Well-posedness of McKean-Vlasov SDEs, related PDE on the Wasserstein space, and some new quantitative estimates of propagation of chaos.

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LSA Autumn Meeting 2020, 21 th October 2020.

Based on joint works with : P.-E. Chaudru de Raynal (Université Savoie Mont Blanc), V. Konakov (HSE, Moscou), L. Li (UNSW, Sydney) and S. Menozzi (Université d'Evry Val d'Essone).

Université de Paris

LPSM
LABORATOIRE DE PROBABILITÉS
STATISTIQUE & MODÉLISATION

Non-linear McKean-Vlasov SDEs

◦ We want to investigate the weak and strong well-posedness of a class of non-linear SDEs :

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◦ Some examples of non-linear interaction :

- ◦ McKean (1960) : $b(x, \mu) := \int_{\mathbb{R}^d} b(x, y)\, \mu(dy), \quad \sigma(x, \mu) := \int_{\mathbb{R}^d} \sigma(x, y)\, \mu(dy).$
- ◦ Scalar interaction : $b(x, \mu) := b\left(x, \int_{\mathbb{R}^d} \bar{b}(x - y)\mu(dy)\right), \quad \sigma(x, \mu) := \sigma\left(x, \int_{\mathbb{R}^d} \bar{\sigma}(x - y)\mu(dy)\right).$
- ◦ Polynomials : $b(x, \mu) := \prod_{i=1}^N \int_{\mathbb{R}} \bar{b}_i(x, y)\mu(dy), \quad \sigma(x, \mu) := \prod_{i=1}^N \int_{\mathbb{R}} \bar{\sigma}_i(x, y)\mu(dy).$

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◦ and obtain some quantitative rates of propagation of chaos :

$$X_t^i = \xi^i + \int_0^t b(s, X_s^i, \frac{1}{N}\sum_{j=1}^N \delta_{X_s^j})ds + \int_0^t \sigma(s, X_s^i, \frac{1}{N}\sum_{j=1}^N \delta_{X_s^j})dW_s^i, \quad i = 1, \cdots, N,$$

$(\xi^i, W^i)_{1\leq i\leq N}$ i.i.d. with same law as (ξ, W).

- Asymptotic synchronization : each particle $(X_t^i)_{0\leq t\leq T}$ converges in law to the same mean-field limit equation $(X_t)_{0\leq t\leq T}$.
- Asymptotic independence : for any fixed k

$$Law\Big((X_t^1, \cdots, X_t^k)_{0\leq t\leq T} \Big) \to Law\Big((X_t^\xi)_{0\leq t\leq T}\Big)^{\otimes k}, \text{ as } N \uparrow \infty.$$

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◦ Numerous Applications :

- Probabilistic representation of non-linear PDEs : Burgers (see e.g. Bossy & Talay (96), Jourdain (97), ...) , Keller-Segel (see e.g. Jabir, Talay, Tomasevic (18-20)), ...
- Economics and Finance : Mean Field Game theory (Carmona & Delarue), systemic risk, ...
- Biology : chemotaxi, neurons, ...

Classical Cauchy-Lipschitz theory for McKean-Vlasov SDE

- Need a suitable distance on the space of probability measures $\mathcal{P}(\mathbb{R}^d)$.
- Usually make use of the Wasserstein metric on $\mathcal{P}_p(\mathbb{R}^d)$

$$\mathcal{P}_p(\mathbb{R}^d) : \text{probability measures with finite } p\text{-moment}$$

$$\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \; W_p(\mu, \nu) = \left(\inf_{\pi} \int_{(\mathbb{R}^d)^2} |x - y|^p \, d\pi(x, y) \right)^{1/p}$$

where π has first and second marginals equals to μ and ν respectively.

- It is important to notice that for any $X, \; X' \in L^p(\mathbb{P})$, it holds

$$W_p([X], [X']) \leq \mathbb{E}[|X - X'|^p]^{1/p}.$$

Classical Cauchy-Lipschitz theory for McKean-Vlasov SDE

- Cauchy-Lipschitz theory : see e.g. Sznitmann (1991), ...
  - Well-posedness : b, σ are Lipschitz-continuous on $\mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$.
    - Unique strong solution for any initial condition $\xi \in L^p(\mathbb{P})$.
    - Proof works as in the standard case of Itô's SDE after noticing

$$\mathbb{E}[|(b,\sigma)(t, X_t, [X_t]) - (b,\sigma)(t, X'_t, [X'_t])|^p] \leq C\mathbb{E}[|X_t - X'_t|^p].$$

  - Propagation of chaos :
    - Relies on the (standard) coupling argument introduced by Sznitmann (91). Take input $(\xi^i, W^i)_{1\leq i\leq N}$ and construct

$$\bar{X}^i_t = \xi^i + \int_0^t b(s, \bar{X}^i_s, [\bar{X}^i_s])\, ds + \int_0^t \sigma(s, \bar{X}^i_s, [\bar{X}^i_s])\, dW^i_s$$

and notice that

$$Law((\bar{X}^i_t)_{0\leq t\leq T}) = Law((X_t)_{0\leq t\leq T}).$$

    - Typical results :

$$\lim_{N\uparrow\infty} \left\{ \mathbb{E}\Big[\sup_{0\leq t\leq T} |X^i_t - \bar{X}^i_t|^p\Big] + \sup_{0\leq t\leq T} \mathbb{E}\Big[\Big(W_p\big(\frac{1}{N}\sum_{i=1}^N \delta_{X^i_t}, [X_t]\big)\Big)^p\Big] \right\} = 0$$

and under some additional integrability condition on the initial measure μ, for any $p \geq 1$ there exists some sequence $(\varepsilon_N)_{N\geq 1}$ s.t. $\varepsilon_N \downarrow 0$ and

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X^i_t - \bar{X}^i_t|^p\Big] + \sup_{0\leq t\leq T} \mathbb{E}\Big[\Big(W_p\big(\frac{1}{N}\sum_{i=1}^N \delta_{X^i_t}, [X_t]\big)\Big)^p\Big] \leq \varepsilon_N.$$

Beyond the Cauchy-Lipschitz theory

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- ∘ Existence follows from a compactness argument under some continuity assumptions on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \ni (t, x, m) \mapsto (b, \sigma)(t, x, m)$ as in the case of standard Itô's SDEs.
  $\rightsquigarrow$ Skorohod (65), Stroock and Varadhan (69).

- ∘ Main issue is Uniqueness analogy with :
  $\rightsquigarrow$ Zvonkin's approach for pathwise uniqueness to standard Itô's SDE.
  $\rightsquigarrow$ Stroock & Varadhan works for weak uniqueness.
  $\rightsquigarrow$ But it is expected to be harder : state space is now $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (X_t, [X_t])$.

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∘ Some counter-examples :

- ∘ Sheutzow : $X_t = \xi + \int_0^t \mathbb{E}[b(X_s)]\, ds$, $\exists b$ bounded, locally Lipschitz $\rightsquigarrow$ uniqueness fail.
- ∘ Delarue : $x_t = x_0 + \int_0^t b(x_s)\, ds$ uniqueness fail then $X_t = x_0 + \int_0^t b(x_s) ds + W_t \rightsquigarrow \mathbb{E}[X_t] = x_t$.

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- Typical examples where uniqueness holds
  $\rightsquigarrow$ Shiga and Tanaka (85), Jourdain (97), Mishura and Veretenikov (2018), Lacker (2018), Röckner and Zhang (2018)

$$X_t = \xi + \int_0^t \int_{\mathbb{R}^d} b(X_s, y) \mu_s(dy)\, ds + \sigma B_t$$

  - b bounded measurable $\rightsquigarrow \mathcal{P}(\mathbb{R}^d) \ni \mu \mapsto b(x, \mu) = \int_{\mathbb{R}^d} b(x, y)\mu(dy) \in \mathbb{R}^d$ Lipschitz w.r.t. T.V. metric
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- As in the case of standard Itô's SDE, uniqueness relies on the non-degeneracy of the noise.
  - Uniqueness should be connected to a Kolmogorov PDE on the space of probability measures.
  - Investigate smoothing properties of McKean-Vlasov SDEs, especially in the measure direction.
  - Expected to be harder : Finite dimensional noise to smooth infinite dimensional variable

Differentiability of functions of measure

For $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. Work with two notions of derivatives
$\rightsquigarrow$ Lions' lectures at Collège de France, Cardaliaguet lecture notes, Carmona & Delarue books.

(1) Flat or linear functional derivative : $\exists$ a continuous map $\delta U/\delta m : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ s.t.

$$\forall m, m' \in \mathcal{P}_2(\mathbb{R}^d), \lim_{\varepsilon \downarrow 0} \frac{U((1-\varepsilon)m + \varepsilon m') - U(m)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m)(y)\, d(m' - m)(y)$$

$\rightsquigarrow$ Defined up to an additive constant. Choose the normalization $\int_{\mathbb{R}^d} [\delta U/\delta m](m_0)(y) dm_0(y) = 0$.

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(2) Lions, L or intrisinc derivative : Work with Lifted version $\mathcal{U} : L_2(\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}, \mathcal{U}(X) = U([X])$

- U is differentiable iif $\mathcal{U}$ is Fréchet differentiable.

- Differential of U
  - Fréchet derivative of $\mathcal{U}$

$$D\mathcal{U}(X) = \partial_\mu U(\mu)(X), \quad \partial_\mu U(\mu) : \mathbb{R} \ni x \mapsto \partial_\mu U(\mu)(x) \in \mathbb{R}^d, \quad \mu = [X].$$

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Link between flat and L-derivatives :

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- Examples :
  - $U(\mu) = \int_{\mathbb{R}^d} h(x)\mu(dx) \rightsquigarrow U((1-\varepsilon)\mu + \varepsilon\mu') - U(\mu) = \varepsilon \int_{\mathbb{R}^d} h(y)\, d(\mu' - \mu)(y)$,

$$\frac{\delta U}{\delta m}(\mu)(y) = h(y), \quad \partial_\mu U(\mu)(y) = \partial h(y).$$

  - $U(\mu) = \int_{(\mathbb{R}^d)^2} h(x,y)\mu(dx)\mu(dy)$

$$\frac{\delta U}{\delta m}(\mu)(y) = \int_{\mathbb{R}^d} h(y,z)\mu(dz) + \int_{\mathbb{R}^d} h(z,y)\mu(dz) \quad \partial_\mu U(\mu)(y) = \int_{\mathbb{R}^d} \partial_1 h(y,z)\mu(dz) + \int_{\mathbb{R}^d} \partial_2 h(z,y)\mu(dz).$$

Weak existence and uniqueness of McKean-Vlasov SDE

- Assumptions on the coefficients :
  - b is bounded and continuous, $\mathcal{P}(\mathbb{R}^d) \ni m \mapsto b(t, x, m)$ is Lipschitz w.r.t. the total variation metric. (unif. in t, x)
  - $a(t, x, m) = (\sigma\sigma^*)(t, x, m)$ is uniformly elliptic.
  - For any $(i, j) \in \{1, \cdots, d\}^2$,
    - $(t, x, m) \mapsto\mapsto a_{i,j}(t, x, m)$ is bounded and η-Hölder in x (unif. in (t, m)),
    - $\mathcal{P}(\mathbb{R}^d) \ni m \mapsto a_{i,j}(t, x, m)$ admits a bounded flat derivative,
    - $(x, y) \mapsto [\delta a_{i,j}/\delta m](t, x, m)(y)$ is η-Hölder (unif. in (t, m)).

Theorem : Well-posedness of the non-linear martingale problem (Chaudru de Raynal, F.)

Under the above set of assumptions, the non-linear martingale problem associated to the McKean-Vlasov SDE is well-posed for any initial distribution $\mu \in \mathcal{P}(\mathbb{R}^d)$.
In particular, weak existence and uniqueness hold for the McKean-Vlasov SDE.
Strong well-posedness under the additional assumption that $x \mapsto \sigma(t, x, m)$ is (uniformly) Lipschitz-continuity.

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◦ Idea :

- Banach fixed point theorem on the space

$$\mathcal{A}_{s,T,\mu} = \left\{\mathbf{P} \in \mathcal{C}([s, T], \mathcal{P}(\mathbb{R}^d)) : \mathbf{P}(s) = \mu\right\}, \ \mu \in \mathcal{P}(\mathbb{R}^d)$$

  which is a complete metric space equipped with $d(\mathbf{P}, \mathbf{Q}) = \sup_{s\le t\le T} d_{TV}(\mathbf{P}(t), \mathbf{Q}(t))$.
- Define a map $\mathscr{T} : \mathcal{A}_{s,T,\mu} \to \mathcal{A}_{s,T,\mu}$, where for $t \in [s, T]$, $\mathscr{T}(\mathbf{P})(t) = [X_t^{\mathbf{P}}]$ with

$$X_t^{\mathbf{P}} = \xi + \int_s^t b(r, X_r^{\mathbf{P}}, \mathbf{P}(r))\, dr + \int_s^t \sigma(r, X_r^{\mathbf{P}}, \mathbf{P}(r))\, dW_r.$$

- Prove that $\mathscr{T}$ is a contraction if T is small enough
  - ⇝ make use of parametrix expansion (Friedman 64) to control $d_{TV}(\mathscr{T}(\mathbf{P}^1)(t), \mathscr{T}(\mathbf{P}^2)(t))$.

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Strong well-posedness under the additional assumption that $x \mapsto \sigma(t,x,m)$ is (uniformly) Lipschitz-continuity.

Example 1 (McKean) : b bounded measurable, $(x,z) \mapsto \sigma(t,x,z)$ bounded, η-Hölder and $a(t,x,m)$ uniformly elliptic, $b(t,x,\mu) = \int_{\mathbb{R}^d} b(t,x,y)\mu(dy)$, $\sigma(t,x,\mu) = \int_{\mathbb{R}^d} \sigma(t,x,y)\mu(dy)$

$$X_t = \xi + \int_0^t b(s, X_s, [X_s])ds + \int_0^t \sigma(s, X_s, [X_s])dW_s.$$

Weak existence and uniqueness of McKean-Vlasov SDE

- Assumptions on the coefficients :
  - b is bounded and continuous, $\mathcal{P}(\mathbb{R}^d) \ni m \mapsto b(t, x, m)$ is Lipschitz w.r.t. the total variation metric. (unif. in t, x)
  - $a(t, x, m) = (\sigma\sigma^*)(t, x, m)$ is uniformly elliptic.
  - For any $(i, j) \in \{1, \cdots, d\}^2$,
    - $(t, x, m) \mapsto\mapsto a_{i,j}(t, x, m)$ is bounded and η-Hölder in x (unif. in (t, m)),
    - $\mathcal{P}(\mathbb{R}^d) \ni m \mapsto a_{i,j}(t, x, m)$ admits a bounded flat derivative,
    - $(x, y) \mapsto [\delta a_{i,j}/\delta m](t, x, m)(y)$ is η-Hölder (unif. in (t, m)).

Theorem : Well-posedness of the non-linear martingale problem (Chaudru de Raynal, F.)

Under the above set of assumptions, the non-linear martingale problem associated to the McKean-Vlasov SDE is well-posed for any initial distribution $\mu \in \mathcal{P}(\mathbb{R}^d)$.
In particular, weak existence and uniqueness hold for the McKean-Vlasov SDE.
Strong well-posedness under the additional assumption that $x \mapsto \sigma(t, x, m)$ is (uniformly) Lipschitz-continuity.

Example 1 (McKean) : b bounded measurable, $(x, z) \mapsto \sigma(t, x, z)$ bounded, η-Hölder and $a(t, x, m)$ uniformly elliptic, $b(t, x, \mu) = \int_{\mathbb{R}^d} b(t, x, y)\mu(dy)$, $\sigma(t, x, \mu) = \int_{\mathbb{R}^d} \sigma(t, x, y)\mu(dy)$

$$X_t = \xi + \int_0^t b(s, X_s, [X_s])ds + \int_0^t \sigma(s, X_s, [X_s])dW_s.$$

Example 2 : b bounded continuous, ψ_i bounded measurable, $x \mapsto a(t, x, z)$ η-Hölder, $z \mapsto a(t, x, z)$ continuously differentiable, φ_i η-Hölder continuous and $a(t, x, \mu)$ uniformly elliptic

$$X_t = \xi + \int_0^t b\Big(s, X_s, \mathbb{E}[\psi_1(X_s)], \cdots, \mathbb{E}[\psi_N(X_s)]\Big)ds + \int_0^t \sigma\Big(s, X_s, \mathbb{E}[\varphi_1(X_s)], \cdots, \mathbb{E}[\varphi_N(X_s)]\Big)dW_s.$$

Smoothness of the semigroup and of the transition density

◦ Back to the McKean-Vlasov SDE :

$$X_t^{s,\xi} = \xi + \int_s^t b(r, X_r^{s,\xi}, [X_r^{s,\xi}])dr + \int_s^t \sigma(r, X_r^{s,\xi}, [X_r^{s,\xi}])dW_r, \quad [\xi] = \mu \in \mathcal{P}_2(\mathbb{R}^d)$$

Introduce the *decoupling field* or *characteristic* defined by :

$$X_t^{s,x,\mu} = x + \int_s^t b(r, X_r^{s,x,\mu}, [X_r^{s,\xi}])dr + \int_s^t \sigma(r, X_r^{s,x,\mu}, [X_r^{s,\xi}])dW_r.$$

By standard arguments, $X_t^{s,\xi}$ admits a density $z \mapsto p(\mu, s, t, z)$ and so does $X_t^{s,x,\mu}$ with $z \mapsto p(\mu, s, t, x, z)$ s.t.

$$p(\mu, s, t, z) = \int p(\mu, s, t, x, z)\mu(dx).$$

For a map $\phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, ansatz for a semigroup on $\mathcal{P}_2(\mathbb{R}^d)$

$$\mathscr{P}_{s,t}\phi(\mu) = \phi([X_t^{s,\xi}]).$$

◦ Important Questions :

- Smoothing properties : regularity of $[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto \mathscr{P}_{s,t}\phi(\mu)$ even if ϕ is irregular ?
- What is the regularity of the map $[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$?
- PDE satisfied by $(s, \mu) \mapsto \mathscr{P}_{s,t}\phi(\mu)$ or $(s, \mu) \mapsto p(\mu, s, t, z)$, notion of fundamental solution on $\mathcal{P}_2(\mathbb{R}^d)$?
- More generally, address the Cauchy problem with *non-smooth* data (φ, f)

$$u(s, x, \mu) = \mathbb{E}\Big[\varphi(X_t^{s,x,\mu}, [X_t^{s,\xi}]) + \int_s^T f(r, X_r^{s,x,\mu}, [X_r^{s,\xi}])\, dr\Big].$$

Need chain rule on $\mathcal{P}_2(\mathbb{R}^d)$. Informal discussion

∘ Choose a smooth map ϕ, set $\mu_s^{\lambda,\varepsilon} = \lambda\mu + (1-\lambda)[X_s^{s-\varepsilon,\xi}]$ and make use of Markov property $[X_t^{s-\varepsilon,\xi}] = [X_t^{s,X_s^{s-\varepsilon,\xi}}]$:

$$\begin{aligned}
\frac{d}{ds}\mathscr{P}_{s,t}\phi(\mu) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}(\phi([X_t^{s,\xi}]) - \phi([X_t^{s-\varepsilon,\xi}]))\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}(\phi([X_t^{s,\xi}]) - \phi([X_t^{s,X_s^{s-\varepsilon,\xi}}]))\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}(\mathscr{P}_{s,t}\phi(\mu) - \mathscr{P}_{s,t}\phi([X_s^{s-\varepsilon,\xi}]))\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^1\int_{\mathbb{R}^d}\frac{\delta}{\delta m}\Big[\mathscr{P}_{s,t}\phi(\mu_s^{\lambda,\varepsilon})\Big](y)\, d(\mu - [X_s^{s-\varepsilon,\xi}])(y)\, d\lambda\\
&= -\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}\Big[\frac{\delta}{\delta m}\Big[\mathscr{P}_{s,t}\phi(\mu)\Big](X_s^{s-\varepsilon,\xi}) - \frac{\delta}{\delta m}\Big[\mathscr{P}_{s,t}\phi(\mu)\Big](\xi)\Big]\\
&= -\mathbb{E}\Big[b(s,\xi,\mu).\partial_y\frac{\delta}{\delta m}\Big[\mathscr{P}_{s,t}\phi(\mu)\Big](\xi) + \frac{1}{2}\mathrm{trace}\Big(a(s,\xi,\mu)\partial_y^2\frac{\delta}{\delta m}\Big[\mathscr{P}_{s,t}\phi(\mu)\Big](\xi)\Big)\Big]\\
&= -\mathbb{E}\Big[b(s,\xi,\mu).\partial_\mu\mathscr{P}_{s,t}\phi(\mu)(\xi) + \frac{1}{2}\mathrm{trace}\Big(a(s,\xi,\mu)\partial_y\Big[\partial_\mu\mathscr{P}_{s,t}\phi(\mu)\Big](\xi)\Big)\Big]\\
&= -\mathscr{L}_s\mathscr{P}_{s,t}\phi(\mu)
\end{aligned}$$

with

$$\mathscr{L}_sU(s,\mu) = \int_{\mathbb{R}^d}\Big\{b(s,y,\mu).\partial_\mu U(s,\mu)(y) + \frac{1}{2}\mathrm{trace}(a(s,y,\mu)\partial_y[\partial_\mu U](s,\mu)(y)\Big\}\,\mu(dy).$$

∘ Require to investigate smoothness of $\mu \mapsto \mathscr{P}_{s,t}\phi(\mu)$ for ϕ possibly irregular.
- ∘ Regularization effect : ϕ Lipschitz in $d_{TV} \rightsquigarrow \mu \mapsto \mathscr{P}_{s,t}\phi(\mu)$ Lipschitz in W_1 metric !

Smoothness of the transition density

- Assumptions : need to strengthen regularity assumptions of well-posedness
  - $x \mapsto b_i(t,x,m)$ is η-Hölder, $i \in \{1,\cdots,d\}$
  - Two bounded and η-Hölder continuous flat derivatives for b_i, $a_{i,j}$, $(i,j) \in \{1,\cdots,d\}^2$.

Theorem : Fundamental sol. of Backward Kolmogorov PDE on $\mathcal{P}_2(\mathbb{R}^d)$ (Chaudru de Raynal, F.)

Under the above set of assumptions, the map $(s,\mu) \mapsto p(\mu,s,t,z) \in \mathcal{C}^{1,2}([0,t) \times \mathcal{P}_2(\mathbb{R}^d))$ and is the unique fundamental sol. of

$$(\partial_s + \mathscr{L}_s)p(\mu,s,t,z) = 0 \text{ on } [0,t) \times \mathcal{P}_2(\mathbb{R}^d), \quad \lim_{s\uparrow t} p(\mu,s,t,z) = \delta_z(.) \star \mu.$$

Its derivatives satisfy some Gaussian upper-estimates, $z \mapsto g(c,z)$ being the density funct. of a r.v. with law $\mathcal{N}(0,cI_d)$:

$$|\partial_s p(\mu,s,t,z)| \leq \frac{C}{t-s}\int_{\mathbb{R}^d} g(c(t-s),z-x)\,\mu(dx)$$

$$|\partial_\mu p(\mu,s,t,z)(y)| \leq \frac{C}{(t-s)^{\frac{1-\eta}{2}}}\int_{\mathbb{R}^d} g(c(t-s),z-x)\,\mu(dx) + \frac{C}{(t-s)^{\frac{1}{2}}} g(c(t-s),z-y)$$

$$|\partial_y\partial_\mu p(\mu,s,t,z)(y)| \leq \frac{C}{(t-s)^{\frac{2-\eta}{2}}}\int_{\mathbb{R}^d} g(c(t-s),z-x)\,\mu(dx) + \frac{C}{t-s} g(c(t-s),z-y).$$

Idea :

- Construct a smooth sequence

$$\{[0,t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s,\mu) \mapsto p^{(m)}(\mu,s,t,z), m \geq 1\} \quad \text{converging to} \quad p(\mu,s,t,z).$$

- Uniform regularity + equi-continuity properties on $p^{(m)}(\mu,s,t,z) \rightsquigarrow$ parametrix method + circular arguments
- Extract a converging subsequence by compactness argument (Arzela-Ascoli).

Related Backward Kolmogorov PDE on the Wasserstein space

◦ Backward PDE associated to the Markov process $(X_T^{t,x,\mu}, [X_T^{t,\xi}])$:

$$\begin{aligned} &\text{On } [0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \quad (\partial_t + \mathcal{L}_t + \mathscr{L}_t)U(t,x,\mu) = f(t,x,\mu),\\ &\text{On } (x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \quad U(T,x,\mu) = h(x,\mu)\end{aligned}$$

for the non-local operator acting on $U \in \mathcal{C}^{1,2,2}(\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\begin{aligned}\mathcal{L}_t U(t,x,\mu) &= b(t,x,\mu)\partial_x U(t,x,\mu) + \frac{1}{2}a(t,x,\mu)\partial_x^2 U(t,x,\mu)\\ \mathscr{L}_t U(t,x,\mu) &= \int \mu(dz)\left\{b(t,z,\mu).\partial_\mu U(t,x,\mu)(z) + \frac{1}{2}a(t,z,\mu)\partial_z[\partial_\mu U(t,x,\mu)](z)\right\}\end{aligned}$$

$\mathcal{L}_t + \mathscr{L}_t \rightsquigarrow$ should be understood as the infinitesimal operator associated to $(X_t^{x,\mu}, [X_t^{\xi}])_{t\geq 0}$.

admits a unique classical solution and we have the Feynman-Kac probabilistic representation formula :

$$\begin{aligned} U(t,x,\mu) &= \mathbb{E}\Big[\varphi(X_T^{t,x,\mu}, [X_T^{t,\xi}]) + \int_t^T f(s, X_s^{t,x,\mu}, [X_s^{t,\xi}])\, ds\Big]\\ &= \int_{\mathbb{R}^d}\varphi(z, [X_T^{t,\xi}])\, p(\mu, t, T, x, z)\, dz + \int_t^T\int_{\mathbb{R}^d} f(s, z, [X_s^{t,\xi}])\, p(\mu, t, s, x, z)\, dz\, ds\end{aligned}$$

Related Backward Kolmogorov PDE on the Wasserstein space

- Backward PDE associated to the Markov process $(X_T^{t,x,\mu}, [X_T^{t,\xi}])$:

$$\begin{aligned} &\text{On } [0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \quad (\partial_t + \mathcal{L}_t + \mathscr{L}_t)U(t,x,\mu) = f(t,x,\mu),\\ &\text{On } (x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \quad U(T,x,\mu) = h(x,\mu)\end{aligned}$$

for the non-local operator acting on $U \in \mathcal{C}^{1,2,2}(\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\mathcal{L}_t U(t,x,\mu) = b(t,x,\mu)\partial_x U(t,x,\mu) + \frac{1}{2}a(t,x,\mu)\partial_x^2 U(t,x,\mu)$$

$$\mathscr{L}_t U(t,x,\mu) = \int \mu(dz)\left\{b(t,z,\mu).\partial_\mu U(t,x,\mu)(z) + \frac{1}{2}a(t,z,\mu)\partial_z[\partial_\mu U(t,x,\mu)](z)\right\}$$

$\mathcal{L}_t + \mathscr{L}_t \rightsquigarrow$ should be understood as the infinitesimal operator associated to $(X_t^{x,\mu}, [X_t^{\xi}])_{t\geq 0}$.

admits a unique classical solution and we have the Feynman-Kac probabilistic representation formula :

$$\begin{aligned} U(t,x,\mu) &= \mathbb{E}\left[\varphi(X_T^{t,x,\mu}, [X_T^{t,\xi}]) + \int_t^T f(s, X_s^{t,x,\mu}, [X_s^{t,\xi}])\, ds\right]\\ &= \int_{\mathbb{R}^d} \varphi(z, [X_T^{t,\xi}])\, p(\mu, t, T, x, z)\, dz + \int_t^T \int_{\mathbb{R}^d} f(s, z, [X_s^{t,\xi}])\, p(\mu, t, s, x, z)\, dz\, ds\end{aligned}$$

under the following additional hypothesis :

- The two maps $[0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)\ni(t,z,m)\mapsto f(t,z,m)$, $\varphi(z,m)$ are continuous,
- The two maps $m\mapsto f(t,x,m)$, $\varphi(x,m)$ admit a flat derivative with suitable exponential growth at infinity.
- The two functions $z\mapsto f(t,z,m)$ and $(z,z')\mapsto[\delta f/\delta m](t,z,m)(z')$ are locally η-Hölder.

Related Backward Kolmogorov PDE on the Wasserstein space

◦ Backward PDE associated to the Markov process $(X_T^{t,x,\mu}, [X_T^{t,\xi}])$:

$$\begin{aligned} &\text{On } [0,T)\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \quad (\partial_t+\mathcal{L}_t+\mathscr{L}_t)U(t,x,\mu)=f(t,x,\mu),\\ &\text{On } (x,\mu)\in\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d), \quad U(T,x,\mu)=h(x,\mu)\end{aligned}$$

for the non-local operator acting on $U\in\mathcal{C}^{1,2,2}(\mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\begin{aligned}\mathcal{L}_tU(t,x,\mu)&=b(t,x,\mu)\partial_xU(t,x,\mu)+\frac{1}{2}a(t,x,\mu)\partial_x^2U(t,x,\mu)\\ \mathscr{L}_tU(t,x,\mu)&=\int\mu(dz)\left\{b(t,z,\mu).\partial_\mu U(t,x,\mu)(z)+\frac{1}{2}a(t,z,\mu)\partial_z[\partial_\mu U(t,x,\mu)](z)\right\}\end{aligned}$$

$\mathcal{L}_t+\mathscr{L}_t \rightsquigarrow$ should be understood as the infinitesimal operator associated to $(X_t^{x,\mu},[X_t^{\xi}])_{t\geq 0}$.

admits a unique classical solution and we have the Feynman-Kac probabilistic representation formula :

$$\begin{aligned}U(t,x,\mu)&=\mathbb{E}\Big[\varphi(X_T^{t,x,\mu},[X_T^{t,\xi}])+\int_t^T f(s,X_s^{t,x,\mu},[X_s^{t,\xi}])\,ds\Big]\\ &=\int_{\mathbb{R}^d}\varphi(z,[X_T^{t,\xi}])\,p(\mu,t,T,x,z)\,dz+\int_t^T\int_{\mathbb{R}^d}f(s,z,[X_s^{t,\xi}])\,p(\mu,t,s,x,z)\,dz\,ds\end{aligned}$$

◦ Related literature :

- ◦ Buckdhan & al. (2017) : same PDE but b,σ,φ are smooth and $f\equiv 0$.
- ◦ Chassagneux & al. (2017) : *Master equation* $\rightsquigarrow$ non-linear PDE but with regular coefficients.
- ◦ Crisan & Murray (2017) : similar PDE with $f=0$, coefficients b,σ are smooth, unif. ellipticity, φ *irregular* $\rightsquigarrow$ Malliavin calculus for McKean-Vlasov SDEs.

From Kolmogorov PDE on $\mathcal{P}_2(\mathbb{R}^d)$ to propagation of chaos

- On the same probability space, consider system of particles + coupling

$$X_t^i = \xi^i + \int_0^t b(s, X_s^i, \mu_s^N)\, ds + \int_0^t \sigma(s, X_s^i, \mu_s^N)\, dW_s^i, \quad \mu_s^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_s^j},$$

$$\bar{X}_t^i = \xi^i + \int_0^t b(s, \bar{X}_s^i, [\bar{X}_s^i])\, ds + \int_0^t \sigma(s, \bar{X}_s^i, [\bar{X}_s^i])\, dW_s^i, \quad [\bar{X}_s^i] = \mu_t.$$

Theorem : Propagation of chaos at the level of paths, (Chaudru de Raynal, F.)

Under the previous set of assumptions, assuming $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \ni (x, \mu) \mapsto \sigma(t, x, \mu)$ Lipschitz and $\xi \in L^{4+}(\mathbb{P})$, it holds

$$\sup_{0\le t\le T} \mathbb{E}[W_2^2(\mu_t, \mu_t^N)] + \max_{i=1,\dots,N} \sup_{0\le t\le T} \mathbb{E}\Big[|X_t^i - \bar{X}_t^i|^2\Big] \le C\varepsilon_N$$

and

$$\mathbb{E}[\sup_{0\le t\le T} W_2^2(\mu_t, \mu_t^N)] + \max_{i=1,\dots,N} \mathbb{E}\Big[\sup_{0\le t\le T} |X_t^i - \bar{X}_t^i|^2\Big] \le C\sqrt{\varepsilon_N}$$

with

$$\mathbb{E}[W_2^2(\frac{1}{N}\sum_{i=1}^N \delta_{\xi^i}, \mu)] = \mathcal{O}(\varepsilon_N), \quad \varepsilon_N := \begin{cases} N^{-1/2} \text{ if } d < 4, \\ N^{-1/2}\log(1+N) \text{ if } d = 4, \\ N^{-2/d} \text{ if } d > 4. \end{cases}$$

- **Idea :** Make use of Zvonkin's technique.
  - Take u solution of

$$(\partial_t + (\mathcal{L}_t + \mathscr{L}_t))u(t, x, \mu) = b(t, x, \mu), \; u(T, .) = 0$$

  - Zvonkin's transforms : $\Phi(t, X_t^i, N^{-1}\sum_{j=1}^N \delta_{X_t^j}) = X_t^i - u(t, X_t^i, N^{-1}\sum_{j=1}^N \delta_{X_t^j})$, $\Phi(t, \bar{X}_t^i, [X_t]) = \bar{X}_t^i - u(t, \bar{X}_t^i, [X_t])$ allows to remove drift.
  - Compare paths X_t^i and $\bar{X}_t^i$.

From Kolmogorov PDE on $\mathcal{P}_2(\mathbb{R}^d)$ to propagation of chaos

- System of interacting particles :

$$X_t^{s,\xi^i} = \xi^i + \int_s^t b(r, X_r^{s,\xi^i}, \frac{1}{N}\sum_{j=1}^N \delta_{X_r^{s,\xi^j}})\, dr + \int_0^t \sigma(r, X_r^{s,\xi^i}, \frac{1}{N}\sum_{j=1}^N \delta_{X_r^{s,\xi^j}})\, dW_r^i.$$

- Denote by $p^{1,N}(\mu, s, t, z)$ the density of one particle.

Theorem : Propagation of chaos at the level of transition densities, (Chaudru de Raynal, F.)

Under the PDE assumptions, an upper-bound holds

$$|(p^{1,N} - p)(\mu, 0, t, z)| \leq \frac{C}{N}\left\{\frac{1}{t^{\frac{1-\eta}{2}}}\int_{\mathbb{R}^d} g(ct, z-x)|x|\mu(dx) + \frac{1}{t^{1-\frac{\eta}{2}}}\int_{\mathbb{R}^d} g(ct, z-x)\mu(dx)\right\}.$$

Under some additional smoothness assumptions of $m \mapsto b(t, x, m)$, $a(t, x, m)$, a first order expansion holds

$$\begin{aligned}(p^{1,N} - p)(\mu, 0, t, z) = {} & \frac{1}{N}\mathbb{E}\Big[\frac{\delta}{\delta m}p(\mu, 0, t, \xi^1, z)(\xi^1) - \frac{\delta}{\delta m}p(\mu, 0, t, \xi^1, z)(\widetilde{\xi})\Big] \\ & + \frac{1}{2N}\mathbb{E}\Big[\frac{\delta^2}{\delta m^2}p(\mu, 0, t, \xi^1, z)(\widetilde{\xi}, \widetilde{\xi}) - \frac{\delta^2}{\delta m^2}p(\mu, 0, t, \xi^1, z)(\widetilde{\xi}, \xi^2)\Big] \\ & + \frac{1}{N}\int_0^t \mathbb{E}[\mathcal{A}_s p(\mu_s, s, t, z)]\, ds + \frac{1}{N}\mathcal{R}_N(\mu, 0, t, z).\end{aligned}$$

- **Idea :** Test the fundamental solution $[0, t) \times \mathcal{P}_2(\mathbb{R}^d) \ni (s, \mu) \mapsto p(\mu, s, t, z)$ on the particle system.
  - ⇝ Strategy is reminiscent of prev. works : Mouhot-Mischler (2011), Cardaliaguet, Delarue, Lasry, Lions (2015).
  - A natural candidate for $p(\mu, 0, t, z)$ is

$$p(\mu_s^N, s, t, z) \approx p(\mu_s, s, t, z) = p(\mu, 0, t, z), \quad s \in [0, t), z \in \mathbb{R}^d, \quad \text{with} \quad \mu_s^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_s^i}.$$

  - Derive an expansion by applying Itô's formula to the map defined by $f(s, X_s^1, \cdots, X_s^N) := p(\mu_s^N, s, t, z)$.

From Kolmogorov PDE on $\mathcal{P}_2(\mathbb{R}^d)$ to propagation of chaos

- System of interacting particles

$$X_t^{s,\xi^i} = \xi^i + \int_s^t b(r, X_r^{s,\xi^i}, \mu_{s,r}^N)\,dr + \int_0^t \sigma(r, X_r^{s,\xi^i}, \mu_{s,r}^N)\,dW_r^i,\ \mu_{s,t}^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{s,\xi^j}},$$
$$X_t^{s,\xi} = \xi^i + \int_s^t b(r, X_r^{s,\xi}, \mu_{s,r})\,dr + \int_s^t \sigma(r, X_r^{s,\xi}, \mu_{s,r})\,dW_r^i,\ \mu_{s,r} := [X_r^{s,\xi}].$$

- Consider a continuous map $\phi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ with two bounded and α-Hölder flat derivatives.

Theorem : Propagation of chaos at the level of semigroup on $\mathcal{P}_2(\mathbb{R}^d)$, (Chaudru de Raynal, F.)

Under the PDE assumptions, an upper-bound holds

$$\mathbb{E}\Big[\Big|\phi(\mu_{0,t}^N) - \mathcal{P}_{0,t}\phi(\mu)\Big|\Big] = \mathbb{E}\Big[\Big|\phi(\mu_{0,t}^N) - \phi(\mu_{0,t})\Big|\Big] \le \frac{C}{t^{\frac{1+\alpha}{2}}} W_1\Big(\frac{1}{N}\sum_{j=1}^N \delta_{\xi^j}, \mu\Big), \quad \Big|\mathbb{E}\Big[\phi(\mu_{0,t}^N) - \phi(\mu_{0,t})\Big]\Big| \le \frac{C}{t^{1-\frac{\alpha}{2}}}\frac{1}{N}.$$

Under some additional smoothness assumptions of $m \mapsto b(t,x,m)$, $a(t,x,m)$, a first order expansion holds.

- Idea :
  - Test the solution of the Backward-Kolmogorov PDE $(s,\mu) \mapsto \mathscr{P}_{s,t}\phi(\mu)$ on the empirical measure $\mu_{0,s}^N$.
  - Applying Itô's formula and using the fact that $(s,\mu) \mapsto \mathscr{P}_{s,t}\phi(\mu)$ solves the Kolmogorov PDE on $\mathcal{P}_2(\mathbb{R}^d)$

$$\mathbb{E}\Big[\Big|\mathscr{P}_{s,t}\phi(\mu_{0,s}^N) - \mathscr{P}_{0,t}\phi(\mu)\Big|\Big] = \mathbb{E}\Big[\Big|\mathscr{P}_{s,t}\phi(\mu_{0,s}^N) - \mathscr{P}_{s,t}\phi(\mu_s)\Big|\Big] \le \mathbb{E}\Big[\Big|\mathscr{P}_{0,t}\phi(\frac{1}{N}\sum_{j=1}^N \delta_{\xi^j}) - \mathscr{P}_{0,t}\phi(\mu)\Big|\Big] + \frac{C}{N}$$

  - Conclude by letting $s \uparrow t$ and use the fact that $y \mapsto [\delta/\delta m]\mathscr{P}_{0,t}\phi(m)(y)$ is Lipschitz-continuous (uniformly in m).