Variational limits for active particles models

Mauro Mariani, Higher School of Economics, Moscow

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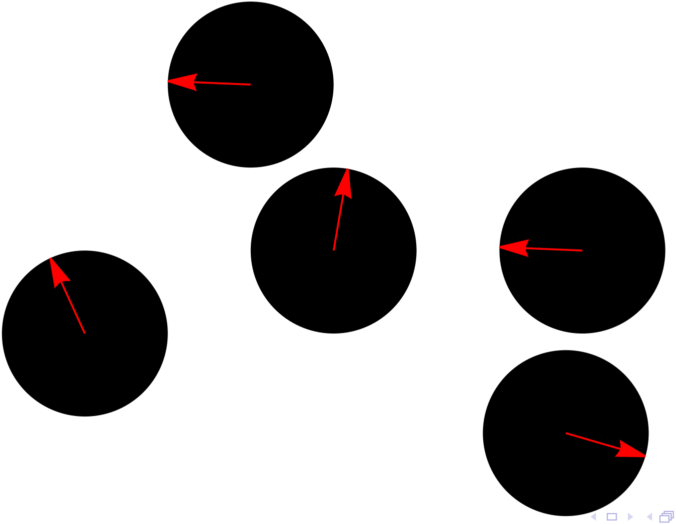

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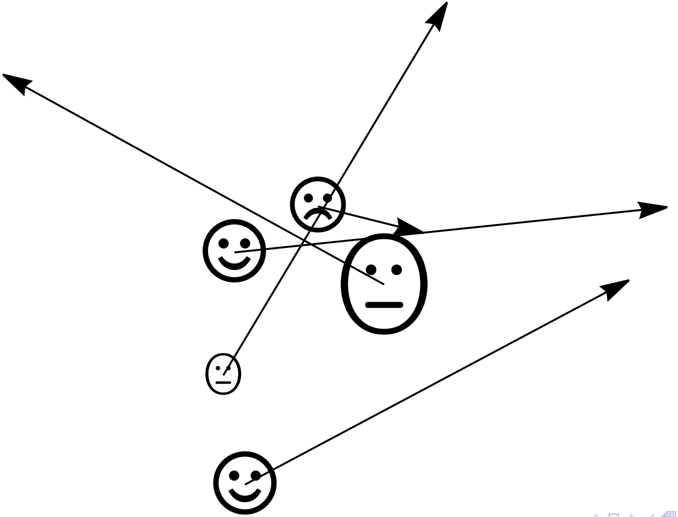

Active particles models

Typical dynamics used to model active interacting systems is

$$\begin{cases} \dot{X}_i = v(X_i, \theta_i) \\ \dot{\theta}_i = F(X_i, \theta_i; \frac{1}{N} \sum_j \delta_{(X_j, \theta_j)}) + \dot{W}_i \end{cases}$$

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X will represent the space of spacial coordinates and Θ will denote the space of attached features.

Active particles models

A reference example is given by $X = \mathbb{R}^d$ and $\Theta = S^{d-1}$ (picture with black disks, sometimes called the Ising model of active particles):

$$\begin{cases} \dot{X}_i = \hat{n}_{\theta_i} \\ \dot{\theta}_i = \frac{\sum_{j=1}^N \chi_r(X_i - X_j) f(\theta_i - \theta_j)}{\sum_{j=1}^N \chi_r(X_i - X_j)} + \dot{W}_i \end{cases}$$

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There are various informal predictions on the behavior of these kind of systems, as $N \to \infty$, the interaction radius r vanishes and a suitable long-time asymptotic is considered. See Degond et al.

Dynamic functional

It is a classical fact that, as $N \to \infty$, the empirical measure $\frac{1}{N}\sum_i \delta_{X_i,\theta_i}$ is *very* close to the solution to

$$\partial_t \mu_t = -d_x^*(\mu_t v) + (\Delta\mu_t + \mathrm{div}(\mu_t F[\mu_t])) + \sqrt{\mu_t/N}\,\mathrm{div}(\dot{W})$$

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or otherwise said, the large deviations rate functional of the law of the empirical measure is given by the control problem

$$\partial_t \mu_t + d_x^*(\mu_t v) - (\Delta \mu_t + \mathrm{div}(\mu_t F[\mu_t])) = \mathrm{div}(\mu_t E)$$

$$\mathrm{Cost}(\mu) = \frac{1}{2} \int dt\, d\mu_t(x)\, E^2(t, x)$$

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Hereafter we assume that X and Θ are smooth, connected, compact manifolds without boundary. Θ is equipped with a Riemann metric and $d\theta$ is the associated normalized volume measure; but we stress that X does not have any natural metric, and it is only equipped with a reference probability measure M.

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Let $\mathcal{U} := C([0,T];\mathcal{P}(X \times \Theta))$. For our aims, the statistical properties of these systems are described by a dynamical free energy functional $I \colon \mathcal{U} \to [0,\infty]$ (see below)

$$I_1(\mu) = \left\| \partial_t \mu_t + d_x^*(\mu_t v) - (\Delta \mu_t + \mathrm{div}(\mu_t F[\mu_t])) \right\|^2_{L_2([0,T];H_\mu^{-1})}$$

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where

- The interaction $F = F[\mu](x, \theta)$ writes as $F[\mu](x, \theta) = -\nabla \psi(x, \theta) + K[\mu](x, \theta)$ with $\mathrm{div}(e^{-\psi} K[\mu]) = 0$.

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- The equilibrium measure is $dm(x,\theta) = e^{-\psi(x,\theta)}dM(x)\,d\theta$.
- $v = v(x,\theta)$ is a vector field on X, such that $d_x^* v = 0$ and $m(v) = 0$.
- The initial data is fixed.

Dynamic functional

As suggested in the literature, we consider the scaling

$$I_\varepsilon(\mu) = \left\| \varepsilon \partial_t \mu_t + d_x^*(\mu_t v) - \varepsilon^{-1}(\Delta \mu_t + \mathrm{div}(\mu_t F[\mu_t])) \right\|^2_{L_2([0,T];H_\mu^{-1})}$$

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and investigate the limit $\varepsilon \downarrow 0$.
The functional I_ε can be understood as the optimal control problem

$$\varepsilon \partial_t \mu_t + d_x^*(\mu_t v) = \varepsilon^{-1} \Delta \mu_t + \operatorname{div}(\mu_t (F[\mu_t] + E))$$

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Most importantly, one is interested in a limit in a variational sense, which is relevant for LD and control.

Understanding the scaling

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For each fixed x, the second equation admits an invariant measure μ_x on Θ. As $\varepsilon \to 0$, X converges to the solution to $\dot{X} = \bar{V}(X)$, where

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But if $\bar{V} = 0$, rather consider the limit of $X_{t/\varepsilon}$:

$$\dot{X} = \frac{1}{\varepsilon} v(X, \theta)$$
$$\dot{\theta} = \frac{1}{\varepsilon^2} F(X, \theta) + \frac{1}{\varepsilon} \dot{W}$$

Γ-limit

We want to identify the Γ-limit of I_ε as $\varepsilon \to 0$. In the optimal control interpretation, this means that we want to calculate the limiting cost of a $\mu \in \mathcal{U}$ as the minimal limiting cost

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From the LD point of view, this means that as $N \to \infty$ and then $\varepsilon \to 0$

$$\mathbb{P}(\pi_{N,\varepsilon} \sim \mu) = \exp(-N\, I(\mu))$$

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This implies that minima and minizers converge, that constrained problems converge, that the inequalities hold in the limit of LD etc.

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Recall that $dm = \exp(-\psi(x,\theta))dM(x)\,d\theta$ we the equilibrium (Gibbs) measure, and $F = -\nabla\psi + K$.
$I\colon \mathcal{U} \to [0,\infty]$ is defined as follows, for suitable tensors A and B.

$$I(\mu) := \begin{cases} \|\partial_t \varrho - d_x^*(B[\varrho]d_x\varrho)\|^2_{L_2([0,T];H^{-1}_{A[\varrho]})} & \text{if } \mu_t = \varrho_t m \\ +\infty & \text{otherwise} \end{cases}$$

This has again a control interpretation, but in the x-variable only. If

$$\partial_t \varrho - d_x^*(B[\varrho]d_x\varrho) = d_x^*(A[\varrho]E)$$

then $I(\varrho m) = \int dt\,dm\,A[\varrho]\,E\cdot E$.

Einstein's relations

For each smooth 1-form ω on X, there is a unique solution φ such that $m(\varphi) = 0$ and

$$\mathrm{div}(e^{-\psi}\nabla\varphi) - e^{-\psi}K[\varrho m]\cdot\nabla\varphi = e^{-\psi}\langle v, \omega\rangle$$

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For ϱ a probability density on X, the tensors A and B are given by

$$\mathcal{A}[\varrho](\omega, \omega') := m(\varrho\nabla\langle\Phi[\varrho], \omega\rangle \cdot \nabla\langle\Phi[\varrho], \omega'\rangle)$$
$$\mathcal{B}[\varrho](\phi, \phi') := m\big(\phi\,\langle v, d_x\langle\Phi[\varrho], d_x\phi'\rangle\rangle\big)$$

where ω, ω' and ϕ, ϕ' are smooth 1-forms and functions on X.

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where ω, ω' and ϕ, ϕ' are smooth 1-forms and functions on X.
The Einstein's relation $B[\varrho] = h''(\varrho)A[\varrho]$ holds, with $h(\varrho) = \varrho\log\varrho$.

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- For instance, on $\mathbb{R}^d$, with constant v and *non-interacting particles*, one gathers $A(\varrho) = \varrho$ and this reduces to the JKO formulation on the heat equation.
- If $d_x^* v \neq 0$, a drift term appears.

Hint of the proof

Consider the rescaled control problem

$$\varepsilon\partial_t\mu_t^\varepsilon + d_x^*(\mu_t^\varepsilon v) = \varepsilon^{-1}\Delta\mu_t^\varepsilon + \varepsilon^{-1}\mathrm{div}(\mu_t F[\mu_t^\varepsilon]) + \mathrm{div}(\mu^\varepsilon E^\varepsilon)$$

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This implies that the marginal in x satisfies a continuity equation

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- taking θ-average $\varepsilon\partial_t\mu_t^\varepsilon \to 0$.
- $d_x^*(\mu_t^\varepsilon v) \to (\partial_t\varrho)m$.
- When tested again $\varphi = \langle\Phi[\varrho], \omega\rangle$, the terms order ε^{-1} converge to the controlled forcing of the limiting functional.

Grazie!


obrigado
Dank U
Merci
mahalo
Köszi
спасибо
Grazie
Thank you
mauruuru
Takk
Gracias
Dziękuję
Děkuju
danke
Kiitos