On the spectrum of the hierarchical Schrödinger-type operator I: the case of locally bounded potentials

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October 26, 2020

This lecture is based on the project "Spectrum of the hierarchical Schrödinger type operator acting on a Cantor like set" joint with A. A. Grigor'yan (Bielefeld University) and S. A. Molchanov (UNC at Charlotte and HSE, Moscow).

1 Introduction

The concept of the hierarchical Laplacian is going back to N. Bogolubov and his school. This concept has been used by F. J. Dyson in his construction of the phase transition in **1D** ferromagnetic model with long range interaction.

- F. J. Dyson, Existence of a phase-transition in a one-dimensional Ising ferromagnet, Comm. Math. Phys., 12: 91-107, 1969.
- S. A. Molchanov, Hierarchical random matrices and operators, Application to Anderson model, Proc. of 6th Lucacs Symposium (1996), 179-194.

The notion of the hierarchical Laplacian acting on general ultrametric space X was developed to the high level of generality in the papers:

- A. D. Bendikov, A. A. Grigoryan, Ch. Pittet, and W. Woess, Isotropic Markov semigroups on ultrametric spaces, Russian Math. Surveys. 69:4, 589-680 (2014).

- A. D. Bendikov, Heat kernels for isotropic like Markov generators on ultrametric spaces: a Survey, p-Adic Numbers, Ultrametric Analysis and Applications, 2018, Vol. 10, No. 1, pp. 1-11

In the case $X = \mathbb{Q}_p$, the field of p-adic numbers, we would like to mention closely related works of S. Albeverio, W. Karwowski, V. S. Vladimirov, I. V. Volovich, E. I. Zelenov, and A. N. Kochubei.

Let us consider (as a simplest example) the Dyson dyadic model. In this model the hierarchical Laplacian L is realized as a self-adjoint integral operator acting in $L^2(0,\infty)$.

The hierarchical structure It is defined by the family of partitians $\{\Pi_r : r \in \mathbb{Z}\}$ of the set $X = [0,\infty)$. Each partitian Π_r is made of dyadic intervals $I = [(i-1)2^r, i2^r)$. We call r the rank of the partitian Π_r (resp. the rank of the dyadic interval I).

Any point x belongs to exactly one interval $I_r(x)$ of rank r, and the whole set X is union of the increasing family of dyadic intervals $I_r(x)$ as $r \nearrow \infty$.

The hierarchical distance $\mathrm{d}(x,y)$ is defined as the Lebesgues measure $|I|$ of the minimal dyadic interval I which contains both x and y.

One can easily see that for all x, y, z in X,

$$\mathrm{d}(x,y) \leq \max\{\mathrm{d}(x,z), \mathrm{d}(z,y)\},$$

that is, $\mathrm{d}(x,y)$ is an *ultrametric* on X.

- The Euclidean metric $|x-y|$ and the introduced ultrametric $\mathrm{d}(x,y)$ define non-equivalent topologies. Indeed, by the very definition

  $$\mathrm{d}(x,y) \geq |x-y|, \ \forall x, y \in X,$$

  but on the other hand

  $$\mathrm{d}(1-\varepsilon, 1) = 2, \ \forall \varepsilon \in (0,1].$$

- The couple (X, d) is a complete locally compact non-compact and separable metric space. In this metric space the set $\mathcal{B}$ of all open balls coincides with the set of all dyadic intervals.

- Each open ball B in (X, d) is a closed compact set, each point $a \in B$ can be regarded as its center, any two balls either do not intersect or one is a subset of another etc. Thus (X, d) is a proper totally disconnected metric space. In particular, (X, d) is homeomorphic to the Cantor set with punctured point.

- The Borel σ-algebra generated by the ultrametric balls coincides with the classical Borel σ-algebra generated by the Euclidean metric.

The hierarchical Laplacian Let $\mathcal{D}$ be the set of all compactly supported locally constant functions. Let $\varkappa \in]0,1[$ be a fixed parameter.

The hierarchical Laplacian L is introduced as sum of (minus) Markov generators L_r of pure jump processes [1]

$$(Lf)(x) = \sum_{r=-\infty}^{+\infty} \underbrace{(1-\varkappa)\varkappa^r \left(f(x) - \frac{1}{|I_r(x)|} \int\limits_{I_r(x)} f dl \right)}_{(L_r f)(x)}, \ \forall f \in \mathcal{D}.$$

As each *elementary Laplacian* L_r can be written in the form

$$L_r f(x) = \int\limits_0^\infty \left(f(x) - f(y)\right) J_r(x,y)dy,$$

$$J_r(x,y)dy = \underbrace{(1-\kappa)\kappa^{r-1}}_{\lambda_r(x)} \cdot \underbrace{\mathbf{1}_{I_r(x)}(y)/|I_r(x)|dy}_{\mathcal{U}_r(x,dy)}$$

the operator L can be represented as a hypersingular integral operator

$$(Lf)(x) = \int_0^\infty (f(x) - f(y))J(x,y)dy,$$

$$J(x,y) = \frac{\kappa^{-1}-1}{1-\kappa/2} \cdot \frac{1}{\mathrm{d}(x,y)^{1+\alpha}}, \ \alpha = \frac{2}{\log_2 1/\varkappa}.$$

The spectrum of L To each dyadic interval $I = [(i-1)2^r, i2^r)$ we associate the Haar function

$$\mathcal{X}_I(x) = \begin{cases} 2^{-r/2} & \text{if} \quad x \in [(i-1)2^r, (i-1/2)2^r) \\ -2^{-r/2} & \text{if} \quad x \in [(i-1/2)2^r, i2^r) \\ 0 & \text{if} \quad x \notin I \end{cases}.$$

[1] A Markov process is called a pure jump process if, starting from any point x, it has all sample paths constant except for isolated jumps, and right-continuous.

The basic data which defines the process are (i) a function $0 < \lambda(x) < \infty$, and (ii) a Markov kernel $\mathcal{U}(x,dy)$ satisfying $\mathcal{U}(x,\{x\}) = 0$. Intuitively a particle starting from x remains there for an exponentialy distributed time with parameter $\lambda(x)$ at which time it "jumps" to a new position x' according to distribution $\mathcal{U}(x,\cdot)$ etc.

The Haar function $\mathcal{X}_I$ is an eigenfunction of the operator L subject to the eigenvalue $\varkappa^r$,

$$L\mathcal{X}_I = \varkappa^r \mathcal{X}_I.$$

It is easy to see that each eigenvalue $\varkappa^r$ has infinite multiplicity.

The set $\{\mathcal{X}_I : I \in \mathcal{B}\}$ is a complete orthonormal basis in $L^2(0,\infty)$. In particular, L is essentially self-adjoint operator having a pure point spectrum

$$Spec(L) = \{\varkappa^r : r \in \mathbb{Z}\} \cup \{0\}.$$

The heat kernel of L The operator L generates a symmetric Markov semigroup $(e^{-tL})_{t>0}$. The semigroup $(e^{-tL})_{t>0}$ admits a continuous heat kernel $p(t,x,y)$ (the fundamental solution of the equation $(\partial_t - L)u = v$) which can be estimated as follows

$$p(t,x,y) \asymp \frac{t}{[t^{1/\alpha} + \mathrm{d}(x,y)]^{1+\alpha}}, \quad \alpha = \frac{2}{\log_2 1/\varkappa}.$$ [2]

The function $p(t,x,x)$ does not depend on x. Setting $\mathfrak{p}(t) := p(t,x,x)$ we get via spectral resolution formula

$$\mathfrak{p}(t) = t^{-1/\alpha}\mathcal{A}(\log_2 t),$$

where $\mathcal{A}(\tau)$ is a continuous non-constant α-periodic function. In particular, in contrary to the classical case (symmetric stable densities), the function $t \to \mathfrak{p}(t)$ does not vary regularly.

The Taibleson-Vladimirov multiplier It is remarkable that the hierarchical Laplacian L introduced above can be identified with the Taibleson-Vladimirov multiplier $\mathfrak{D}^\alpha, \alpha > 0$, acting in $L^2(\mathbb{Q}_2)$, where $\mathbb{Q}_2$ is the field of 2-adic numbers,

$$\widehat{\mathfrak{D}^\alpha f}(\zeta) = \|\zeta\|_2^\alpha \widehat{f}(\zeta).$$

[2] The kernel $p(t,x,y)$ is a continuous (and even locally Lipschitz continuous) function in the introduced d-topology but it is discontinuous function in the Euclidean topology. This fact follows from the representation

$$p(t,x,y) = t\int_0^{1/d_*(x,y)} e^{-t\tau} N(\tau)d\tau,$$

where $d_*(x,y)$ is the ultrametric which defines topology equivalent to d-topology and which is intrinsically related to L, and $N(\tau)$ is the so-called spectral distribution function related to the Laplacian L, see [2]

In particular, $-\mathfrak{D}^\alpha$ is a symmetric α-stable Lévy generator acting on the Abelian group $\mathbb{Q}_2$ whose heat kernel $p_\alpha(t,x,y)$ can be estimated as

$$p_\alpha(t,x,y) \asymp \frac{t}{[t^{1/\alpha}+\|x-y\|_2]^{1+\alpha}}.$$

2 The Schrödinger type operator

Let V be a locally bounded function and $V : u \to V \cdot u$ a multiplicator. The operator $H = L + V$ with domain $\mathcal{D}$ is a densly defined symmetric operator acting in $L^2(0,\infty)$.

Theorem 2.1 *The following properties hold true:*

1. The operator H is essentially self-adjoint.

2. If $V(x) \to +\infty$ as $x \to \infty$, then the self-adjoint operator H has a compact resolvent. (Thus, its spectrum is discrete).

3. If $V(x) \to 0$ as $x \to \infty$, then the essential spectrum of H coincides with the spectrum of L. (Thus, the spectrum of H is pure point and the negative part of the spectrum consists of isolated eigenvalues of finite multiplicity).

Remark 2.2 *In the case of Schrödinger operator $H = -\Delta + V$ in $\mathbb{R}^D$ the statement about essential self-adjointness of H does not hold in such a great generality. Indeed, in the case of Schrödinger operator*

$$H\psi = -\psi'' + V \cdot \psi, \;\; \psi \in C_c^\infty(0,\infty),$$

with $V(x) = -x^\gamma$, $\gamma > 2$, there is continuum *of self-adjoint extensions of H.*

Furthermore, due to S. Kotani the spectrum of the operator H may contain non-trivial absolutely continuous and singular continuous parts.[3]

3 Splitting lemma

In some cases spectral properties of the operator $H = L + V$ can be reduced to the spectral properties of certain operator $\mathcal{H} = \mathcal{L} + \mathcal{V}$ defined on a discrete ultrametric space $\mathcal{X}$, say $\mathcal{X} = \{0,1,2,...\}$.

[3]Whether this result holds true in the setting of the hierarchical Laplacian (e.g. Taibleson-Vladimirov multiplier) is an interesting open at present writing question.

The association $L \leftrightarrow \mathcal{L}$ Consider the family of dyadic partitians $\{\Pi_r\}$ of the set $\mathcal{X}$:

$$\begin{aligned} &\Pi_0 = \{0, 1, 2, ..., n, ...\} \text{ - single points} \\ &\Pi_1 = \{(0, 1), (2, 3), (4, 5,), (6, 7), ...\} \\ &\Pi_2 = \{(0, 1, 2, 3), (4, 5, 6, 7), ...\} \\ &\quad \text{..............................} \end{aligned}$$

and define in the evident way the hierarchical structure (the ultrametric, the set of ultrametric balls etc)

Definition 3.1 *Set* $\mathcal{I}_r = \{(i-1)2^r, ..., i2^r - 1\}$ *and let* $\mathcal{I}_r(x)$ *be the unique ultrametric ball* $\mathcal{I}_r$ *which contains* x*. The hierarchical Laplacian* $\mathcal{L}$ *associated with* $\mathcal{X}$ *we define pointwise as follows*

$$(\mathcal{L}f)(x) = \sum_{r=1}^{\infty} (1-\varkappa)\varkappa^r \left(f(x) - \frac{1}{2^r} \sum_{y \in \mathcal{I}_r(x)} f(y) \right).$$

The operator $\mathcal{L}$ is a *bounded symmetric operator* having eigenvalues $\varkappa^r$, $r = 1, 2, ...$ The corresponding eigenfunctions are discrete versions of the Haar functions $\mathcal{X}_I$ as defined in the continuous case setting.

The association $V \leftrightarrow \mathcal{V}$ Consider the potential $V = \sum_{i=0}^{\infty} \sigma_i \mathbf{1}_{[i,i+1)}$ and define its discrete version $\mathcal{V} = \sum_{i=0}^{\infty} \sigma_i \delta_i$. Clearly the operator $\mathcal{V} : f \to \mathcal{V} \cdot f$ can be written in the form

$$\mathcal{V}f = \sum_{i=0}^{\infty} \sigma_i (f, \delta_i) \delta_i.$$

The association $H \leftrightarrow \mathcal{H}$ Along with the operator $Hf = Lf + Vf$ let us consider its discrete counterpart the operator

$$\mathcal{H}f := \mathcal{L}f + \mathcal{V}f.$$

To describe the association $H \leftrightarrow \mathcal{H}$ we define two subspaces of $L^2(0, \infty)$:

- $L^2_- = span\{\mathcal{X}_{I_r} : r \leq 0\}$, and
- $L^2_+ = span\{\mathbf{1}_{I_r} : r \geq 0\}$.

Lemma 3.2 *(Splitting Lemma) In the notation from above*

1. $L^2(0,\infty) = L^2_- \oplus L^2_+$.

2. The spaces L^2_- *and* L^2_+ *reduce the operator* H.

3. If $I \subset [i, i+1)$ *has rank* r *then*

$$H\mathcal{X}_I = (\varkappa^r + \sigma_i)\mathcal{X}_I.$$

4. The operator H *restricted to* L^2_+ *can be identified with the operator* $\mathcal{H}$.

4 Rank one perturbations

Let us assume that the homogeneous ultrametric measure space (X, d, m) is *countably infinite*. In this case X can be identified with a countable Abelian group G (e.g. weak sum of cyclic groups) equipped with the sequence $\{G_n\}_{n\in\mathbb{N}}$ of its small subgroups. Each ball in G is a set of the form $g + G_n$ for some g and n.

Let L be a homogeneous hierarchical Laplacian and

$$Hf(x) = Lf(x) - \sigma(f, \delta_a)\delta_a(x),$$

a rank one perturbation of the operator L.

Let us recall the Simon-Wolff theorem about pure point spectrum of rank one perturbation of operators having simple spectrum.

Definition 4.1 *One says that a self-adjoint operator* A *acting on a Hilbert space* $\mathcal{H}$ *has a simple spectrum if there exists a vector* φ *(a cyclic vector) such that* $\{(A-\lambda)^{-1}\varphi \mid \operatorname{Im}\lambda > 0\}$ *is a total set for* $\mathcal{H}$.

The Simon-Wolff criterion Let A be a self-adjoint operator with simple spectrum on a Hilbert space $\mathcal{H}$, and let φ be a cyclic vector. By the spectral theorem, $\mathcal{H}$ is unitary equivalent to $L^2(\mathbb{R}, \mu)$ in such a way that A is the multiplication by x with cyclic vector $\varphi \equiv 1$. Let $H_\sigma = A - \sigma(\varphi, \cdot)\varphi$ be a rank one perturbation of the operator A. Set

$$F(x) := \int (x-y)^{-2} d\mu(y) = \lim_{\epsilon \to 0} \left\| (A - (x + i\epsilon)\mathrm{I})^{-1}\varphi \right\|^2.$$

Theorem 4.2 *Fix an interval* (a, b). *The following properties are equivalent:*
(i) H_σ *has only pure point spectrum in* (a, b) $\sigma - a.e.$.
(ii) $F(x) < \infty$ $x - a.e$ *in* (a, b).

In general, $\mathcal{H}_0 := \{(A - \lambda \mathrm{I})^{-1}\varphi \mid \mathrm{Im}\,\lambda > 0\}$ is the closed subspace, and its orthogonal complement $(\mathcal{H}_0)^{\perp}$ is an invariant space for H and $H = A$ on $(\mathcal{H}_0)^{\perp}$. Thus, the extension from the cyclic to general case is clear.

The function $\varphi = \delta_a$ is not a cyclic vector for L because the operator L has many compactly supported outside of a eigenfunctions.

To show that the spectrum of the operator $H_\sigma = L - \sigma\delta_a$ is pure point for all σ we use the Krein type identity

$$\mathcal{R}_V(\lambda, x, y) = \mathcal{R}(\lambda, x, y) + \frac{\sigma\mathcal{R}(\lambda, x, a)\mathcal{R}(\lambda, a, y)}{1 - \sigma\mathcal{R}(\lambda, a, a)}$$

where $\mathcal{R}(\lambda, x, y) = (L - \lambda I)^{-1}\delta_y(x)$ and $\mathcal{R}_V(\lambda, x, y) = (H_\sigma - \lambda I)^{-1}\delta_y(x)$ are the resolvent kernels.

Theorem 4.3 *$Spec(H_\sigma)$ is pure point for all σ, it consists of at most one negative eigenvalue and countably many positive eigenvalues.*

If $\sigma > 0$, then H_σ has precisely one negative eigenvalue

$$\lambda_-^\sigma < 0 < \dots < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < \dots < \lambda_2 < \lambda_1^\sigma < \lambda_1.$$

if and only if one of the following two conditions holds

- *the Markov semigroup $(e^{-tL})_{t>0}$ is recurrent, i.e. $\mathcal{R}(0, a, a) = \infty$,*
- *the Markov semigroup $(e^{-tL})_{t>0}$ is transient, i.e. $\mathcal{R}(0, a, a) < \infty$, and*

$$\mathcal{R}(0, a, a) > 1/\sigma.$$

If $\sigma < 0$, then all eigenvalues of H_σ are positive

$$0 < \dots < \lambda_{k+1} < \lambda_k^\sigma < \lambda_k < \dots < \lambda_2 < \lambda_1^\sigma < \lambda_1 < \lambda_+^\sigma.$$

The eigenvalues λ_k are eigenvalues of the operator L. All λ_k have infinite multiplicity and compactly supported eigenfunctions, the eigenfunctions of the operator L whose supports do not contain a.

The eigenvalue λ_k^σ (resp. λ_-^σ, λ_+^σ) is the unique solution of the equation

$$\mathcal{R}(\lambda, a, a) = 1/\sigma$$

in the interval $]\lambda_{k+1}, \lambda_k[$ (resp. $]-\infty, 0[$, $]\lambda_1, +\infty[$). Each λ_k^σ (resp. λ_-^σ, λ_+^σ) has multiplicity one and non-compactly supported eigenfunction

$$\psi_k(x) = \mathcal{R}(\lambda_k^\sigma, x, a) \ (resp. \psi_-(x) = \mathcal{R}(\lambda_-^\sigma, x, a), \psi_+(x) = \mathcal{R}(\lambda_+^\sigma, x, a)).$$

5 Sparse potentials

We assume that the ultrametric measure space (X, d, m) is *countably infinite.* Analysis of the finite dimensional perturbations $V = \sum_{i=1}^{n} \sigma_i \delta_{a_i}$ indicates that in the case of increasing distances between locations $\{a_i\}$ of the *bumps* $V_i = -\sigma_i \delta_{a_i}$ their contributions to the spectrum is close to the union of the contributions of the individual bumps V_i (each bump contributes one eigenvalue in each gap $(\varkappa^{m+1}, \varkappa^m)$ of the spectrum of the operator L). The development of this idea leads to consideration of the class of *sparse potentials*

$$V = \sum_{i=1}^{\infty} \sigma_i \delta_{a_i}$$

where distances between locations $\{a_i : i = 1, 2, ...\}$ form an increasing to infinity sequence. In the classical spectral theory this idea goes back to D. B. Pearson, S. Molchanov, and to A. Kiselev, J. Last, S. and B. Simon.

Notation. Let us set

- $\mathcal{I}_*$ is the set of limit points of the sequence $\{\sigma_i\}$.
- $1/\mathcal{I}_* := \{1/\sigma_* : \sigma_* \in \mathcal{I}_*\}$.
- $\mathcal{R}^{-1}(1/\mathcal{I}_*) := \{\lambda : \mathcal{R}(\lambda, a, a) \in 1/\mathcal{I}_*\}$.

Theorem 5.1 *Assume that* $\alpha < \sigma_i < \beta$ *for some* $\alpha, \beta > 0$ *and that*

$$\lim_{n\to\infty} \sup_{i \geq n} \sum_{j \geq n:\ j \neq i} \frac{1}{\mathrm{d}(a_i, a_j)} = 0, \tag{5.1}$$

then

$$Spec_{ess}(H) = Spec(L) \cup \mathcal{R}^{-1}(1/\mathcal{I}_*). \tag{5.2}$$

6 Spectral localization

Theorem 5.1 does not contain any information about $Spec_{ac}(H)$ and $Spec_{sc}(H)$, absolutely continuous and singular continuous parts of $Spec(H)$. Our next theorem shows that under more restrictive assumption $Spec_{ac}(H) = \varnothing$ and $Spec_{sc}(H) = \varnothing$, that is, $Spec(H)$ is pure point. Moreover, the eigenfunctions of H decay at infinity exponentially in certain ultrametric - this is the so-called *spectral localization property.*

Consider a Schrödinger type operator with random potential

$$H^{\omega} = L + V^{\omega},\ \omega \in (\Omega, \mathcal{F}, P).$$

Here L, the deterministic part of H^{ω}, is the hierarchical Laplacian and

$$V^{\omega} = \sum_{a \in \mathcal{I}} \sigma(a, \omega) 1_{B(a)}$$

is a random potential defined by a family of open balls $\{B(a) : a \in \mathcal{I}\}$ and a family $\{\sigma(a, \omega) : a \in \mathcal{I}\}$ of i.i.d. random variables. As the set of all open balls is countably infinite the set $\mathcal{I}$ of locations is at most countable.

Henceforth we assume that all $B(a)$, $a \in \mathcal{I}$, belong to the same horocycle (have the same diameter). Splitting Lemma reduces then the study of the set $Spec_{ess}(H^{\omega})$ to the case where the ultrametric measure space (X, d, m) is *countably infinite* and the potential V^{ω} is of the form

$$V^{\omega} = \sum_{a \in \mathcal{I}} \sigma(a, \omega) \delta_a.$$

When $\mathcal{I} = X$ the operator

$$H^{\omega} = L + \sum_{a \in X} \sigma(a, \omega) \delta_a$$

has a pure point spectrum for $P-$a.s. ω provided the distribution function $\mathcal{F}_{\sigma}(\tau)$ satisfies certain regularity conditions. This statement (*the spectral localization theorem*) appeares first in the paper of Molchanov ($\mathcal{F}_{\sigma}(\tau)$ is the Cauchy distribution) and later in a more general form in two papers of Kritchevski. The proof essentially uses *self-similarity of* H^{ω}.

Denote by $\{a_i\}$ the set of locations, set $\sigma_i(\omega) := \sigma(a_i, \omega)$, and assume that the distribution function $\mathcal{F}_{\sigma}(x)$ is absolutely continuous and has a bounded density supported by a finite interval $\mathcal{I}$. Assume further that $V^{\omega} = \sum_i \sigma_i(\omega) \delta_{a_i}$ is a sparse potential, that is,

$$\lim_{n \to \infty} \sup_{i \geq n} \sum_{j \geq n:\ j \neq i} \frac{1}{\mathrm{d}(a_i, a_j)} = 0.$$

The Spectral localization theorem below complements Theorem 5.1 about structure of the set $Spec(H^{\omega})$. The proof of this theorem is based on the abstract form of Simon-Wolff theorem for pure point spectrum, the technique of fractional moments, the decoupling lemma of Molchanov and Borel-Cantelly type arguments.

Theorem 6.1 *$Spec(H^\omega)$ is pure point a.s. ω (i.e. $Spec_{ac}(H)$ and $Spec_{sc}(H)$ are empty sets a.s. ω) provided*

$$\lim_{n\to\infty} \sup_{i\geq n} \sum_{j\geq n:\ j\neq i} \frac{1}{\mathrm{d}(a_i, a_j)^r} = 0 \tag{6.1}$$

for some small enough r (say, $0 < r < 1/3$). Furthermore, we have

$$Spec_{ess}(H^\omega) = Spec(L) \cup \mathcal{I}_1 \cup \mathcal{I}_2 \cup \dots$$

where $\mathcal{I}_k$ are intervals $\{\lambda \in (\lambda_{k+1}, \lambda_k) : \mathcal{R}(\lambda, a, a) \in 1/\mathcal{I}\}$.

References

[1] A. Bendikov, A. Grigor'yan, S. Molchanov, On the spectrum of the hierarchical Schrödinger type operators, arXiv: 2006.02263v1 02.06.2020.

[2] A. Bendikov, Ch. Pittet, R. Sauer, Spectral distribution and L^2-isoperimetric profile of Laplace operators on groups, Math. Ann. (2012) 354:43-72

Thank you for your attention!