\varepsilon,\delta}_{\kappa} - \vartheta_{\ell,\kappa}) = h_{\ell}, \text{ in } \mathcal{O},$$

$$\text{s.t. } u^{\varepsilon,\delta}_{\ell} = 0, \text{ on } \partial\mathcal{O}.$$

Using integration by parts and Itô's formula in $\mathrm{e}^{-r(t)}\, u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_t)$ on $[\tilde{\tau}_i, \tilde{\tau}_{i+1})$, $i \geq 0$,taking expected value on it,

$$\begin{aligned}
\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\tilde{\tau}_i)}\, u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_{\tilde{\tau}_i})\Big] &= \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[[u_{\ell_1}^{\varepsilon,\delta_{\hat{n}}}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\
&+ [u_{\ell_0}^{\varepsilon,\delta_{\hat{n}}}(X_{\tau_1}) - u_{\ell_1}^{\varepsilon,\delta_{\hat{n}}}(X_{\tau_1}) - \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\
&+ \Big[\mathrm{e}^{-r(t\wedge\tilde{\tau}_1)}\, u_{\ell_0}^{\varepsilon,\delta_{\hat{n}}}(X_{t\wedge\tilde{\tau}_1}) - \widetilde{\mathcal{M}}_{\ell_0}[0, t\wedge\tilde{\tau}_1; X, u^{\varepsilon,\delta_{\hat{n}}}] \\
&+ \int_0^{t\wedge\tilde{\tau}_1} \mathrm{e}^{-r(s)}[c_{\ell_0}(X_s)u_{\ell_0}^{\varepsilon,\delta_{\hat{n}}}(X_s) - \mathcal{L}_{\ell_0} u_{\ell_0}^{\varepsilon,\delta_{\hat{n}}}(X_s) + \langle \mathrm{D}^1 u_{\ell_0}^{\varepsilon,\delta_{\hat{n}}}(X_s), \dot{\zeta}_s \mathtt{n}_s\rangle]\mathrm{d}s\Big]\mathbb{1}_{\{\tau_1\neq 0, i=0\}} \\
&+ \Big[\mathrm{e}^{-r(t\wedge\tilde{\tau}_{i+1})}\, u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_{t\wedge\tilde{\tau}_{i+1}}) - \widetilde{\mathcal{M}}_{\ell_i}[\tilde{\tau}_i, t\wedge\tilde{\tau}_{i+1}; X, u^{\varepsilon,\delta_{\hat{n}}}] \\
&+ \int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}[c_{\ell_i}(X_s)u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_s) - \mathcal{L}_{\ell_i} u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_s) + \langle \mathrm{D}^1 u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_s), \dot{\zeta}_s \mathtt{n}_s\rangle]\mathrm{d}s\Big]\mathbb{1}_{\{i\neq 0\}}\Big],
\end{aligned} \tag{13}$$

where

$$\widetilde{\mathcal{M}}_{\ell_i}[\tilde{\tau}_i, t\wedge\tilde{\tau}_{i+1}; X, u^{\varepsilon,\delta_{\hat{n}}}] \coloneqq \int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}\langle \mathrm{D}^1 u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_s), \sigma_{\ell_i}(X_s)\rangle \mathrm{d}W_s.$$

Since $\widetilde{\mathcal{M}}_{\ell_i}[\tilde{\tau}_i, t \wedge \tilde{\tau}_{i+1}; X, u^{\varepsilon,\delta_{\hat{n}}}]$ is a square integrable martingale,

$$\langle \gamma, y \rangle \leq \psi_\varepsilon(|\gamma|^2 - g_\ell(x)^2) + l_\ell^\varepsilon(y, x) \quad \text{and} \quad [c_\ell - \mathcal{L}_\ell] u_\ell^{\varepsilon,\delta_{\hat{n}}} + \psi_\varepsilon(|\, \mathrm{D}^1 u_\ell^{\varepsilon,\delta_{\hat{n}}}|^2 - g_\ell^2) \leq h_\ell,$$

it follows that

$$\begin{aligned}
&\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\tilde{\tau}_i)} u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_{\tilde{\tau}_i})\Big] \\
&\leq \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[[u_{\ell_1}^{\varepsilon,\delta_{\hat{n}}}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}] \mathbb{1}_{\{\tau_1=0, \tau_1<\tau, i=0\}} + \mathcal{D}[\tau_1, \ell_0, \ell_1; X, u^{\varepsilon,\delta_{\hat{n}}}] \mathbb{1}_{\{\tau_1=0, \tau_1<\tau, i=0\}} \\
&\quad + \mathrm{e}^{-r(t\wedge\tilde{\tau}_1)}[u_{\ell_1}^{\varepsilon,\delta_{\hat{n}}}(X_{t\wedge\tilde{\tau}_1}) + \vartheta_{\ell_0,\ell_1}] \mathbb{1}_{\{\tau_1\neq 0, i=0\}} + \mathrm{e}^{-r(t\wedge\tilde{\tau}_{i+1})}[u_{\ell_{i+1}}^{\varepsilon,\delta_{\hat{n}}}(X_{t\wedge\tilde{\tau}_{i+1}}) + \vartheta_{\ell_i,\ell_{i+1}}] \mathbb{1}_{\{i\neq 0\}} \\
&\quad + \mathcal{D}[t\wedge\tilde{\tau}_1, \ell_0, \ell_1; X, u^{\varepsilon,\delta_{\hat{n}}}] \mathbb{1}_{\{\tau_1\neq 0, i=0\}} + \mathcal{D}[t\wedge\tilde{\tau}_{i+1}, \ell_i, \ell_{i+1}; X, u^{\varepsilon,\delta_{\hat{n}}}] \mathbb{1}_{\{i\neq 0\}} \\
&\quad + \mathbb{1}_{\{\tau_1\neq 0, i=0\}} \int_0^{t\wedge\tilde{\tau}_1} \mathrm{e}^{-r(s)}[h_{\ell_0}(X_s) + l_{\ell_0}^\varepsilon(\dot{\zeta}_s \mathfrak{m}_s, X_s)] \mathrm{d}s \\
&\quad + \mathbb{1}_{\{i\neq 0\}} \int_{\tilde{\tau}_i}^{t\wedge\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}[h_{\ell_i}(X_s) + l_{\ell_i}^\varepsilon(\dot{\zeta}_s \mathfrak{m}_s, X_s)] \mathrm{d}s \bigg],
\end{aligned} \tag{14}$$

where

$$\mathcal{D}[t\wedge\tilde{\tau}_{i+1}, \ell_i, \ell_{i+1}; X, u^{\varepsilon,\delta_{\hat{n}}}] \coloneqq \mathrm{e}^{-r(t\wedge\tilde{\tau}_{i+1})}[u_{\ell_i}^{\varepsilon,\delta_{\hat{n}}}(X_{t\wedge\tilde{\tau}_{i+1}}) - [u_{\ell_{i+1}}^{\varepsilon,\delta_{\hat{n}}}(X_{t\wedge\tilde{\tau}_{i+1}}) + \vartheta_{\ell_i,\ell_{i+1}}]]. \tag{15}$$

It is known

$$\max\{\mathrm{e}^{-r(t\wedge\tilde{\tau}_i)}\, u^{\varepsilon,\delta_{\hat{n}}}_{\ell_i}(X_{t\wedge\tilde{\tau}_i}), |\mathcal{D}[t\wedge\tilde{\tau}_{i+1},\ell_i,\ell_{i+1},X,u^{\varepsilon,\delta_{\hat{n}}}]|\}\leq 2C_1+\max_{(x,\ell)\in\mathcal{O}\times\mathbb{I}} c_\ell(x),$$

$$\max_{(x,\ell)\in\mathcal{O}\times\mathbb{I}}|u^{\varepsilon,\delta_{\hat{n}}}_{\ell}(x)-u^{\varepsilon}_{\ell}(x)|\underset{\delta_{\hat{n}}\to 0}{\longrightarrow}0.$$

Then, letting first $\delta_{\hat{n}}\to 0$ and after $t\to\infty$ in (14) and by the dominated convergence theorem, it follows that

$$\begin{aligned}\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\tilde{\tau}_i)}\, u^{\varepsilon}_{\ell_i}(X_{\tilde{\tau}_i})\Big]\leq \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[&\{\mathrm{e}^{-r(\tau_{i+1})}\, u^{\varepsilon}_{\ell_{i+1}}(X_{\tau_{i+1}})+\mathcal{D}[\tau_{i+1},\ell_i,\ell_{i+1};X,u^{\varepsilon}]\}\mathbb{1}_{\{\tau_{i+1}<\tau\}}\\ &+\int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}}\mathrm{e}^{-r(s)}[h_{\ell_i}(X_s)+l^{\varepsilon}_{\ell_i}(\dot{\zeta}_s\mathrm{n}_s,X_s)]\mathrm{d}s+\mathrm{e}^{-r(\tau_{i+1})}\,\vartheta_{\ell_i,\ell_{i+1}}\mathbb{1}_{\{\tau_{i+1}<\tau\}}\Big],\quad \text{for } i\geq 0, \qquad (16)\end{aligned}$$

Observe that $\mathcal{D}[\tau_{i+1},\ell_i,\ell_{i+1},X,u^{\varepsilon}]\mathbb{1}_{\{\tau_{i+1}<\tau\}}\leq 0$ for $i\geq 0$ because of $u^{\varepsilon}_{\ell}-(u^{\varepsilon}_{\kappa}+\vartheta_{\ell,\kappa})\leq u^{\varepsilon}_{\ell}-\mathcal{M}_{\ell}u^{\varepsilon}\leq 0$. With this remark and (16), we conclude the statement of the lemma above.

For any $\ell \in \mathbb{I}$, let $\mathcal{S}_\ell^\varepsilon$ be the set defined by

$$\mathcal{S}_\ell^\varepsilon = \{x \in \mathcal{O} : u_\ell^\varepsilon(x) - \mathcal{M}_\ell u^\varepsilon(x) = 0\}.$$

Notice that $\mathcal{S}_\ell^\varepsilon$ is a closed subset of $\mathcal{O}$ and corresponds with the region where it is optimal to switch regimes. The complement $\mathcal{C}_\ell^\varepsilon$ of $\mathcal{S}_\ell^\varepsilon$ in $\mathcal{O}$, where is optimal to stay in the regime ℓ, is the so-called continuation region

$$\mathcal{C}_\ell^\varepsilon = \{x \in \mathcal{O} : u_\ell^\varepsilon(x) - \mathcal{M}_\ell u^\varepsilon(x) < 0\}.$$

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Observe that $u_\ell^\varepsilon \in \mathrm{W}^{2,\infty}_{\mathrm{loc}}(\mathcal{C}_\ell^\varepsilon) = \mathrm{C}^{1,1}_{\mathrm{loc}}(\mathcal{C}_\ell^\varepsilon)$. Then, it can be proven that $u_\ell^\varepsilon \in \mathrm{C}^{2,\alpha'}_{\mathrm{loc}}(\mathcal{C}_\ell^\varepsilon)$.

For any $\ell \in \mathbb{I}$, let $\mathcal{S}^{\varepsilon}_{\ell}$ be the set defined by

$$\mathcal{S}^{\varepsilon}_{\ell} = \{x \in \mathcal{O} : u^{\varepsilon}_{\ell}(x) - \mathcal{M}_{\ell} u^{\varepsilon}(x) = 0\}.$$

Notice that $\mathcal{S}^{\varepsilon}_{\ell}$ is a closed subset of $\mathcal{O}$ and corresponds with the region where it is optimal to switch regimes. The complement $\mathcal{C}^{\varepsilon}_{\ell}$ of $\mathcal{S}^{\varepsilon}_{\ell}$ in $\mathcal{O}$, where is optimal to stay in the regime ℓ, is the so-called continuation region

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Lemma 5

Let ℓ be in $\mathbb{I}$. Then, $\mathcal{S}^{\varepsilon}_{\ell} = \widetilde{\mathcal{S}}^{\varepsilon}_{\ell} := \bigcup_{\kappa \in \mathbb{I} \setminus \{\ell\}} \mathcal{S}^{\varepsilon}_{\ell,\kappa}$ where

$$\mathcal{S}^{\varepsilon}_{\ell,\kappa} := \{x \in \mathcal{C}^{\varepsilon}_{\kappa} : u^{\varepsilon}_{\ell}(x) = u^{\varepsilon}_{\kappa}(x) + \vartheta_{\ell,\kappa}\}.$$

Construction of the optimal stochastic control for (12)

(i) Define $\tau_0^* = 0$ and $\ell_{0-}^* = \tilde{\ell}$. If $\tilde{x} \notin \mathcal{C}_{\tilde{\ell}}^{\varepsilon}$, take $\tau_1^* \coloneqq 0$ and pass to item (ii) because of Lemma 5. Otherwise, the process $X^{\varepsilon,*}$ evolves as

$$X_{t\wedge\tilde{\tau}_1^*}^{\varepsilon,*} = \tilde{x} - \int_0^{t\wedge\tilde{\tau}_1^*} [b(X_s^{\varepsilon,*}, \ell_0^*) + \mathfrak{n}_s^{\varepsilon,*}\zeta_s^{\varepsilon,*}]\mathrm{d}s + \int_0^{t\wedge\tilde{\tau}_1^*} \sigma(X_s^{\ell,\varepsilon,*}, \ell_0^*)\mathrm{d}W_s, \quad \text{for } t > 0, \tag{17}$$

with $X_0^{\varepsilon,*} = \tilde{x}$, $\tau^* \coloneqq \inf\{t > 0 : X_t^{\varepsilon,*} \notin \mathcal{O}\}$,

$$\tilde{\tau}_1^* \coloneqq \tau_1^* \wedge \tau^* \quad \text{and} \quad \tau_1^* \coloneqq \inf\left\{t \geq 0 : X_t^{\varepsilon,*} \in \mathcal{S}_{\ell_0^*}^{\varepsilon}\right\}. \tag{18}$$

The control process $\xi^{\varepsilon,*} = (\mathfrak{n}^{\varepsilon,*}, \zeta^{\varepsilon,*})$ is defined by

$$\mathfrak{n}_t^{\varepsilon,*} = \begin{cases} \dfrac{\mathrm{D}^1 u_{\ell_0^*}^{\varepsilon}(X_t^{\varepsilon,*})}{|\mathrm{D}^1 u_{\ell_0^*}^{\varepsilon}(X_t^{\varepsilon,*})|}, & \text{if } |\mathrm{D}^1 u_{\ell_0^*}^{\varepsilon}(X_t^{\varepsilon,*})| \neq 0 \text{ and } t \in [0, \tilde{\tau}_1^*), \\ \gamma_0, & \text{if } |\mathrm{D}^1 u_{\ell_0^*}^{\varepsilon}(X_t^{\varepsilon,*})| = 0 \text{ and } t \in [0, \tilde{\tau}_1^*), \end{cases} \tag{19}$$

where $\gamma_0 \in \mathbb{R}^d$ is a unit vector fixed, and $\zeta_t^{\varepsilon,*} = \int_0^t \dot{\zeta}_s^{\varepsilon,*}\mathrm{d}s$, with $t \in [0, \tilde{\tau}_1^*)$ and

$$\dot{\zeta}_s^{\varepsilon,*} = 2\psi_\varepsilon'(|\mathrm{D}^1 u_{\ell_0^*}^{\varepsilon}(X_s^{\varepsilon,*})|^2 - g_{\ell_0^*}(X_s^{\varepsilon,*})^2)|\mathrm{D}^1 u_{\ell_0^*}^{\varepsilon}(X_s^{\varepsilon,*})|. \tag{20}$$

(ii) Recursively, letting $i \geq 1$ and defining

$$\begin{aligned} &\ell_i^* \in \underset{\kappa \in \mathbb{I} \setminus \{\ell_{i-1}^*\}}{\arg\min} \left\{ u_\kappa^\varepsilon(X_{\tau_i^*}^{\varepsilon,*}) + \vartheta_{\ell_{i-1}^*,\kappa} \right\}, \\ &\tilde{\tau}_{i+1}^* = \tau_{i+1}^* \wedge \tau^*, \quad \tau_{i+1}^* = \inf\left\{ t > \tau_i^* : X_t^{\varepsilon,*} \in \mathcal{S}_{\ell_i^*}^\varepsilon \right\}, \end{aligned} \tag{21}$$

if $\tau_i^* < \tau^*$, the process $X^{\varepsilon,*}$ evolves as

$$X_{t \wedge \tilde{\tau}_{i+1}^*}^{\varepsilon,*} = X_{\tau_i^*}^{\varepsilon,*} - \int_{\tau_i^*}^{t \wedge \tilde{\tau}_{i+1}^*} [b(X_s^{\varepsilon,*}, \ell_i^*) + \mathfrak{n}_s^{\varepsilon,*} \zeta_s^{\varepsilon,*}] \mathrm{d}s + \int_{\tau_i^*}^{t \wedge \tilde{\tau}_{i+1}^*} \sigma(X_s^{\varepsilon,*}, \ell_i^*) \mathrm{d}W_s, \quad \text{for } t \geq \tau_i^*, \tag{22}$$

where

$$\mathfrak{n}_t^{\varepsilon,*} = \begin{cases} \dfrac{\mathrm{D}^1 u_{\ell_i^*}^\varepsilon(X_t^{\varepsilon,*})}{|\mathrm{D}^1 u_{\ell_i^*}^\varepsilon(X_t^{\varepsilon,*})|}, & \text{if } |\mathrm{D}^1 u_{\ell_i^*}^\varepsilon(X_t^{\varepsilon,*})| \neq 0 \text{ and } t \in [\tau_i^*, \tilde{\tau}_{i+1}^*), \\ \gamma_0, & \text{if } |\mathrm{D}^1 u_{\ell_i^*}^\varepsilon(X_t^{\varepsilon,*})| = 0 \text{ and } t \in [\tau_i^*, \tilde{\tau}_{i+1}^*), \end{cases} \tag{23}$$

with $\gamma_0 \in \mathbb{R}^d$ is a unit vector fixed, and $\zeta_t^{\varepsilon,*} = \int_{\tau_i^{\varepsilon,*}}^t \dot{\zeta}_s^{\varepsilon,*} \mathrm{d}s$, with $t \in [\tau_i^*, \tilde{\tau}_{i+1}^*)$ and

$$\dot{\zeta}_s^{\varepsilon,*} = 2\psi_\varepsilon'(|\mathrm{D}^1 u_{\ell_i^*}^\varepsilon(X_s^{\varepsilon,*})|^2 - g_{\ell_i^*}(X_s^{\varepsilon,*})^2) |\mathrm{D}^1 u_{\ell_i^*}^\varepsilon(X_s^{\varepsilon,*})|. \tag{24}$$

Lemma 6 (Verification Lemma. Second part)

Let $\varepsilon \in (0,1)$ be fixed and let $(X^{\varepsilon,*}, I^{\varepsilon,*})$ be the process that is governed by (17)–(24). Then, $u^{\varepsilon}_{\tilde{\ell}}(\tilde{x}) = \mathcal{V}_{\xi^{\varepsilon,*},\varsigma^{\varepsilon,*}}(\tilde{x},\tilde{\ell}) = V^{\varepsilon}_{\tilde{\ell}}(\tilde{x})$ for each $(\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$.

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Take $\hat{\tau}_i^{*,q} := \tau_i^* \wedge \inf\{t > \tau^*_{i-1} : X^{\varepsilon,*}_t \notin \mathcal{O}_q\}$, with

$$\mathcal{O}_q := \{x \in \mathcal{O} : \mathrm{dist}(x, \partial\mathcal{O}) > 1/q\}$$

and q a positive integer large enough. By the seen two slides above, it is known that u^{ε}_{ℓ} is a C^2-function on $\mathcal{C}^{\varepsilon}_{\ell}$ and that $X^{\varepsilon,*}_t \in \mathcal{C}^{\varepsilon}_{\ell^*_i}$ if $t \in [\hat{\tau}^{*,q}_i, \hat{\tau}^{*,q}_{i+1})$.

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Take $\hat{\tau}^{*,q}_i := \tau^*_i \wedge \inf\{t > \tau^*_{i-1} : X^{\varepsilon,*}_t \notin \mathcal{O}_q\}$, with

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Then, using integration by parts and Itô's formula in $\mathrm{e}^{-r(t)}\, u^{\varepsilon}_{\ell^*_i}(X^{\varepsilon,*}_t)$ on the interval $[\hat{\tau}^{*,q}_i, \hat{\tau}^{*,q}_{i+1})$, taking expected value on it, , we obtain a similar expression as in (13).

Key steps in the proof of Theorem 2

Let $\{u^{\varepsilon_n}\}_{\varepsilon_n\in(0,1)}$ be the sequence of unique strong solutions to the HJB equation (5), when $\varepsilon=\varepsilon_n$, which satisfy (8). From Lemma 6, we know that

$$u^{\varepsilon_n}_{\tilde{\ell}}(\tilde{x})=\mathcal{V}_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\tilde{x},\tilde{\ell})=V^{\varepsilon_n}(\tilde{x},\tilde{\ell})\quad\text{for } (\tilde{x},\tilde{\ell})\in\overline{\mathcal{O}}\times\mathbb{I},$$

with $(\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*})$ as in (18)–(21) and (23)–(24), when $\varepsilon=\varepsilon_n$.

Key steps in the proof of Theorem 2

Let $\{u^{\varepsilon_n}\}_{\varepsilon_n \in (0,1)}$ be the sequence of unique strong solutions to the HJB equation (5), when $\varepsilon = \varepsilon_n$, which satisfy (8). From Lemma 6, we know that

$$u^{\varepsilon_n}_{\tilde{\ell}}(\tilde{x}) = \mathcal{V}_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\tilde{x},\tilde{\ell}) = V^{\varepsilon_n}(\tilde{x},\tilde{\ell}) \quad \text{for } (\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I},$$

with $(\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*})$ as in (18)–(21) and (23)–(24), when $\varepsilon = \varepsilon_n$.

Notice that

$$l_{\varepsilon_n}(x,\beta\gamma) \geq \langle \beta\gamma, g_{\ell_i^*}(x)\gamma\rangle - \psi_{\varepsilon_n}(|g_{\ell_i^*}(x)\gamma|^2 - g_{\ell_i^*}(x)^2) = \beta g_{\ell_i^*}(x),$$

with $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$ a unit vector.

Now, considering that $\widetilde{\mathcal{M}}_{\ell_i^*}[\hat{\tau}_i^{*,q}, t \wedge \hat{\tau}_{i+1}^{*,q}; X^{\varepsilon,*}, u^{\varepsilon}]$ is a square integrable martingale and since

$$[c_{\ell_i^*} - \mathcal{L}_{\ell_i^*}]u^{\varepsilon}_{\ell_i^*} = h_{\ell_i^*} - \psi_{\varepsilon}(|\,\mathrm{D}^1\, u^{\varepsilon}_{\ell_i^*}|^2 - g^2_{\ell_i^*}) \quad \text{on} \quad \mathcal{C}^{\varepsilon}_{\ell_i^*}$$

and the supremum of $l^{\varepsilon}_{\ell}(\eta, x)$ is attained if γ is related to η by $\eta = 2\psi'_{\varepsilon}(|\gamma|^2 - g_{\ell}(x)^2)\gamma$, i.e.,

$$l^{\varepsilon}_{\ell}(2\psi'_{\varepsilon}(|\gamma|^2 - g_{\ell}(x)^2)\gamma, x) = 2\psi'_{\varepsilon}(|\gamma|^2 - g_{\ell}(x)^2)|\gamma|^2 - \psi_{\varepsilon}(|\gamma|^2 - g_{\ell}(x)^2),$$

it can be checked

$$\begin{aligned}
\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\hat{\tau}_i^{*,q})}\, u^{\varepsilon}_{\ell_i^*}(X^{\varepsilon,*}_{\hat{\tau}_i^{*,q}})\Big] = \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[&[u^{\varepsilon}_{\ell_1^*}(X^{\varepsilon,*}_{\tau_1^*}) + \vartheta_{\ell_0^*,\ell_1^*}]\mathbb{1}_{\{\tau_1^*=0,\tau_1^*<\tau^*,i=0\}} \\
&+ [u^{\varepsilon}_{\ell_0^*}(X^{\varepsilon,*}_{\tau_1^*}) - u^{\varepsilon}_{\ell_1^*}(X^{\varepsilon,*}_{\tau_1^*}) - \vartheta_{\ell_0^*,\ell_1^*}]\mathbb{1}_{\{\tau_1^*=0,\tau_1^*<\tau^*,i=0\}} \\
&+ \mathrm{e}^{-r(t\wedge\hat{\tau}_1^{*,q})}\, u^{\varepsilon}_{\ell_0^*}(X^{\varepsilon,*}_{t\wedge\hat{\tau}_1^{*,q}})\mathbb{1}_{\{\tau_1^*\neq 0,i=0\}} + \mathrm{e}^{-r(t\wedge\hat{\tau}_{i+1}^{*,q})}\, u^{\varepsilon}_{\ell_i^*}(X^{\varepsilon,*}_{t\wedge\hat{\tau}_{i+1}^{*,q}})\mathbb{1}_{\{i\neq 0\}} \\
&+ \mathbb{1}_{\{\tau_1^*\neq 0,i=0\}}\int_0^{t\wedge\hat{\tau}_1^{*,q}} \mathrm{e}^{-r(s)}[h_{\ell_0^*}(X^{\varepsilon,*}_s) + l^{\varepsilon}_{\ell_0^*}(\dot{\zeta}^*_s\,\mathrm{n}^*_s, X^{\varepsilon,*}_s)]\mathrm{d}s \\
&+ \mathbb{1}_{\{i\neq 0\}}\int_{\hat{\tau}_i^{*,q}}^{t\wedge\hat{\tau}_{i+1}^{*,q}} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s) + l^{\varepsilon}_{\ell_i^*}(\dot{\zeta}^*_s\,\mathrm{n}^*_s, X^{\varepsilon,*}_s)]\mathrm{d}s\Big],
\end{aligned} \tag{25}$$

Notice that $\hat{\tau}_i^{*,q} \uparrow \tilde{\tau}_i^*$ as $q \to \infty$, $\mathbb{P}_{\tilde{x}}$-a.s.. Consequently, letting first $q \to \infty$ and after $t \to \infty$ in (25), we see that

$$\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\tilde{\tau}_i^*)}\, u^{\varepsilon}_{\ell_i^*}(X^{\varepsilon,*}_{\tilde{\tau}_i^*})\Big] = \mathbb{E}_{\tilde{x},\tilde{\ell}}\Bigg[\{\mathrm{e}^{-r(\tau_{i+1}^*)}\, u^{\varepsilon}_{\ell_{i+1}^*}(X^{\varepsilon,*}_{\tau_{i+1}^*}) + \mathcal{D}[\tau_{i+1}^*, \ell_i^*, \ell_{i+1}^*, X^{\varepsilon,*}, u^{\varepsilon}]\}\mathbb{1}_{\{\tau_{i+1}^* < \tau^*\}}$$
$$+ \int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon,*}) + l^{\varepsilon}_{\ell_i^*}(\dot{\zeta}_s^*\mathfrak{n}_s^*, X_s^{\varepsilon,*})]\mathrm{d}s + \mathrm{e}^{-r(\tau_{i+1}^*)}\,\vartheta_{\ell_i^*,\ell_{i+1}^*}\mathbb{1}_{\{\tau_{i+1}^* < \tau^*\}}\Bigg] \quad \text{for } i \geq 0, \qquad (26)$$

with $\mathcal{D}[\tau_{i+1}^*, \ell_i^*, \ell_{i+1}^*, X^{\varepsilon,*}, u^{\varepsilon}]$ as in (15). By (18) and (21),

$$\mathcal{D}[\tau_{i+1}^*, \ell_i^*, \ell_{i+1}^*, X^{\varepsilon,*}, u^{\varepsilon}]\mathbb{1}_{\{\tau_{i+1}^* < \tau^*\}} = 0.$$

Therefore, from here and (26), we obtain the desired result that was given in the lemma above.

Then, from here and considering $(X^{\varepsilon_n,*}, I^{\varepsilon_n,*})$ governed by (17) and (22), it follows that

$$
\begin{aligned}
V_{\tilde{\ell}}(\tilde{x}) &\leq V_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\tilde{x},\tilde{\ell}) \\
&= \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon_n,*})\mathrm{d}s + \dot{\zeta}_s^* g_{\ell_i^*}(X_{s-}^{\varepsilon_n,*})]\mathrm{d}s \\
&\quad + \mathrm{e}^{-r(\tau_{i+1}^*)}\,\vartheta_{\ell_i^*,\ell_{i+1}^*}\mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\bigg] \\
&\leq \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon_n,*}) + l_{\ell_i^*}^{\varepsilon_n}(\dot{\zeta}_s^*\mathbb{n}_s^*, X_s^{\varepsilon_n,*})]\mathrm{d}s \\
&\quad + \mathrm{e}^{-r(\tau_{i+1}^*)}\,\vartheta_{\ell_i^*,\ell_{i+1}^*}\mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\bigg] \\
&= u_{\tilde{\ell}}^{\varepsilon_n}(\tilde{x}). \qquad (27)
\end{aligned}
$$

Then, from here and considering $(X^{\varepsilon_n,*}, I^{\varepsilon_n,*})$ governed by (17) and (22), it follows that

$$
\begin{aligned}
V_{\tilde{\ell}}(\tilde{x}) &\leq V_{\xi^{\varepsilon_n,*},\varsigma^{\varepsilon_n,*}}(\tilde{x},\tilde{\ell}) \\
&= \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon_n,*})\mathrm{d}s + \dot{\zeta}_s^* g_{\ell_i^*}(X_{s-}^{\varepsilon_n,*})]\mathrm{d}s \\
&\quad + \mathrm{e}^{-r(\tau_{i+1}^*)}\,\vartheta_{\ell_i^*,\ell_{i+1}^*}\mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\bigg] \\
&\leq \sum_{i\geq 0} \mathbb{E}_{\tilde{x},\tilde{\ell}}\bigg[\int_{\tilde{\tau}_i^*}^{\tilde{\tau}_{i+1}^*} \mathrm{e}^{-r(s)}[h_{\ell_i^*}(X_s^{\varepsilon_n,*}) + l_{\ell_i^*}^{\varepsilon_n}(\dot{\zeta}_s^*\mathbb{n}_s^*, X_s^{\varepsilon_n,*})]\mathrm{d}s \\
&\quad + \mathrm{e}^{-r(\tau_{i+1}^*)}\,\vartheta_{\ell_i^*,\ell_{i+1}^*}\mathbb{1}_{\{\tau_{i+1}^*<\tau^*\}}\bigg] \\
&= u_{\tilde{\ell}}^{\varepsilon_n}(\tilde{x}). \qquad (27)
\end{aligned}
$$

Letting $\varepsilon_n \to 0$ in (27), it yields $V_{\tilde{\ell}} \leq u_{\tilde{\ell}}$ on $\overline{\mathcal{O}}$.

Let $\{u^{\varepsilon_n,\delta_{\hat{n}}}\}_{n,\hat{n}\geq 1} \subset \mathrm{C}^{4,\alpha'}(\overline{\mathcal{O}})$ be the sequence of unique solutions to the NPDS

$$[c_\ell - \mathcal{L}_\ell]u_\ell^{\varepsilon,\delta} + \psi_\varepsilon(|\,\mathrm{D}^1\, u_\ell^{\varepsilon,\delta}|^2 - g_\ell^2) + \sum_{\kappa\in\mathbb{I}\setminus\{\ell\}} \psi_\delta(u_\ell^{\varepsilon,\delta} - u_\kappa^{\varepsilon,\delta} - \vartheta_{\ell,\kappa}) = h_\ell, \text{ in } \mathcal{O},$$
$$\text{s.t. } u_\ell^{\varepsilon,\delta} = 0, \text{ on } \partial\mathcal{O}$$

when $\varepsilon = \varepsilon_n$ and $\delta = \delta_{\hat{n}}$, which satisfy

$$u_\ell^{\varepsilon,\delta_{\hat{n}}} \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} u_\ell^\varepsilon \text{ in } \mathrm{C}(\overline{\mathcal{O}}),\ \partial_i u_\ell^{\varepsilon,\delta_{\hat{n}}} \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} \partial_i u_\ell^\varepsilon \text{ in } \mathrm{C}_{\mathrm{loc}}(\mathcal{O}),$$
$$\partial_{ij} u_\ell^{\varepsilon,\delta_{\hat{n}}} \underset{\delta_{\hat{n}}\to 0}{\longrightarrow} \partial_{ij} u_\ell^\varepsilon, \text{ weakly } \mathrm{L}^p_{\mathrm{loc}}(\mathcal{O}), \text{ for each } p \in (1,\infty)$$

and

$$u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} u_\ell \text{ in } \mathrm{C}(\overline{\mathcal{O}}),\ \partial_i u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_i u_\ell \text{ in } \mathrm{C}_{\mathrm{loc}}(\mathcal{O}),$$
$$\partial_{ij} u_\ell^{\varepsilon_n} \underset{\varepsilon_n\to 0}{\longrightarrow} \partial_{ij} u_\ell, \text{ weakly } \mathrm{L}^p_{\mathrm{loc}}(\mathcal{O}), \text{ for each } p \in (1,\infty).$$

Let us consider (X, I) evolves as in (2) with initial state $(\tilde{x}, \tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$ and the control process (ξ, ς) belongs to $\mathcal{U} \times \mathcal{S}$.

Take $\hat{\tau}_i^q := \tau_i \wedge \inf\{t > \tau_{i-1} : X_t \notin \mathcal{O}_q\}$, with $i \geq 1$,

$$\mathcal{O}_q := \{x \in \mathcal{O} : \mathrm{dist}(x, \partial\mathcal{O}) > 1/q\}$$

and q a positive integer large enough.

Using integration by parts and Itô's formula in $\mathrm{e}^{-r(t)} u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_t)$ on $[\hat{\tau}^q_i, \hat{\tau}^q_{i+1})$, $i \geq 0$, we get that

$$\begin{aligned}
\mathrm{e}^{-r(\hat{\tau}^q_i)} u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_{\hat{\tau}^q_i}) &= [u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_1}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}}\\
&+ [u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_0}(X_{\tau_1}) - u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_1}(X_{\tau_1}) - \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}}\\
&+ \mathrm{e}^{-r(t\wedge\hat{\tau}^q_1)} u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_0}(X_{t\wedge\hat{\tau}^q_1})\mathbb{1}_{\{\tau_1\neq0,i=0\}} + \mathrm{e}^{-r(t\wedge\hat{\tau}^q_{i+1})} u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_{t\wedge\hat{\tau}^q_{i+1}})\mathbb{1}_{\{i\neq0\}}\\
&+ \mathbb{1}_{\{\tau_1\neq0,i=0\}}\int_0^{t\wedge\hat{\tau}^q_1} \mathrm{e}^{-r(s)}[[c_{\ell_0}(X_s)u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_0}(X_s) - \mathcal{L}_{\ell_0}u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_0}(X_s)]\mathrm{d}s + \langle \mathrm{D}^1 u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_0}(X_s), \mathfrak{n}_s\rangle \mathrm{d}\zeta^{\mathrm{c}}_s]\\
&+ \mathbb{1}_{\{i\neq0\}}\int_{\hat{\tau}^q_i}^{t\wedge\hat{\tau}^q_{i+1}} \mathrm{e}^{-r(s)}[[c_{\ell_i}(X_s)u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_s) - \mathcal{L}_{\ell_i}u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_s)]\mathrm{d}s + \langle \mathrm{D}^1 u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_s), \mathfrak{n}_s\rangle \mathrm{d}\zeta^{\mathrm{c}}_s]\\
&- \sum_{0\leq s<t\wedge\hat{\tau}^q_1} \mathrm{e}^{-r(s)}\mathcal{J}[s;\ell_0,X,u^{\varepsilon_n,\delta_{\hat{n}}}]\mathbb{1}_{\{\tau_1\neq0,i=0\}} - \sum_{\hat{\tau}^q_i\leq s<t\wedge\hat{\tau}^q_{i+1}} \mathrm{e}^{-r(s)}\mathcal{J}[s;\ell_i,X,u^{\varepsilon_n,\delta_{\hat{n}}}]\mathbb{1}_{\{i\neq0\}}\\
&- \widetilde{\mathcal{M}}_{\ell_0}[0,t\wedge\hat{\tau}^q_1;X,u^{\varepsilon_n,\delta_{\hat{n}}}]\mathbb{1}_{\{\tau_1\neq0,i=0\}} - \widetilde{\mathcal{M}}_{\ell_i}[\hat{\tau}^q_i,t\wedge\hat{\tau}^q_{i+1};X,u^{\varepsilon_n,\delta_{\hat{n}}}]\mathbb{1}_{\{i\neq0\}},
\end{aligned} \tag{28}$$

where

$$\mathcal{J}[s;\ell_i,X,u^{\varepsilon_n,\delta_{\hat{n}}}] \coloneqq u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_{s-} - \mathfrak{n}_s\Delta\zeta_s) - u^{\varepsilon_n,\delta_{\hat{n}}}_{\ell_i}(X_{s-}), \quad \text{for } i \geq 0.$$

Since $X_{s-} - \mathfrak{n}_s \Delta \xi_t \in \mathcal{O}$ for $s \in [\hat{\tau}_i^q, t \wedge \hat{\tau}_{i+1}^q)$, $i \geq 0$, and $\mathcal{O}$ is a convex set, by Mean Value Theorem, it implies

$$-\mathcal{J}[s; \ell_i, X, u^{\varepsilon n, \delta_{\hat{n}}}] \leq |u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_{s-} - \mathfrak{n}_s \Delta \zeta_s) - u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_{s-})|$$
$$\leq \Delta \zeta_s \int_0^1 |\, \mathrm{D}^1 u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_{s-} - \lambda \mathfrak{n}_s \Delta \zeta_s)| \mathrm{d}\lambda.$$

Taking expected value in (28) and since $\widetilde{\mathcal{M}}_{\ell_i}[\hat{\tau}_i^q, t \wedge \hat{\tau}_{i+1}^q; X, u^{\varepsilon n, \delta_{\hat{n}}}]$ is a square integrable martingale and

$$[c_{\ell_i} - \mathcal{L}_{\ell_i}] u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}} \leq h_{\ell_i},$$

it follows that

$$\begin{aligned}
\mathbb{E}_{\tilde{x}, \tilde{\ell}}\big[\, \mathrm{e}^{-r(\hat{\tau}_i^q)}\, u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_{\hat{\tau}_i^q})\big] \leq \mathbb{E}_{\tilde{x}, \tilde{\ell}}\bigg[& [u_{\ell_1}^{\varepsilon n, \delta_{\hat{n}}}(X_{\tau_1}) + \vartheta_{\ell_0, \ell_1}] \mathbb{1}_{\{\tau_1 = 0, \tau_1 < \tau, i=0\}} \\
& + [u_{\ell_0}^{\varepsilon n, \delta_{\hat{n}}}(X_{\tau_1}) - u_{\ell_1}^{\varepsilon n, \delta_{\hat{n}}}(X_{\tau_1}) - \vartheta_{\ell_0, \ell_1}] \mathbb{1}_{\{\tau_1 = 0, \tau_1 < \tau, i=0\}} \\
& + \mathrm{e}^{-r(t \wedge \hat{\tau}_1^q)}\, u_{\ell_0}^{\varepsilon n, \delta_{\hat{n}}}(X_{t \wedge \hat{\tau}_1^q}) \mathbb{1}_{\{\tau_1 \neq 0, i=0\}} + \mathrm{e}^{-r(t \wedge \hat{\tau}_{i+1}^q)}\, u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_{t \wedge \hat{\tau}_{i+1}^q}) \mathbb{1}_{\{i \neq 0\}} \\
& + \mathbb{1}_{\{\tau_1 \neq 0, i=0\}} \int_0^{t \wedge \hat{\tau}_1^q} \mathrm{e}^{-r(s)} [h_{\ell_0}(X_s) \mathrm{d}s + |\, \mathrm{D}^1 u_{\ell_0}^{\varepsilon n, \delta_{\hat{n}}}(X_s)| \mathrm{d}\zeta_s^{\mathrm{c}}] \\
& + \mathbb{1}_{\{i \neq 0\}} \int_{\hat{\tau}_i^q}^{t \wedge \hat{\tau}_{i+1}^q} \mathrm{e}^{-r(s)} [h_{\ell_i}(X_s) \mathrm{d}s + |\, \mathrm{D}^1 u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_s)| \mathrm{d}\zeta_s^{\mathrm{c}}] \\
& + \sum_{0 \leq s < t \wedge \hat{\tau}^q} \mathrm{e}^{-r(s)}\, \Delta \zeta_s \int_0^1 |\, \mathrm{D}^1 u_{\ell_0}^{\varepsilon n, \delta_{\hat{n}}}(X_{s-} - \lambda \mathfrak{n}_s \Delta \zeta_s)| \mathrm{d}\lambda \mathbb{1}_{\{\tau_1 \neq 0, i=0\}} \\
& + \sum_{\hat{\tau}_i^q \leq s < t \wedge \hat{\tau}_{i+1}^q} \mathrm{e}^{-r(s)}\, \Delta \zeta_s \int_0^1 |\, \mathrm{D}^1 u_{\ell_i}^{\varepsilon n, \delta_{\hat{n}}}(X_{s-} - \lambda \mathfrak{n}_s \Delta \zeta_s)| \mathrm{d}\lambda \mathbb{1}_{\{i \neq 0\}} \bigg].
\end{aligned} \tag{29}$$

Notice that for $s \in [\hat{\tau}_i^q, t \wedge \hat{\tau}_{i+1}^q)$,

$$|u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s) - u_{\ell_i}(X_s)| \leq \max_{(x,\ell)\in\mathcal{O}_q\times\mathbb{I}} |u_{\ell}^{\varepsilon_n,\delta_{\hat{n}}}(x) - u_\ell(x)| \underset{\varepsilon_n,\delta_{\hat{n}}\to 0}{\longrightarrow} 0,$$

$$|\partial_j u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s) - \partial_j u_{\ell_i}(X_s)| \leq \max_{(x,\ell)\in\mathcal{O}_q\times\mathbb{I}} |\partial_j u_{\ell}^{\varepsilon_n,\delta_{\hat{n}}}(x) - \partial_j u_\ell(x)| \underset{\varepsilon_n,\delta_{\hat{n}}\to 0}{\longrightarrow} 0,$$

$$|\partial_j u_{\ell_i}^{\varepsilon_n,\delta_{\hat{n}}}(X_s - \lambda \mathrm{n}_s\Delta\zeta_s) - \partial_j u_{\ell_i}(X_s - \lambda \mathrm{n}_s\Delta\zeta_s)| \leq \max_{(x,\ell)\in\mathcal{O}_q\times\mathbb{I}} |\partial_j u_{\ell}^{\varepsilon_n,\delta_{\hat{n}}}(x) - \partial_j u_\ell(x)| \underset{\varepsilon_n,\delta_{\hat{n}}\to 0}{\longrightarrow} 0,$$

with $\lambda \in [0,1]$. Then, letting $\varepsilon_n, \delta_{\hat{n}} \to 0$ in (29), by Dominated Convergence Theorem and using $|\,\mathrm{D}^1 u_{\ell_i}| \leq g_{\ell_i}$,

$$\begin{aligned}
\mathbb{E}_{\tilde{x},\tilde{\ell}}\big[\mathrm{e}^{-r(\hat{\tau}_i^q)}\, u_{\ell_i}(X_{\hat{\tau}_i^q})\big] \leq \mathbb{E}_{\tilde{x},\tilde{\ell}}\Bigg[&[u_{\ell_1}(X_{\tau_1}) + \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\
&+ [u_{\ell_0}(X_{\tau_1}) - u_{\ell_1}(X_{\tau_1}) - \vartheta_{\ell_0,\ell_1}]\mathbb{1}_{\{\tau_1=0,\tau_1<\tau,i=0\}} \\
&+ \mathrm{e}^{-r(t\wedge\hat{\tau}_1^q)}\, u_{\ell_0}(X_{t\wedge\hat{\tau}_1^q})\mathbb{1}_{\{\tau_1\neq 0,i=0\}} + \mathrm{e}^{-r(t\wedge\hat{\tau}_{i+1}^q)}\, u_{\ell_i}(X_{t\wedge\hat{\tau}_{i+1}^q})\mathbb{1}_{\{i\neq 0\}} \\
&+ \mathbb{1}_{\{\tau_1\neq 0,i=0\}} \int_0^{t\wedge\hat{\tau}_1^q} \mathrm{e}^{-r(s)}[h_{\ell_0}(X_s)\mathrm{d}s + g_{\ell_0}(X_{s-})\circ \mathrm{d}\zeta_s^{\mathrm{c}}] \\
&+ \mathbb{1}_{\{i\neq 0\}} \int_{\hat{\tau}_i^q}^{t\wedge\hat{\tau}_{i+1}^q} \mathrm{e}^{-r(s)}[h_{\ell_i}(X_s)\mathrm{d}s + g_{\ell_i}(X_{s-})\circ \mathrm{d}\zeta_s]\Bigg],
\end{aligned} \tag{30}$$

Again, letting $q \to \infty$ and $t \to \infty$ in (30), it implies that

$$\mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\mathrm{e}^{-r(\tilde{\tau}_i)}\, u_{\ell_i}(X_{\tilde{\tau}_i})\Big] \leq \mathbb{E}_{\tilde{x},\tilde{\ell}}\Big[\{\mathrm{e}^{-r(\tau_{i+1})}\, u_{\ell_{i+1}}(X_{\tau_{i+1}}) + \mathcal{D}[\tau_{i+1}, \ell_i, \ell_{i+1}, X, u]\}\mathbb{1}_{\{\tau_{i+1}<\tau\}} + \int_{\tilde{\tau}_i}^{\tilde{\tau}_{i+1}} \mathrm{e}^{-r(s)}[h_{\ell_i}(X_s)\mathrm{d}s + g_{\ell_i}(X_{s-}) \circ \mathrm{d}\zeta_s] + \mathrm{e}^{-r(\tau_{i+1})}\, \vartheta_{\ell_i,\ell_{i+1}}\mathbb{1}_{\{\tau_{i+1}<\tau\}}\Big] \quad \text{for } i \geq 0, \tag{31}$$

with $\tilde{\tau}_i = \tau_i \wedge \tau$ and $\mathcal{D}[\tau_{i+1}, \ell_i, \ell_{i+1}, X, u]$ as in (15). Noticing that

$$\mathcal{D}[\tau_{i+1}, \ell_i, \ell_{i+1}, X, u]\mathbb{1}_{\{\tau_{i+1}<\tau\}} \leq 0$$

and using definition of $V_{\xi,\varsigma}$, (31), it yields that $u_{\tilde{\ell}}(\tilde{x}) \leq V_{\xi,\varsigma}(\tilde{x},\tilde{\ell})$ for each $(\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$. Since the previous property is true for each control $(\xi,\varsigma) \in \mathcal{U} \times \mathcal{S}$, we conclude that $u_{\tilde{\ell}}(\tilde{x}) \leq V_{\tilde{\ell}}(\tilde{x})$ for each $(\tilde{x},\tilde{\ell}) \in \overline{\mathcal{O}} \times \mathbb{I}$.

Taking ε_n small enough and considering that the process $X_t^{\varepsilon_n,*}$ is on the regime ℓ_i^* at time $t \in [\tau_i^*, \tilde{\tau}_{i+1}^*)$, we have that $X_t^{\varepsilon_n,*} \in \mathcal{C}_{\ell_i^*}^{\varepsilon_n}$ and the control $(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*})$ will be exercised as follows:

Taking ε_n small enough and considering that the process $X_t^{\varepsilon_n,*}$ is on the regime ℓ_i^* at time $t \in [\tau_i^*, \tilde{\tau}_{i+1}^*)$, we have that $X_t^{\varepsilon_n,*} \in \mathcal{C}_{\ell_i^*}^{\varepsilon_n}$ and the control $(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*})$ will be exercised as follows:

(i) if the controlled process $X^{\varepsilon_n,*}$ satisfies $|\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})| \leq g_{\ell_i^*}(X_t^{\varepsilon_n,*})$, then $\zeta_t^{\varepsilon_n,*} \equiv 0$ and $X_t^{\varepsilon_n,*}$ will stay in $\mathcal{C}_{\ell_i^*}$;

Taking ε_n small enough and considering that the process $X_t^{\varepsilon_n,*}$ is on the regime ℓ_i^* at time $t \in [\tau_i^*, \tilde{\tau}_{i+1}^*)$, we have that $X_t^{\varepsilon_n,*} \in \mathcal{C}_{\ell_i^*}^{\varepsilon_n}$ and the control $(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*})$ will be exercised as follows:

(i) if the controlled process $X^{\varepsilon_n,*}$ satisfies $|\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})| \leq g_{\ell_i^*}(X_t^{\varepsilon_n,*})$, then $\zeta_t^{\varepsilon_n,*} \equiv 0$ and $X_t^{\varepsilon_n,*}$ will stay in $\mathcal{C}_{\ell_i^*}$;

(ii) if $0 < |\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|^2 - g_{\ell_i^*}(X_t^{\varepsilon_n,*})^2 < 2\varepsilon_n$, the process $X_t^{\varepsilon_n,*}$ will be crossing $\partial\mathcal{C}_{\ell_i^*}$ persistently;

Taking ε_n small enough and considering that the process $X_t^{\varepsilon_n,*}$ is on the regime ℓ_i^* at time $t \in [\tau_i^*, \tilde{\tau}_{i+1}^*)$, we have that $X_t^{\varepsilon_n,*} \in \mathcal{C}_{\ell_i^*}^{\varepsilon_n}$ and the control $(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*})$ will be exercised as follows:

(i) if the controlled process $X^{\varepsilon_n,*}$ satisfies $|\operatorname{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})| \leq g_{\ell_i^*}(X_t^{\varepsilon_n,*})$, then $\zeta_t^{\varepsilon_n,*} \equiv 0$ and $X_t^{\varepsilon_n,*}$ will stay in $\mathcal{C}_{\ell_i^*}$;

(ii) if $0 < |\operatorname{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|^2 - g_{\ell_i^*}(X_t^{\varepsilon_n,*})^2 < 2\varepsilon_n$, the process $X_t^{\varepsilon_n,*}$ will be crossing $\partial\mathcal{C}_{\ell_i^*}$ persistently;

(iii) otherwise, $\xi_t^{\varepsilon_n,*} = (\mathbb{n}_t^{\varepsilon_n,*}, \zeta_t^{\varepsilon_n,*})$ will exercise a force $\frac{2}{\varepsilon_n}|\operatorname{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|$ and in the direction $-\frac{\operatorname{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})}{|\operatorname{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|}$ at $X_t^{\varepsilon_n,*}$ in such a way that it will be pushed back to $\partial\mathcal{C}_{\ell_i^*}$.

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Taking ε_n small enough and considering that the process $X_t^{\varepsilon_n,*}$ is on the regime ℓ_i^* at time $t \in [\tau_i^*, \tilde{\tau}_{i+1}^*)$, we have that $X_t^{\varepsilon_n,*} \in \mathcal{C}_{\ell_i^*}^{\varepsilon_n}$ and the control $(\xi^{\varepsilon_n,*}, \varsigma^{\varepsilon_n,*})$ will be exercised as follows:

(i) if the controlled process $X^{\varepsilon_n,*}$ satisfies $|\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})| \leq g_{\ell_i^*}(X_t^{\varepsilon_n,*})$, then $\zeta_t^{\varepsilon_n,*} \equiv 0$ and $X_t^{\varepsilon_n,*}$ will stay in $\mathcal{C}_{\ell_i^*}$;

(ii) if $0 < |\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|^2 - g_{\ell_i^*}(X_t^{\varepsilon_n,*})^2 < 2\varepsilon_n$, the process $X_t^{\varepsilon_n,*}$ will be crossing $\partial\mathcal{C}_{\ell_i^*}$ persistently;

(iii) otherwise, $\xi_t^{\varepsilon_n,*} = (\mathbb{n}_t^{\varepsilon_n,*}, \zeta_t^{\varepsilon_n,*})$ will exercise a force $\frac{2}{\varepsilon_n}|\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|$ and in the direction $-\frac{\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})}{|\,\mathrm{D}^1 V_{\ell_i^*}^{\varepsilon_n}(X_t^{\varepsilon_n,*})|}$ at $X_t^{\varepsilon_n,*}$ in such a way that it will be pushed back to $\partial\mathcal{C}_{\ell_i^*}$.

(iv) At time $t = \tau_{i+1}^* < \tau^*$, $X_{\tau_{i+1}^*}^{\varepsilon_n,*}$ will be for the first time in $\mathcal{S}_{\ell_i^*}^{\varepsilon_n}$ and will switch to the regime ℓ_{i+1}^*.

This procedure will be repeated until the time τ^* which represents the first exit time of the process $X^{\varepsilon_n,*}$ from the set $\mathcal{O}$.

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Thank you for your attention