On the simulation of tempered infinitely divisible distributions and associated processes

Michael Grabchak

University of North Carolina at Charlotte
Department of Mathematics and Statistics

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Part I: Simulation of Tempered Lévy Processes

Lévy Process

A stochastic process $\{X_t : t \geq 0\}$ is a Lévy process if $X_0 = 0$ and

- (Independent Increments) For any $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$, the random variables X_{t_0}, $X_{t_1} - X_{t_0}$,...,$X_{t_n} - X_{t_{n-1}}$ are independent.
- (Stationary Increments) The distribution of $X_{s+t} - X_s$ does not depend on s.
- Stochastic Continuity
- Càdlàg Paths

Infinitely Divisible Distributions

The characteristic function of an infinitely divisible distribution μ can be written in the form

$$\begin{aligned}\hat{\mu}(z) &= \mathrm{E}\left[e^{izX}\right] = \exp\left\{-\frac{z^2}{2}A + ibz \right.\\ &\left. + \int_{\mathbb{R}} \left(e^{ixz} - 1 - izxh(x)\right) L(\mathrm{d}x)\right\},\ z \in \mathbb{R}\end{aligned}$$

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where $A \geq 0$, $b \in \mathbb{R}$, L is a Lévy measure satisfying

$$L(\{0\}) = 0 \text{ and } \int_{\mathbb{R}}(x^2 \wedge 1)L(\mathrm{d}x) < \infty.$$

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For fixed h, the parameters A, L, b uniquely determine the distribution and we write

$$\mu = ID(A, L, b)_h.$$

ID Distributions and Lévy Process

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The characteristic function of X_t is $(\hat{\mu}(z))^t$.

We denote the distribution of X_t by μ^t

The Lévy measure governs the jumps of the process, specifically

$$L(B) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in B\right\}\right], \quad B \in \mathfrak{B}(\mathbb{R}).$$

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Tails of the distribution μ are intimately related to the tails of the Lévy measure L.

In particular for $\beta > 0$, if $X \sim \mu = ID(A, L, b)_h$ then

$$\mathbb{E}|X|^\beta < \infty \quad \text{if and only if} \quad \int_{|x|>1} |x|^\beta L(\mathrm{d}x) < \infty.$$

α-Stable Distributions

For $\alpha \in (0, 2)$, α-Stable Distributions are a class of infinitely divisible distributions with Lévy measure

$$M_\alpha(\mathrm{d}x) = c_- x^{-1-\alpha} 1_{x<0} \mathrm{d}x + c_+ x^{-1-\alpha} 1_{x>0} \mathrm{d}x.$$

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These models are used for many applications and often provide a good fit to data.

However, they have an infinite variance, which is not realistic for many applications.

In practice there are often real-world frictions preventing such heavy tails.

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For $\alpha \in (0,2)$, classical tempered α-Stable Distributions are a class of infinitely divisible distributions with Lévy measure

$$M(\mathrm{d}x) = c_- e^{-b_- x} x^{-1-\alpha} 1_{x<0} \mathrm{d}x + c_+ e^{-b_+ x} x^{-1-\alpha} 1_{x>0} \mathrm{d}x.$$

Tweedie (1984), Hougaard (1986), Koponen (1995), Boyarchenko and Levendorskii (2000), Carr, Geman, Madan, and Yor (2002)

General Tempered Stable Distributions

Rosiński (2007) and Rosiński and Sinclair (2010) suggested the more general form

$$M(\mathrm{d}x) = c_- g(x) x^{-1-\alpha} 1_{x<0} \mathrm{d}x + c_+ g(x) x^{-1-\alpha} 1_{x>0} \mathrm{d}x.$$

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We extend this idea to more general infinitely divisible distributions

Tempered Lévy Measures

Let L be a Lévy measure and let g be a Borel function such that

A1. $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}$, and

A2.

$$\int_{\mathbb{R}} (|xh(x)| \vee 1)\,(1 - g(x))\,L(\mathrm{d}x) < \infty.$$

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Now set

$$\tilde{L}(\mathrm{d}x) = g(x)L(\mathrm{d}x).$$

We call $\tilde{L}$ the tempered Lévy measure and we call g the tempering function.

Tempered Infinitely Divisible Distributions

Let $\mu = ID(A, L, b)_h$ and let

$$\tilde{\mu} = ID(A, \tilde{L}, \tilde{b})_h,$$

where

$$\tilde{b} = b - \int_{\mathbb{R}} x h(x) \left(1 - g(x)\right) L(\mathrm{d}x).$$

We call $\tilde{\mu}$ the tempering of μ.

Let $X = \{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim \mu = ID(A, L, b)_h$.

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Note that

$$L(\mathrm{d}x) = g(x)L(\mathrm{d}x) + (1 - g(x))\, L(\mathrm{d}x) = \tilde{L}(\mathrm{d}x) + \rho(\mathrm{d}x)$$

where

$$\rho(\mathrm{d}x) = (1 - g(x))\, L(\mathrm{d}x).$$

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$$\rho(\mathrm{d}x) = (1 - g(x))\, L(\mathrm{d}x).$$

We must understand the process governed by ρ.

Removed Jumps

Let

$$\rho(\mathrm{d}x) = (1 - g(x))\, L(\mathrm{d}x).$$

and

$$\eta := \rho(\mathbb{R}) < \infty.$$

Define the probability measure

$$\rho_1(B) = \frac{\rho(B)}{\eta}, \quad B \in \mathfrak{B}(\mathbb{R}).$$

Let $Z_1, Z_2, \ldots \stackrel{\text{iid}}{\sim} \rho_1$ and let $\{N_t : t \geq 0\}$ be an independent Poisson process with intensity η. Now set

$$V_t = \sum_{i=1}^{N_t} Z_i, \quad t \geq 0.$$

Process of Removed Jumps

The characteristic function of the compound Poisson process

$$V_t = \sum_{i=1}^{N_t} Z_i, \quad t \geq 0.$$

is

$$\mathbb{E}\left[e^{i\langle V_t z\rangle}\right] = \exp\left\{t \int_{\mathbb{R}} \left(e^{ixz} - 1\right) \rho(\mathrm{d}x)\right\}, \quad z \in \mathbb{R}.$$

Let $T = \inf\{t : N_t > 0\}$ and note that $T \sim \exp(\eta)$, i.e.

$$P(T > t) = e^{-t\eta}, \quad t > 0.$$

Main Theorem

Theorem

Let $\mu = ID(A, L, b)_h$, $\tilde{\mu} = ID(A, \tilde{L}, \tilde{b})_h$, $V = \{V_t : t \geq 0\}$, *and* T *be as described above. Let* $\tilde{X} = \{\tilde{X}_t : t \geq 0\}$ *be a Lévy process, independent of* V*, with* $\tilde{X}_1 \sim \tilde{\mu}$ *and set*

$$X_t = \tilde{X}_t + V_t, \quad t \geq 0.$$

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2. *If* $0 \leq t < T$, *then* $X_t = \tilde{X}_t$.
3. *For any* $B \in \mathfrak{B}(\mathbb{R})$ *and* $t \geq 0$ *we have*

$$P(\tilde{X}_t \in B) \leq e^{t\eta} P(X_t \in B).$$

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1. *The process $X = \{X_t : t \geq 0\}$ is a Lévy process with $X_1 \sim \mu$.*
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For relativistic stable distributions, a variant of this was given in Ryznar (2002).

Proof (sketch)

We have

$$X_t = \tilde{X}_t + V_t, \quad t \geq 0.$$

1. Since the sum of independent Lévy processes is a Lévy process, we just need to check that the characteristic function of μ^t is the product of the characteristic function of $\tilde{X}_t$ and V_t.

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2. Since $T = \inf\{t : N_t > 0\}$, $V_t = 0$ for $t < T$. Hence, if $0 \leq t < T$, then $X_t = \tilde{X}_t$.
3. Since $\tilde{X}$ and T are independent, by Part 2, we have

$$\begin{aligned} P(\tilde{X}_t \in B)e^{-t\eta} &= P(\tilde{X}_t \in B)P(T > t) = P(\tilde{X}_t \in B, T > t) \\ &= P(X_t \in B, T > t) \leq P(X_t \in B). \end{aligned}$$

Hence

$$P(\tilde{X}_t \in B) \leq e^{t\eta} P(X_t \in B).$$

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If we know how to simulate from μ^t, then this gives us a way to use rejection sampling to simulate from $\tilde{\mu}^t$.

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Theorem 1 implies that, for Lebesgue almost every x,

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Theorem 1 implies that, for Lebesgue almost every x,

$$\tilde{f}_t(x) \leq e^{t\eta} f_t(x),$$

Thus, we can set up a rejection sampling algorithm to sample from $\tilde{f}_t$ as follows.

Algorithm

Algorithm 1.
Step 1. Independently simulate $U \sim U(0,1)$ and $Y \sim f_t$.
Step 2. If $U \le e^{-\eta t}\tilde{f}_t(Y)/f_t(Y)$ accept, otherwise reject and go back to Step 1.

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The probability of acceptance on a given iteration is $p_t = e^{-\eta t}$.

The expected number of iterations before the first acceptance is $I_t = e^{\eta t}$.

When $t \to 0$ then $p_t \to 1$ and $I_t \to 1$.

Thus the algorithm works well for small t.

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The probability of acceptance on a given iteration is $p_t = e^{-\eta t}$.

The expected number of iterations before the first acceptance is $I_t = e^{\eta t}$.

However, when $t \to \infty$ then $p_t \to 0$ and $I_t \to \infty$.

Thus the algorithm fails for large t.

Medium Size t

When t is “medium” size, we can do the following:

For some $n \in \mathbb{N}$, use Algorithm 1 to sample

$$Y_1, Y_2, \ldots, Y_n \stackrel{\text{iid}}{\sim} \tilde{f}_{t/n}.$$

Then

$$\tilde{X}_t = Y_1 + Y_2 + \cdots + Y_n \sim \tilde{f}_t.$$

In this case, we only expect to need $nI_{t/n} = ne^{\eta t/n}$ iterations.

The optimal choice of n is near ηt.

Medium Size t (Example)

Say we want to simulate one observation when $t = 10$ and $\eta = 1$.

To do this directly, we expect to need $I_{10} = e^{10} \approx 22026$ iterations.

To simulate 10 observations when $t = 1$ and $\eta = 1$ we only expect to need $10I_1 = 10 * e^1 \approx 27$ iterations.

Simulations

Fix $\alpha \in (0,1)$. Let $\mu = ID(0, M_\alpha, 0)_0$ be a symmetric α-stable distribution with

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Fix $\ell > 0$, consider the tempering function

$$g(x) = \alpha\frac{\alpha+\ell+1}{2\Gamma(1-\alpha)}\int_0^\infty e^{-|x|u}(1+1/u)^{-2-\alpha-\ell}u^{-2-\alpha}\mathrm{d}u.$$

and let $\tilde{\mu} = \mathrm{ID}(0, \tilde{M}_\alpha, 0)_0$.

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$$g(x) = \alpha \frac{\alpha+\ell+1}{2\Gamma(1-\alpha)} \int_0^\infty e^{-|x|u}(1+1/u)^{-2-\alpha-\ell}u^{-2-\alpha}\mathrm{d}u.$$

and let $\tilde{\mu} = \mathrm{ID}(0, \tilde{M}_\alpha, 0)_0$.

$\tilde{\mu}$ is a tempered stable distribution for which, up to now, there has been no exact simulation technique beyond the inversion method.

Simulations

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We obtained 1110 observations.

KDE for Simulated Data

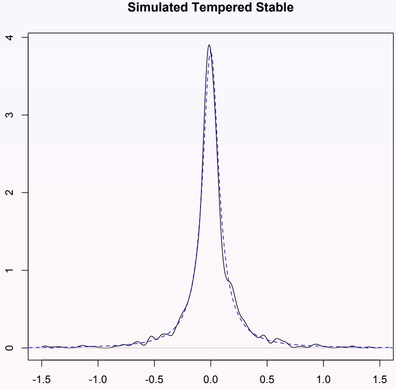


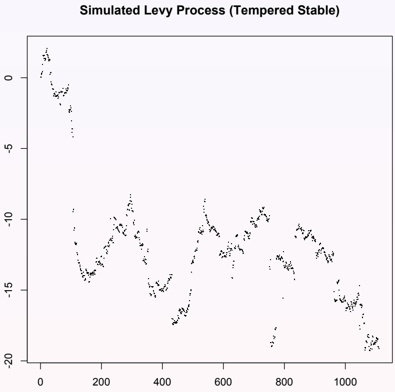
Figure: Tempered Stable KDE

Lévy Process

We convert the 1110 observations $\{Y_1, Y_2, \ldots, Y_{1110}\}$ into a simulated Lévy process by taking

$$X_t = \sum_{i=1}^{t} Y_i, \quad t = 1, 2, \ldots, 1110.$$

Simulated Lévy Processes



Comparison with Inversion Method

	Algorithm 1			Inversion Method		
t	run time	iterations	obs	run time	obs	ratio
1	24.71	3000	1121	285.21	1000	0.09
2	47.40	6000	1105	255.88	1000	0.19
5	118.56	15000	1104	224.23	1000	0.53
10	237.75	30000	1084	193.32	1000	1.23
20	443.29	56000	1024	191.01	1000	2.32

Here Algorithm 1 is performed without the modification.

Comparison with Inversion Method

		Algorithm 1		Inversion Method		
α	ℓ	run time	obs	run time	obs	ratio
.50	0.5	66.841 sec	1057	785.870 sec	1000	0.085
.50	1.0	30.531 sec	1116	338.867 sec	1000	0.090
.50	5.0	47.035 sec	1090	509.506 sec	1000	0.092
.75	0.5	53.346 sec	1124	668.588 sec	1000	0.080
.75	1.0	23.844 sec	1082	282.788 sec	1000	0.084
.75	5.0	37.353 sec	1093	448.820 sec	1000	0.083
.95	0.5	51.121 sec	1101	601.804 sec	1000	0.084
.95	1.0	23.444 sec	1096	278.804 sec	1000	0.084
.95	5.0	38.190 sec	1100	470.784 sec	1000	0.081

Table: Here $t = 1$ and $\eta = 1$. For Algorithm 1, the obs column gives the number of random variables obtained based on a sample for 3000 from the stable distribution.

Part II: Simulation of Tempered Stable OU-Processes

TSOU-Processes

We can use the results of the previous section to simulate other related processes.

In this part we discuss simulation of Tempered Stable Processes of Ornstein-Uhlenbeck-type or TSOU-processes

Such processes are mean reverting, which makes them useful for a variety of application.

In math finance that are often used to model:

- Stochastic volatility
- Stochastic interest rate
- Conversion rates

Generalized TS Distributions

Recall that a general tempered stable distribution has a Lévy measure of the form

$$M(\mathrm{d}x) = c_- g(x) x^{-1-\alpha} 1_{x<0} \mathrm{d}x + c_+ g(x) x^{-1-\alpha} 1_{x>0} \mathrm{d}x.$$

where $\alpha \in (0, 2)$ and g is a tempering function. We denote the corresponding distribution $\mathrm{TS}_\alpha(g, c_-, c_+)$.

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Recall that

A1. $0 \leq g(x) \leq 1$ for all $x \in \mathbb{R}$, and

A2.

$$\int_{\mathbb{R}} (|xh(x)| \vee 1) (1 - g(x)) L(\mathrm{d}x) < \infty.$$

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In addition, we assume that

A3. $g(x)$ is monotonely increasing on $(-\infty, 0)$ and monotonely decreasing on $(0, \infty)$.

Generalized TS Distributions are Selfdecomposable

The additional condition ensure that the distribution is selfdecomposable.

Generalized TS Distributions are Selfdecomposable

The additional condition ensure that the distribution is selfdecomposable.

This implies that

- The distribution has a unimodal pdf
- The distribution is the stationary distribution of some OU-process

OU-Processes

Let $Z = \{Z_t : t \geq 0\}$ be a Lévy process with $Z_1 = \mathrm{ID}(A, M, b)$, let $\lambda > 0$, and define a process Y_t by the stochastic diff eq

$$\mathrm{d}Y_t = -\lambda Y_t \mathrm{d}t + \mathrm{d}Z_t$$

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We call this an OU-process and we call Z the BDLP.

For $A = 0$ and an appropriate choice of M, the stationary distribution of this process is tempered stable

We call it a TSOU-Process

Transition Laws of TSOU-Processes

Theorem

Let $\{Y_t : t \geq 0\}$ be a TSOU-process with parameter $\lambda > 0$ and limiting distribution $\mathrm{TS}_\alpha(g, c_-, c_+)$. Under mild conditions on g, for $t > 0$ if we are given $Y_s = y$, then

$$Y_{s+t} \stackrel{d}{=} e^{-\lambda t} y + d_{\alpha,\lambda,t} + X_0 + \sum_{j=1}^{N} X_j.$$

where $N, X_0, X_1, X_2, X_3, \ldots$ are indep. random variables with:

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2. N has a Poisson distribution with mean $Ke^{-\alpha\lambda t}$.

3. $X_1, X_2, X_3, \ldots$ are iid random variables with pdf

$$h(u) = \frac{1}{K}\left(g(u) - g(ue^{\lambda t})\right) u^{-1-\alpha}$$

Modified Log-Laplace Distribution

A modified log-Laplace distribution has pdf

$$\begin{aligned} g(x;\alpha,p,\delta) &= \frac{\alpha\delta^p}{\alpha\delta^p+p-\alpha}\delta^{-(p-\alpha)}(p-\alpha)x^{p-\alpha-1}1_{[0<x\leq\delta]} \\ &\quad +\frac{p-\alpha}{\alpha\delta^p+p-\alpha}\alpha\delta^{\alpha}x^{-1-\alpha}1_{[x>\delta]}. \end{aligned}$$

We denote this by $\mathrm{MLL}(\alpha,p,\delta)$.

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We denote this by $\mathrm{MLL}(\alpha,p,\delta)$.

If $U_1, U_2 \overset{\mathrm{iid}}{\sim} U(0,1)$, then

$$Y = U_1^{1/(p-\alpha)}\delta 1_{[U_2<q]} + U_1^{-1/\alpha}\delta 1_{[U_2\ge q]} \sim \mathrm{MLL}(\alpha,p,\delta).$$

Class F

Without loss of generality, we assume that $g(u) = 0$ for $u < 0$ and the general case is a mixture of such cases.

We assume that in a neighborhood of 0, g satisfied a Lipschitz-type condition. Specifically that for some $M, \epsilon, p > 0$, if $u, v \in (0, \epsilon)$

$$|g(u) - g(v)| \leq M \left| u^p - v^p \right|.$$

In this case, we say that g belongs to Class F and write $g \in \mathrm{CF}_\alpha(\epsilon, M, p)$.

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The TS distribution studied in Rosiński (2007) or G. (2012, 2016) belong to Class F.

Simulation From h

We want to simulate from

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If $g \in \mathrm{CF}_\alpha(\epsilon, M, p)$ and f is the pdf of $\mathrm{MLL}(\alpha, p, \delta_0)$, then

$$h(u) \le V f(u), \quad u > 0,$$

Simulation From h

Algorithm 2.
1. Independently simulate $U \sim U(0,1)$ and $Y \sim \mathrm{MLL}(\alpha, p, \delta_0)$.
2. If $U \le \varphi(Y)$ return Y, otherwise go back to step 1.

Where

$$\varphi(u) = \frac{g(u) - g(ue^{\lambda t})}{\left(u^p 1_{[0<u\le\delta_0]} + 1_{[u>\delta_0]}\right) \max\left\{1, M\left(e^{p\lambda t} - 1\right)\right\}}.$$

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Under addition assumptions, we can take the proposed distribution to be a generalized gamma distribution, which improves performance.

Simulations

We perform a series of simulations.

We again focus on the case where

$$g(x) = \alpha \frac{\alpha + \ell + 1}{2\Gamma(1-\alpha)} \int_0^\infty e^{-|x|u}(1+1/u)^{-2-\alpha-\ell} u^{-2-\alpha} \mathrm{d}u.$$

Simulation Results

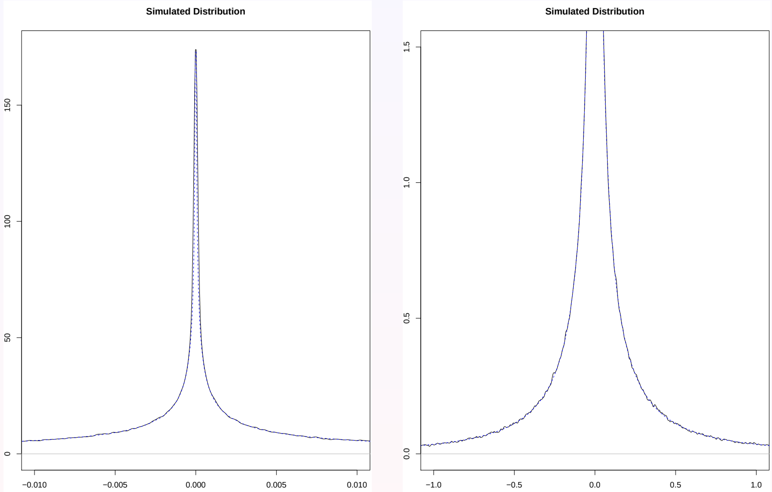
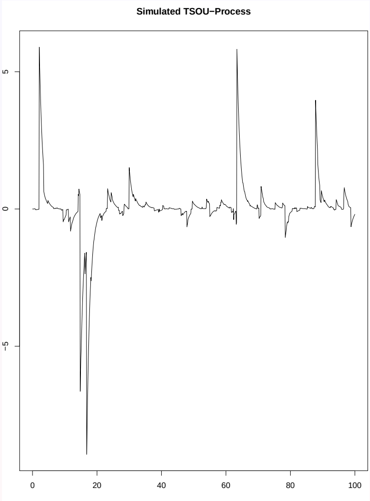
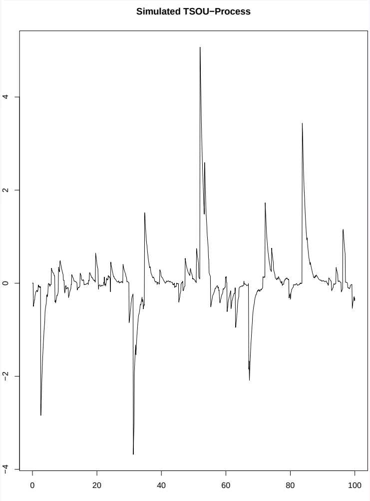


Figure: KDE (solid line) with the true pdf (dashed line) overlaid. These are presented at two scales.

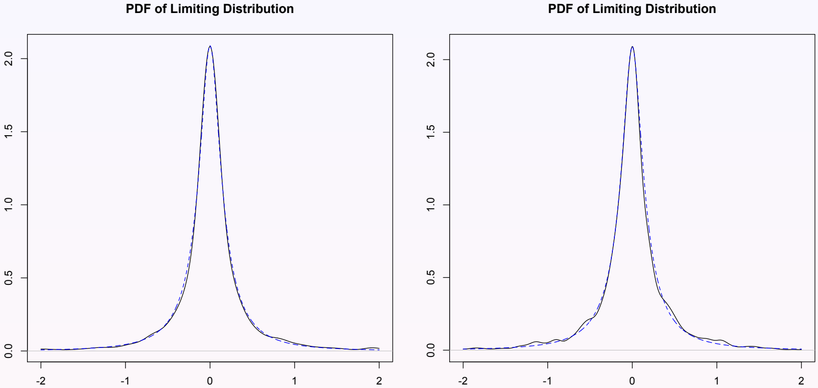
Simulated TSOU-processes



$\alpha = .55, \ell = 1, \lambda = 1$



$\alpha = .75, \ell = 1, \lambda = 1$



$\alpha = .55, \ell = 1, \lambda = 1$ $\qquad$ $\alpha = .75, \ell = 1, \lambda = 1$

Figure: We simulated 3000 TSOU-processes (with the same parameters) up to time $T = 100$. We then consider the last observation for each process and plot the KDE these observations (solid line) overlaid with the true pdf of the stationary distribution.

Extensions

In the papers we consider a more general situation:

1. All results of both parts are extended to the multivariate case
2. Cases with less strict conditions on g are considered

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Plot of μ and $\tilde{\mu}$

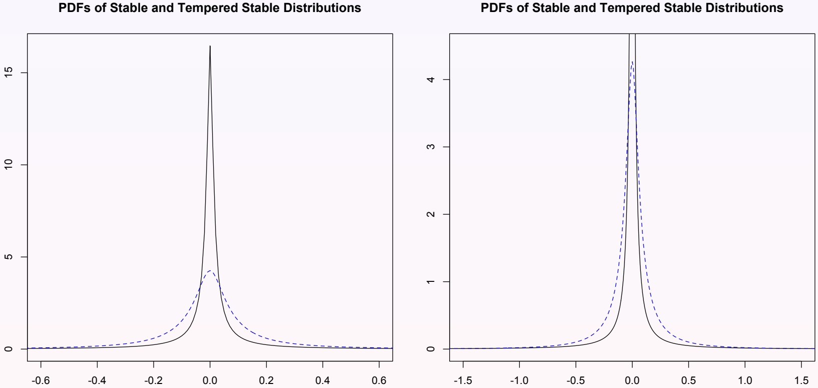

Figure: The solid line is the pdf of the stable distribution μ and the dashed line is the pdf of the tempered stable distribution $\tilde{\mu}$.

$t = 10$

Next we want to simulate from $\tilde{\mu}^{10}$.

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To get one such observations, it is easier to simulate 10 observations from $\tilde{\mu}$ and aggregate.

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This gave 11130 observations from $\tilde{\mu}$.

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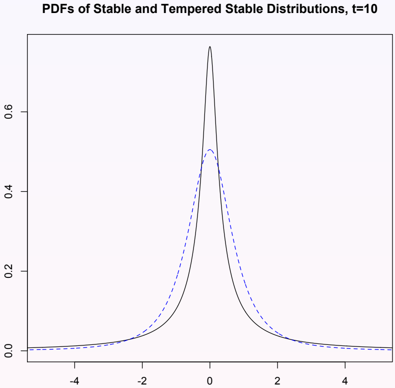
We convert these into 1113 observations from $\tilde{\mu}^{10}$.

Plots for $t = 10$



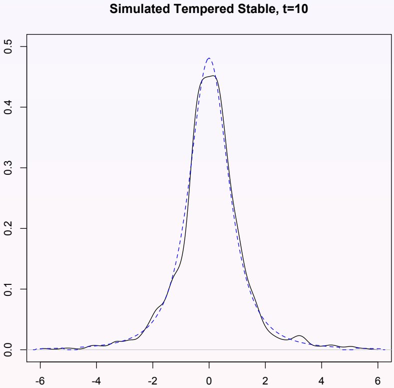


Figure: On the left, the solid line is the pdf of μ^{10} and the dashed line is the pdf of $\tilde{\mu}^{10}$. On the right, the solid line is the KDE of the simulated values from $\tilde{\mu}^{10}$, and the dashed line is the smoothed pdf of $\tilde{\mu}^{10}$.

Comparison with Inversion Method

		Algorithm		Inversion Method		
α	ℓ	run time	obs	run time	obs	ratio
.50	0.5	66.841	1057	785.870	1000	0.085
.50	1.0	30.531	1116	338.867	1000	0.090
.50	5.0	47.035	1090	509.506	1000	0.092
.75	0.5	53.346	1124	668.588	1000	0.080
.75	1.0	23.844	1082	282.788	1000	0.084
.75	5.0	37.353	1093	448.820	1000	0.083
.95	0.5	51.121	1101	601.804	1000	0.084
.95	1.0	23.444	1096	278.804	1000	0.084
.95	5.0	38.190	1100	470.784	1000	0.081