Locally integrable increasing processes with continuous compensators

Dmitriy Borzykh

NRU HSE

October 23, 2020

Introducion

This presentation is based on a published work

(Borzykh, 2018) Borzykh D. On a property of joint terminal distributions of locally integrable increasing processes and their compensators // Theory of Stochastic Processes. 2018. Vol. 23. No. 39 (2). P. 7-20).

I would like to thank my supervisor Prof. A. A. Gushchin for setting the problem and useful advices.

In this presentation we will give a sketch of the proof of the following statement.

A joint distribution of a locally integrable increasing process X° and its compensator A° at a terminal moment of time can be realized as a joint terminal distribution of another locally integrable increasing process $X^\star$ and its compensator $A^\star$, $A^\star$ being continuous.

Our work is essentially based on A. A. Gushchin's article:

(Gushchin, 2018) A. A. Gushchin, The Joint Law of Terminal Values of a Nonnegative Submartingale and Its Compensator, Theory of Probability and Its Applications 62 (2018), no. 2, 216–235.

In (Gushchin, 2018) a class $\mathbb{W}$ of probability measures on the space $(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2))$ is introduced.
It includes all measures μ satisfying the following conditions:

1) $\int_{\mathbb{R}_+^2} (x + y)\, \mu(dx, dy) < \infty$,
2) $\int_{\mathbb{R}_+^2} x\, \mu(dx, dy) = \int_{\mathbb{R}_+^2} y\, \mu(dx, dy)$,
3) $\forall \lambda \geq 0: \ \int_{\{y \leq \lambda\}} x\, \mu(dx, dy) \leq \int_{\mathbb{R}_+^2} (y \wedge \lambda)\, \mu(dx, dy)$.

It is shown in (Gushchin, 2018) that the joint distribution of terminal values of an integrable increasing process and its compensator belongs to the class $\mathbb{W}$.

Conversely, given $\mu \in \mathbb{W}$ there is constructed an increasing integrable process such that the joint distribution of terminal values of the process and its compensator is μ and, moreover, the compensator is continuous.

Thus, if $X^\circ = (X_t^\circ)_{t\in[0;\,\infty)}$ is an integrable increasing process with a compensator $A^\circ = (A_t^\circ)_{t\in[0;\,\infty)}$, one can define on a certain stochastic basis another integrable increasing process $X^\star = (X_t^\star)_{t\in[0;\,\infty)}$ with a compensator $A^\star = (A_t^\star)_{t\in[0;\,\infty)}$, such that

$$\mathrm{Law}\,(X_\infty^\star,\, A_\infty^\star) = \mathrm{Law}\,(X_\infty^\circ,\, A_\infty^\circ)\,. \tag{1}$$

Moreover, the compensator $A^\star$ is continuous.

The main goal of the article is to extend the last statement to the locally integrable case. Namely, we state the following theorem.

Theorem (Main Theorem: Borzykh, 2018, Theorem 1.1)

For any locally integrable increasing process $X^\circ = (X^\circ_t)_{t\in[0;\,\infty)}$ with a compensator $A^\circ = (A^\circ_t)_{t\in[0;\,\infty)}$ on some stochastic basis there exists another locally integrable increasing process $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ with a compensator $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$, such that relation (1) holds, as well as $A^\star$ is continuous.

For more clear exposition we need to introduce you to the main constructions from (Gushchin, 2018).

Theorem (Gushchin, 2018, Proposition 3.6)

Assume that a probability measure $\nu = \nu(dy, dx)$ *on* $\left(\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2)\right)$ *satisfies the inequalities*

$$\int_{\mathbb{R}_+^2} (x-y)^+ \, \nu(dy, dx) \geq \int_{\mathbb{R}_+^2} (y-x)^+ \, \nu(dy, dx) \tag{2}$$

and

$$\forall \lambda \geq 0 \colon \quad \int_{\{y \leq \lambda\}} x \, \nu(dy, dx) \leq \int_{\mathbb{R}_+^2} (y \wedge \lambda) \, \nu(dy, dx). \tag{3}$$

Theorem (Gushchin, 2018, Proposition 3.6)

Let us define one special probability space

- $\Omega^{\flat} := [0;\ \infty] \times [0;\ \infty] \times [0;\ 1]$,
- $\mathcal{F}^{\flat} = \mathcal{B}(\Omega)$,
- $\mathbb{P}^{\flat} := \bar{\nu} \otimes \Lambda$, *where* $\bar{\nu}(B) := \nu(B \cap \mathbb{R}^2_+)$ *and* Λ *is the Lebesgue measure on* $[0;\ 1]$,

and two random variables $\xi(\omega) := x$, $\eta(\omega) := y$, $\omega = (y,\ x,\ u) \in \Omega$.

Theorem (Gushchin, 2018, Proposition 3.6)

Then $\mathrm{Law}(\eta, \xi) = \nu$ *and there exists a random variable* ζ *such that* $0 \le \zeta \le \xi \wedge \eta$, *and*

$$\forall \lambda \geq 0: \quad \int_{\{\eta - \zeta \leq \lambda\}} (\xi - \eta + \lambda) \, d\mathbb{P}^{\mathfrak{b}} = \lambda. \tag{4}$$

Lemma (Borzykh, 2018, Lemma 2.1)

Consider an arbitrary measurable space $(\Omega^{\mathfrak{a}}, \mathcal{F}^{\mathfrak{a}})$ *and a Markov kernel* Q *acting from* $(\Omega^{\mathfrak{a}}, \mathcal{F}^{\mathfrak{a}})$ *to* $(\mathbb{R}^2_+, \mathcal{B}(\mathbb{R}^2_+))$*, and satisfying* $\mathrm{Q}(\omega^{\mathfrak{a}}; \cdot\,) \in \mathbb{W}$ *for all* $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$*. We put*

$$(\Omega, \mathcal{F}) := (\Omega^{\mathfrak{a}} \times \Omega^{\mathfrak{b}}, \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}). \tag{5}$$

Further, on the set

$$\Omega = \Big\{(\omega^{\mathfrak{a}}, \underbrace{y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}}_{=\omega^{\mathfrak{b}}})\colon\ \omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}},\ \ \omega^{\mathfrak{b}} = (y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}) \in \Omega^{\mathfrak{b}}\Big\},$$

we define functions

$$\xi(\omega^{\mathfrak{a}}, \underbrace{y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}}_{=\omega^{\mathfrak{b}}}) = x^{\mathfrak{b}} \quad \text{and} \quad \eta(\omega^{\mathfrak{a}}, \underbrace{y^{\mathfrak{b}}, x^{\mathfrak{b}}, u^{\mathfrak{b}}}_{=\omega^{\mathfrak{b}}}) = y^{\mathfrak{b}},$$

which are $\mathcal{F}$*-measurable as marginal projections.*

Lemma (Borzykh, 2018, Lemma 2.1)

Let us define a Markov kernel $\mathbb{P}^{\mathfrak{a},\mathfrak{b}}$ from $\Omega^{\mathfrak{a}}$ to $(\Omega^{\mathfrak{b}}, \mathcal{F}^{\mathfrak{b}})$ by

$$\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\big(\omega^{\mathfrak{a}};\ B_1 \times B_2 \times B_3\big) := \mathrm{Q}\big(\omega^{\mathfrak{a}};\ (B_2 \times B_1) \cap \mathbb{R}^2_+\big) \cdot \Lambda(B_3), \qquad (6)$$

where $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, $B_1 \times B_2 \times B_3 \in \mathcal{F}^{\mathfrak{b}}$, and Λ is the standard Lebesgue measure.

Then ξ and η satisfy the following property:

$$\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\omega^{\mathfrak{a}};\ \left\{\omega^{\mathfrak{b}}\colon \begin{bmatrix} \xi(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \\ \eta(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \end{bmatrix} \in C\right\}\right) = \mathrm{Q}(\omega^{\mathfrak{a}};\ C), \quad C \in \mathcal{B}(\mathbb{R}^2_+). \qquad (7)$$

Lemma (Borzykh, 2018, Lemma 2.1)

In addition, we can find an $\mathcal{F}$-measurable function $\zeta\colon \Omega \to [0;\infty]$, $\zeta = \zeta(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}})$, $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, $\omega^{\mathfrak{b}} \in \Omega^{\mathfrak{b}}$, which meets, for any $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$, the following two requirements:

$$\forall\, \omega^{\mathfrak{b}} \in \Omega^{\mathfrak{b}}\colon\ 0 \le \zeta(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \le \xi(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) \wedge \eta(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}), \tag{8}$$

and, for all $\lambda \ge 0$,

$$\int_{\{\omega^{\mathfrak{b}}\,:\ \eta(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) - \zeta(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \le \lambda\}} \big(\xi(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) - \eta(\omega^{\mathfrak{a}}, \omega^{\mathfrak{b}}) + \lambda\big)\, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}) = \lambda. \tag{9}$$

Proof.

For any fixed point $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ we define $\mathcal{F}^{\mathfrak{b}}$-measurable function $\zeta(\omega^{\mathfrak{a}}, \cdot\,)$ as in the proof of Proposition 3.6 (Gushchin, 2018), taking measure

$$\nu(B_y \times B_x) = Q(\omega^{\mathfrak{a}};\, B_x \times B_y), \quad B_x \times B_y \in \mathcal{B}(\mathbb{R}^2_+),$$

and probability space

$$\left(\Omega^{\mathfrak{b}}, \mathcal{F}^{\mathfrak{b}}, \mathbb{P}^{\mathfrak{b}}\right) = \left(\Omega^{\mathfrak{b}}, \mathcal{F}^{\mathfrak{b}}, \mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, \cdot\,)\right).$$

Referring again to Proposition 3.6 (Gushchin, 2018), we see that ζ satisfies conditions (8) and (9), as well as functions ξ and η satisfy (7).

Difficult part is to prove that function ζ is measurable not only as a function of variable $\omega^{\mathfrak{b}}$ for fixed $\omega^{\mathfrak{a}}$, but it is also measurable as a function of two variables with respect to the σ-field $\mathcal{F} = \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}}$. We will omit the discussion this question here. For details see (Borzykh, 2018; Lemma 2.1). □

Theorem (Gushchin, 2018, Proposition 3.4)

Let V and W be random variables with values in $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$, respectively, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We also assume that $\{W = \infty\} \subseteq \{V = 0\}$ a.s. and

$$\forall \lambda \geq 0\colon \quad \mathbb{E}\big[V \mathbb{1}_{\{W \leq \lambda\}}\big] = \mathbb{E}[W \wedge \lambda]. \tag{10}$$

For $t \geq 0$ we define

$$\mathcal{G}_t := \Big\{ C \in \mathcal{F}\colon \ C \cap \{W > t\} = \emptyset \ \text{ or } \ C \cap \{W > t\} = \{W > t\} \Big\}.$$

We set

$$X_t := V \mathbb{1}_{\{W \leq t\}}, \qquad A_t := W \wedge t, \qquad t \geq 0.$$

Then $X = (X_t)_{t \geq 0}$ is an $(\mathcal{G}_t)$-adapted locally integrable increasing process, $A = (A_t)_{t \geq 0}$ is its $(\mathcal{G}_t)$-compensator, and $(X_\infty, A_\infty) = (V, W)$ a.s.

Theorem (Gushchin, 2018, Theorem 2.1)

(i) Let X be a nonnegative submartingale of class (D), $X_0 = 0$, with the Doob–Meyer decomposition $X = M + A$ into a sum of a uniformly integrable martingale M and a predictable integrable increasing process A, and let T be a stopping time. Then $\mathrm{Law}(X_T, A_T) \in \mathbb{W}$.

(ii) Let $\mu \in \mathbb{W}$. Then on some stochastic basis there exists an increasing process X with compensator A such that $\mathrm{Law}(X_\infty, A_\infty) = \mu$.

Let us discuss **the proof of statement (ii)** of this theorem.

Let $\mu \in \mathbb{W}$.

Put $\nu(B_y \times B_x) := \mu(B_x \times B_y)$, $B_x \times B_y \in \mathcal{B}(\mathbb{R}_+^2)$.

This measure ν satisfies the requirements of Proposition 3.6 (Gushchin, 2018).

In force of Proposition 3.6 (Gushchin, 2018) there are probability space $\left(\Omega^\mathfrak{b}, \mathcal{F}^\mathfrak{b}, \mathbb{P}^\mathfrak{b}\right)$ and random variables ξ, η, ζ such that $\mathrm{Law}(\eta, \xi) = \nu$, $0 \leq \zeta \leq \xi \wedge \eta$ and

$$\forall \lambda \geq 0\colon \quad \int_{\{\eta - \zeta \leq \lambda\}} (\xi - \eta + \lambda)\, d\mathbb{P}^\mathfrak{b} = \lambda.$$

From definition of ν and condition $\mathrm{Law}(\eta, \xi) = \nu$ it follows, that $\mathrm{Law}(\xi, \eta) = \mu$.

Now, put $V := \xi - \zeta$ and $W := \eta - \zeta$.

It is easy to check that V and W meet the requirements of Proposition 3.4 (Gushchin, 2018).

Thus, we can introduce a filtration

$$\mathcal{G}_t^\flat := \Big\{ C \in \mathcal{F}^\flat : \ C \cap \{W > t\} = \emptyset \ \text{ or } \ C \cap \{W > t\} = \{W > t\} \Big\}$$

$t \geq 0$, on probability space $\big(\Omega^\flat, \mathcal{F}^\flat, \mathbb{P}^\flat\big)$, and the processes

$$V \mathbb{1}_{\{W \leq t\}}, \quad \text{and} \quad W \wedge t, \quad t \geq 0,$$

will be an $(\mathcal{G}_t^\flat)$-adapted locally integrable increasing process, and its $(\mathcal{G}_t^\flat)$-compensator correspondingly.

Let us set

$$\mathcal{F}^{\flat}_t := \begin{cases} \mathcal{G}^{\flat}_{\frac{t}{1-t}}, & \text{if } t < 1, \\ \mathcal{F}^{\flat}, & \text{if } t \geq 1, \end{cases}$$

$$X_t := \begin{cases} V\mathbb{1}_{\{W \leq \frac{t}{1-t}\}}, & \text{if } t < 1, \\ V + ((t \wedge 2) - 1)\zeta, & \text{if } t \geq 1, \end{cases}$$

$$A_t := \begin{cases} \frac{t}{1-t} \wedge W, & \text{if } t < 1, \\ W + ((t \wedge 2) - 1)\zeta, & \text{if } t \geq 1. \end{cases}$$

It can be shown that

- the process X is an integrable increasing process,
- the process $M := X - A$ is an $(\mathcal{F}^{\flat}_t)$-martingale.

As the process A is continuous and $(\mathcal{F}^{\flat}_t)$-adapted, the process A is predictable. Thus, A is a compensator of X.

Lemma (Borzykh, 2018, Lemma 2.2)

Suppose all the conditions of Lemma 2.1 (Borzykh, 2018) are satisfied. Then one can define a filtration $(\mathcal{F}_t)_{t\in[0;\,1]}$ *on the measurable space* $(\Omega,\,\mathcal{F}) = (\Omega^{\mathfrak{a}} \times \Omega^{\mathfrak{b}},\, \mathcal{F}^{\mathfrak{a}} \otimes \mathcal{F}^{\mathfrak{b}})$*, and a pair of increasing processes* $X = (X_t)_{t\in[0;\,1]}$ *and* $A = (A_t)_{t\in[0;\,1]}$*,* $X_0 = 0$*,* $A_0 = 0$*, such that*

(i) *the processes* X *and* A *are adapted, as well as* A *is continuous, and*

$$\mathbb{P}^{\mathfrak{a},\mathfrak{b}}\left(\omega^{\mathfrak{a}};\,\left\{\omega^{\mathfrak{b}}\colon\ \begin{bmatrix} X_1(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \\ A_1(\omega^{\mathfrak{a}},\omega^{\mathfrak{b}}) \end{bmatrix} \in C\right\}\right) = \mathrm{Q}(\omega^{\mathfrak{a}};\, C),\quad C \in \mathcal{B}(\mathbb{R}^2_+), \quad (11)$$

(ii) *the process* $M_t := X_t - A_t$*,* $t \in [0;\,1]$*, satisfies the following condition: for all* $0 \le s \le t \le 1$*,* $\omega^{\mathfrak{a}} \in \Omega^{\mathfrak{a}}$ *and* $B \in \mathcal{F}_s$*,*

$$\int_{\Omega^{\mathfrak{b}}} \left(M_t(\omega^{\mathfrak{a}},\,\omega^{\mathfrak{b}}) - M_s(\omega^{\mathfrak{a}},\,\omega^{\mathfrak{b}})\right) \mathbb{1}_B(\omega^{\mathfrak{a}},\,\omega^{\mathfrak{b}})\,\mathbb{P}^{\mathfrak{a},\mathfrak{b}}(\omega^{\mathfrak{a}};\, d\omega^{\mathfrak{b}}) = 0. \quad (12)$$

Lemma (Borzykh, 2018, Lemma 3.1)

Let a locally integrable increasing process $X^\circ = (X_t^\circ)_{t\in[0;\,\infty)}$ *such that* $\mathbb{E}[X_n^\circ] < \infty$, *for any* $n \in \mathbb{N}$, *be given on a stochastic basis* $(\Omega^\circ, \mathcal{F}^\circ, \mathbb{P}^\circ, (\mathcal{F}_t^\circ)_{t\in[0;\,\infty)})$; $A^\circ = (A_t^\circ)_{t\in[0;\,\infty)}$ *being its compensator. Let also another integrable increasing process* $X^{[n]} = (X_t^{[n]})_{t\in[0;\,n]}$ *on a different stochastic basis* $(\Omega^{[n]}, \mathcal{F}^{[n]}, \mathbb{P}^{[n]}, (\mathcal{F}_t^{[n]})_{t\in[0;\,n]})$, $n \in \mathbb{N}$, *with a compensator* $A^{[n]} = (A_t^{[n]})_{t\in[0;\,n]}$ *be given. Moreover,* $\mathrm{Law}\begin{bmatrix} X_n^{[n]} \\ A_n^{[n]} \end{bmatrix} = \mathrm{Law}\begin{bmatrix} X_n^\circ \\ A_n^\circ \end{bmatrix}$.

Lemma (Borzykh, 2018, Lemma 3.1)

Then one can define a pair of processes $X^{[n+1]} = (X^{[n+1]}_t)_{t\in[0;\, n+1]}$ *and* $A^{[n+1]} = (A^{[n+1]}_t)_{t\in[0;\, n+1]}$ *on a certain extension* $(\Omega^{[n+1]},\ \mathcal{F}^{[n+1]},\ \mathbb{P}^{[n+1]},\ (\mathcal{F}^{[n+1]}_t)_{t\in[0;\, n+1]})$ *of a stochastic basis* $(\Omega^{[n]},\ \mathcal{F}^{[n]},\ \mathbb{P}^{[n]},\ (\mathcal{F}^{[n]}_t)_{t\in[0;\, n]})$*, satisfying the following conditions:*

(i) $X^{[n+1]}$ *is an integrable increasing process, and process* $A^{[n+1]}$ *is its compensator,*

(ii) *the processes* $(X^{[n]}_t)_{t\in[0;\, n]}$ *and* $(X^{[n+1]}_t)_{t\in[0;\, n]}$ *coincide,*

(iii) *the processes* $(A^{[n]}_t)_{t\in[0;\, n]}$ *and* $(A^{[n+1]}_t)_{t\in[0;\, n]}$ *coincide,*

(iv) $\mathrm{Law}\begin{bmatrix} X^{[n+1]}_n \\ A^{[n+1]}_n \end{bmatrix} = \mathrm{Law}\begin{bmatrix} X^{\circ}_n \\ A^{\circ}_n \end{bmatrix}$ *and* $\mathrm{Law}\begin{bmatrix} X^{[n+1]}_{n+1} \\ A^{[n+1]}_{n+1} \end{bmatrix} = \mathrm{Law}\begin{bmatrix} X^{\circ}_{n+1} \\ A^{\circ}_{n+1} \end{bmatrix}$,

(v) *process* $(A^{[n+1]}_t)_{t\in[n;\, n+1]}$ *is continuous.*

Proof of Main Theorem

Let a locally integrable increasing process $X^\circ = (X_t^\circ)_{t\in[0;\infty)}$ and a localizing sequence of finite stopping times $(T_n)_{n=1}^{\infty}$ be given. It can be shown that without loss of generality one can assume that $T_n = n$, $n \in \mathbb{N}$ (for details see (Borzykh, 2018)).

We start with the following recursive procedure.

Step 1. Applying Theorem 2.1 (i) (Gushchin, 2018) to the integrable increasing process $(X_t^\circ)_{t\in[0;1]}$, as well as its compensator $(A_t^\circ)_{t\in[0;1]}$ and a stopping time $T=1$, we get $\mathrm{Law}\,(X_1^\circ, A_1^\circ) \in \mathbb{W}$.

Then by Theorem 2.1 (ii) (Gushchin, 2018), there exists a stochastic basis $\mathbb{B}^{[1]} := \big(\Omega^{[1]}, \mathcal{F}^{[1]}, \mathbb{P}^{[1]}, (\mathcal{F}_t^{[1]})_{t\in[0;1]}\big)$, and an integrable process $(X_t^{[1]})_{t\in[0;1]}$ on it with a continuous compensator $(A_t^{[1]})_{t\in[0;1]}$, such that $\mathrm{Law}\left(X_1^{[1]}, A_1^{[1]}\right) = \mathrm{Law}\,(X_1^\circ, A_1^\circ)$.

All the steps starting from the second are performed similarly.

Step $n+1$, $n \geq 1$. Remark that the pair of processes $(X_t^\circ)_{t\in[0;\,\infty)}$ and $(A_t^\circ)_{t\in[0;\,\infty)}$ and the pair of processes $(X_t^{[n]})_{t\in[0;\,n]}$ and $(A_t^{[n]})_{t\in[0;\,n]}$ fit the requirements of Lemma 3.1 (Borzykh, 2018).

So, applying this lemma, we build a stochastic basis

$$\mathbb{B}^{[n+1]} := \big(\Omega^{[n+1]},\, \mathcal{F}^{[n+1]},\, \mathbb{P}^{[n+1]},\, (\mathcal{F}_t^{[n+1]})_{t\in[0;\,n+1]}\big),$$

and an integrable increasing process $(X_t^{[n+1]})_{t\in[0;\,n+1]}$ with a continuous compensator $(A_t^{[n+1]})_{t\in[0;\,n+1]}$, satisfying the condition

$$\mathrm{Law}\left(X_{n+1}^{[n+1]},\, A_{n+1}^{[n+1]}\right) = \mathrm{Law}\left(X_{n+1}^\circ,\, A_{n+1}^\circ\right).$$

Now, we are ready to define the required stochastic basis

$$\mathbb{B}^\star := \big(\Omega^\star,\, \mathcal{F}^\star,\, \mathbb{P}^\star,\, (\mathcal{F}^\star_t)_{t\in[0;\,\infty)}\big)$$

and a locally integrable increasing process $X^\star = (X^\star_t)_{t\in[0;\,\infty)}$ on it with a continuous compensator $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$. Put:

$$\Omega^\star := \Omega^{[1]} \times (\Omega)^\infty, \quad \mathcal{F}^\star := \mathcal{F}^{[1]} \otimes \bigotimes_{i=2}^{\infty} \mathcal{F},$$

$$\mathcal{F}^\star_t := \begin{cases} \mathcal{F}^{[1]}_t \otimes \{\emptyset,\, \Omega\}^\infty, & t \in [0;\, 1], \\ \mathcal{F}^{[1]}_1 \otimes \mathcal{F}_{t-1} \otimes \{\emptyset,\, \Omega\}^\infty, & t \in (1;\, 2], \\ \mathcal{F}^{[1]}_1 \otimes \left(\bigotimes_{i=2}^{n-1} \mathcal{F}_1\right) \otimes \mathcal{F}_{t-n+1} \otimes \{\emptyset,\, \Omega\}^\infty, & t \in (n-1;\, n], \quad n \geq 3. \end{cases}$$

Next, in view of the Ionescu-Tulcea theorem (see e.g. (Shiryaev, Probability, 2016, vol. 1)) on the measurable space $(\Omega^\star, \mathcal{F}^\star)$ there exists a unique probability measure $\mathbb{P}^\star$, such that

$$\forall n \in \mathbb{N} \quad \forall B^{[n]} \in \mathcal{F}^{[n]} : \quad \mathbb{P}^\star\big(B^{[n]} \times (\Omega)^\infty\big) = \mathbb{P}^{[n]}\big(B^{[n]}\big).$$

Further, let $\omega^\star = \big(\omega^{[1]}, \omega_2, \dots, \omega_n, \dots\big) \in \Omega^\star$. Set

$$X_t^\star(\omega^\star) := \begin{cases} X_t^{[1]}\left(\omega^{[1]}\right), & t \in [0;\, 1], \\ X_t^{[n]}\left(\omega^{[1]}, \omega_2, \dots, \omega_n\right), & t \in (n-1;\, n], \quad n \geq 2, \end{cases}$$

$$A_t^\star(\omega^\star) := \begin{cases} A_t^{[1]}\left(\omega^{[1]}\right), & t \in [0;\, 1], \\ A_t^{[n]}\left(\omega^{[1]}, \omega_2, \dots, \omega_n\right), & t \in (n-1;\, n], \quad n \geq 2, \end{cases}$$

$$M_t^\star(\omega^\star) := X_t^\star(\omega^\star) - A_t^\star(\omega^\star), \quad t \geq 0.$$

It can be shown that $M^\star = (M^\star_t)_{t\in[0;\,\infty)}$ is a martingale on $\left(\Omega^\star,\, \mathcal{F}^\star,\, \mathbb{P}^\star,\, (\mathcal{F}^\star_t)_{t\in[0;\,\infty)}\right)$ (for details see (Borzykh, 2018)).

The process $A^\star = (A^\star_t)_{t\in[0;\,\infty)}$ is a predictable (by continuity) increasing process.

Finally, formula (1) is obtained from the relations

$$\lim_{n\to\infty} (X^\star_n,\, A^\star_n) = (X^\star_\infty,\, A^\star_\infty)\,, \qquad \lim_{n\to\infty} (X^\circ_n,\, A^\circ_n) = (X^\circ_\infty,\, A^\circ_\infty)\,,$$

$$\mathrm{Law}\,(X^\star_n,\, A^\star_n) = \mathrm{Law}\,(X^\circ_n,\, A^\circ_n)\,, \quad n \in \mathbb{N},$$

and the fact that almost sure convergence implies weak convergence. □

Thank you for your attention!