A Yaglom type theorem for a branching process in Markovian environnement

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Outline

1. Formulation of the model.
2. Main results.
3. Intermediate results.
4. Elements of the proofs.

What is the talk about

We deal with a branching process whose behaviour depend on a sequence called environnement.
The environment can be defined by

- an i.i.d. sequence of r.v.'s with values in some abstract space
- a stationary sequence of r.v.'s
- a deterministic sequence (so called varying environment)

The goal:

1. We shall consider the case when the environment is defined by a Markov chain with values in a finite spate space.
2. We shall prove a Yaglom type theorem.

The Markovian environment

The environment is given on $(\Omega, \mathscr{F}, \mathbb{P})$ by a homogeneous Markov chain $(X_n)_{n\geq 0}$ with values in the finite state space $\mathbb{X}$ and with the transition probabilities $P(i,j),\ \ i,j \in \mathbb{X}$.

1. The corresponding transition operator is denoted by $\mathbf{P}$. $\mathbb{P}_i$ is the probability law on $\mathbb{X}^{\mathbb{N}}$ and $\mathbb{E}_i$ the associated expectation, for a starting point $i \in \mathbb{X}$.
2. We suppose the following: The Markov chain $(X_n)_{n\geq 0}$ is irreducible and aperiodic. $\Longrightarrow$ the matrix $\mathbf{P}^{k_0} > 0$ for some k_0 (is primitive).
3. Then, by the Perron-Frobenius theorem we have the spectral gap property: there exist positive constants c_1, c_2, a unique positive $\mathbf{P}$-invariant probability $\boldsymbol{\nu}$ on $\mathbb{X}$ ($\boldsymbol{\nu}(\mathbf{P}) = \boldsymbol{\nu}$) and an operator $\mathbf{Q}$ on $\mathscr{C}(\mathbb{X})$ such that, for any $g \in \mathscr{C}(\mathbb{X})$ and $n \geq 1$, $i \in \mathbb{X}$,

$$\mathbf{P}g(i) = \boldsymbol{\nu}(g) + \mathbf{Q}(g)(i) \quad \text{and} \quad \|\mathbf{Q}^n(g)\|_\infty \leq c_1\, \mathrm{e}^{-c_2 n}\, \|g\|_\infty\,, \qquad (1)$$

where $\mathbf{Q}\,(1) = 0$ and $\boldsymbol{\nu}\,(\mathbf{Q}(g)) = 0$ ($\boldsymbol{\nu} \perp \mathbf{Q}$).

The branching process

The branching process $(Z_n)_{n\geq 0}$ in the Markovian environment $(X_n)_{n\geq 0}$ is defined as follows:

1. The initial population size is $Z_0 = z \in \mathbb{N}$.
2. For $n \geq 1$, we let Z_{n-1} be the population size at time $n-1$ and assume that at time n, provided the environment takes value $X_n = i \in \mathbb{X}$, the parent $j \in \{1, \ldots Z_{n-1}\}$ generates $\xi_i^{n,j}$ children.
3. Then the population size at time n is given by:

$$Z_n = \sum_{j=1}^{Z_{n-1}} \xi_{X_n}^{n,j}, \tag{2}$$

   where the empty sum is equal to 0.
4. $(\xi_i^{n,j})_{j,n\geq 1}$ are i.i.d. with the same probability generating function

$$f_i(s) := \mathbb{E}\left(s^{\xi_i}\right), \quad s \in [0,1]. \tag{3}$$

5. $\xi_i^{n,j}$, $j \geq 1$, $i \in \mathbb{X}$ are independent of $X_0, \ldots, X_n$, and $Z_0, \ldots, Z_n$

Assumption

1. For any $i \in \mathbb{X}$, the random variable ξ_i satisfies the following: $\mathbb{E}(\xi_i) > 0$ and $\mathbb{E}(\xi_i^2) < +\infty$.

2. Then, it follows that $0 < f_i'(1) < +\infty$ and $f_i''(1) < +\infty$.

The associated Markov walk

1. Introduce the function $\rho : \mathbb{X} \mapsto \mathbb{R}$ satisfying

$$\rho(i) = \ln f_i'(1), \quad i \in \mathbb{X}.$$

2. Along with $(Z_n)_{n\geq 0}$ consider the Markov walk $(S_n)_{n\geq 0}$ such that $S_0 = 0$ and, for $n \geq 1$,

$$S_n = \ln \left(f_{X_1}'(1) \cdots f_{X_n}'(1) \right) = \sum_{k=1}^{n} \rho\left(X_k\right). \tag{4}$$

Notation for the expectation

Conside the couple (X_n, Z_n), where
X_n represents the environment and Z_n the branching process.

1. The couple $(X_n, Z_n)_{n\geq 0}$ is a Markov chain with values in $\mathbb{X} \times \mathbb{N}$,
2. Let $\mathbb{P}_{i,z}$ be the probability law on $(\mathbb{X} \times \mathbb{N})^{\mathbb{N}}$ and $\mathbb{E}_{i,z}$ the associated expectation generated by the finite dimensional distributions of the Markov chain $(X_n, Z_n)_{n\geq 0}$ starting at $X_0 = i$ and $Z_0 = z$ (z initial particles).

Non-lattice condition

1. The following non-lattice condition is needed to ensure that the local limit theorem for the Markov walk S_n holds true:

   For any $\theta, a \in \mathbb{R}$, there exist $n \geq 1$ and a path $x_0, \ldots, x_n$ in $\mathbb{X}$ such that $\mathbf{P}(x_0, x_1) \cdots \mathbf{P}(x_{n-1}, x_n)\mathbf{P}(x_n, x_0) > 0$ and

   $$\rho(x_0) + \cdots + \rho(x_n) - (n+1)\theta \notin a\mathbb{Z}.$$

   Recall that $\rho(i) = \ln f_i'(1), \quad i \in \mathbb{X}$.

2. The above condition is equivalent to the condition that the Fourier transform operator

   $$\mathbf{P}_{it} g(i) := \mathbf{P}\left(e^{it\rho} g\right)(i) = \mathbb{E}_i\left(e^{itS_1} g(X_1)\right), \quad g \in \mathscr{C}(\mathbb{X}), i \in \mathbb{X}, \quad (5)$$

   has a spectral radius strictly less than 1 for $t \neq 0$.

Non-latticity for Markov chains with not necessarily finite state spaces is considered, for instance, in Shurenkov 1984 and Alsmeyer 1994.

Spectral radius of the transfer operator

1. From the spectral gap property of the operator **P** it follows that, for any $\lambda \in \mathbb{R}$ and any $i \in \mathbb{X}$, the limit

$$k(\lambda) := \lim_{n \to +\infty} \mathbb{E}_i^{1/n}\left(e^{\lambda S_n}\right)$$

exists and does not depend on the initial state of the Markov chain $X_0 = i$.

2. Moreover, the number $k(\lambda)$ is the spectral radius of the transfer operator $\mathbf{P}_\lambda$:

$$\mathbf{P}_\lambda g(i) := \mathbf{P}\left(e^{\lambda\rho} g\right)(i) = \mathbb{E}_i\left(e^{\lambda S_1} g(X_1)\right), \quad g \in \mathscr{C}(\mathbb{X}), i \in \mathbb{X}. \quad (6)$$

In particular, under the spectral gap and non-lattice conditions $k(\lambda)$ is a simple eigenvalue of the operator $\mathbf{P}_\lambda$ and there is no other eigenvalue of modulus $k(\lambda)$.

3. In addition, the function $k(\lambda)$ is analytic on $\mathbb{R}$.

Classification

1. The branching process in Markovian environment is said to be:
   subcritical if $k'(0) < 0$;
   critical if $k'(0) = 0$;
   supercritical if $k'(0) > 0$.

2. The following identity, has been established in Grama-Lauvergnat LePage 2018-SPA:

$$k'(0) = \boldsymbol{\nu}(\rho) = \mathbb{E}_{\boldsymbol{\nu}}\left(\rho(X_1)\right) = \mathbb{E}_{\boldsymbol{\nu}}\left(\ln f'_{X_1}(1)\right) = \varphi'(0), \qquad (7)$$

   where $\mathbb{E}_{\boldsymbol{\nu}}$ is the expectation generated by the finite dimensional distributions of the Markov chain $(X_n)_{n\geq 0}$ in the stationary regime and $\varphi(\lambda) = \mathbb{E}_{\boldsymbol{\nu}}(\exp\{\lambda \ln f'_{X_1}(1)\})$, $\lambda \in \mathbb{R}$.

3. Relation (7) proves that the classification made in the case of branching processes with Markovian environment and that for i.i.d. environment are coherent: when the random variables $(X_n)_{n\geq 1}$ are i.i.d. with common law $\boldsymbol{\nu}$, from (7) it follows that the two classifications coincide.

Survival probability jointly with X_n

1. In the present paper we will focus on the critical case:

$$k'(0) = 0.$$

2. Our first result establishes the exact asymptotic of the survival probability of Z_n jointly with the event $\{X_n = j\}$ when the branching process starts with z particles.

Theorem 1

Assume that $k'(0) = 0$. Then, there exists a positive function $u(i,z) : \mathbb{X} \times \mathbb{N} \mapsto \mathbb{R}_+^*$ such that for any $(i,j) \in \mathbb{X}^2$ and $z \in \mathbb{N}$, $z \neq 0$,

$$\mathbb{P}_{i,z}\left(Z_n > 0\,,\; X_n = j\right) \underset{n \to +\infty}{\sim} \frac{u(i,z)\nu(j)}{\sqrt{n}}.$$

In the case $z = 1$, Theorem 1 has been proved in Grama Lauvergnat LePage 2018-SPA.

The proof for the case $z > 1$ is not a direct consequence of the case $z = 1$.

Additional condition

1. We shall complement the previous statement by studying the asymptotic behavior of Z_n given $Z_n > 0$ under the following condition:

   The random variables ξ_i, $i \in \mathbb{X}$ satisfy:

   $$\inf_{i \in \mathbb{X}} \mathbb{P}(\xi_i \geq 2) > 0.$$

2. The above condition is quite natural - it tells that each parent can generate more than 1 child with positive probability.
   In the present paper is used to prove the non-degeneracy of the limit of the martingale $(\frac{Z_n}{e^{S_n}})_{n \geq 0}$.

Non-degeneracy of the limit law of the martingale $\left(\frac{Z_n}{e^{S_n}}\right)_{n\geq 0}$

Theorem 1

Assume $k'(0) = 0$. Then, for any $i \in \mathbb{X}$, $z \in \mathbb{N}$, $z \neq 0$, there exists a probability measure $\mu_{i,z}$ on $\mathbb{R}_+$ such that, for any continuity point $t \geq 0$ of the distribution function $\mu_{i,z}([0,\cdot])$ and $j \in \mathbb{X}$, it holds that

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}_{i,z}\left(\frac{Z_n}{e^{S_n}} \leq t, X_n = j, Z_n > 0\right) = \mu_{i,z}([0,t])\boldsymbol{\nu}(j)u(i,z)$$

and

$$\lim_{n\to\infty} \mathbb{P}_{i,z}\left(\frac{Z_n}{e^{S_n}} \leq t, X_n = j \middle| Z_n > 0\right) = \mu_{i,z}([0,t])\boldsymbol{\nu}(j).$$

Moreover, it holds that $\mu_{i,z}(\{0\}) = 0$.

The limit variance

1. Under adopted assumptions, the quantity

$$\sigma^2 := \boldsymbol{\nu}\left(\rho^2\right) - \boldsymbol{\nu}(\rho)^2 + 2\sum_{n=1}^{+\infty}\left[\boldsymbol{\nu}(\rho \mathbf{P}^n \rho) - \boldsymbol{\nu}(\rho)^2\right] \tag{8}$$

is finite and positive, i.e. $0 < \sigma < \infty$.

See for example From Grama Lauvergnat Le Page 2017.

Asymptotic of the normalized random walk S_n

1. Let
$$\Phi^+(t) = (1 - e^{-\frac{t^2}{2}})\mathbb{1}(t \geq 0), \quad t \in \mathbb{R},$$
be the Rayleigh distribution function.
2. The following assertion gives the asymptotic behavior of the normalized Markov walk S_n jointly with X_n provided $Z_n > 0$.

Theorem 2

Assume $k'(0) = 0$. Then, for any $i, j \in \mathbb{X}$, $z \in \mathbb{N}$, $z \neq 0$ and $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}_{i,z}\left(\frac{S_n}{\sigma\sqrt{n}} \leq t, X_n = j, Z_n > 0\right) = \Phi^+(t)\nu(j)u(i,z)$$

and

$$\lim_{n\to\infty} \mathbb{P}_{i,z}\left(\frac{S_n}{\sigma\sqrt{n}} \leq t, X_n = j \middle| Z_n > 0\right) = \Phi^+(t)\nu(j).$$

The Yaglom type theorem

1. The following assertion is the Yaglom-type limit theorem for $\log Z_n$ jointly with X_n.

Theorem 3

Assume $k'(0) = 0$. Then, for any $i \in \mathbb{X}$, $z \in \mathbb{N}$, $z \neq 0$, $j \in \mathbb{X}$ and $t \geq 0$,

$$\lim_{n\to\infty} \sqrt{n}\, \mathbb{P}_{i,z}\left(\frac{\log Z_n}{\sigma\sqrt{n}} \leq t, X_n = j, Z_n > 0\right) = \Phi^+(t)\nu(j)u(i,z)$$

and

$$\lim_{n\to\infty} \mathbb{P}_{i,z}\left(\frac{\log Z_n}{\sigma\sqrt{n}} \leq t, X_n = j \middle| Z_n > 0\right) = \Phi^+(t)\nu(j).$$

Intermediate results

For the proofs we need a number of intermediate results.

1. Introduce the first time when the Markov walk $(y + S_n)_{n\geq 0}$ becomes non-positive: for any $y \in \mathbb{R}$, set

$$\tau_y := \inf\{k \geq 1 : y + S_k \leq 0\}. \tag{9}$$

Theorem 4

Assume that $k'(0) = 0$.

1. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $j \in \mathbb{X}$, we have

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}_i\left(X_n = j\,,\, \tau_y > n\right) = \frac{2V(i,y)\boldsymbol{\nu}(j)}{\sqrt{2\pi}\sigma}.$$

2. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,

$$\mathbb{P}_i\left(X_n = j, \tau_y > n\right) \leq c\frac{1 + \max(y, 0)}{\sqrt{n}}.$$

Harmonic function V

1. The function V appearing in the previous theorem is the harmonic function (Doob transform) related to Markov walk $(y + S_n)_{n \geq 0}$. We state some of its properties to be used in the proofs.

Proposition (Harmonic function)

There exists a non-negative function V on $\mathbb{X} \times \mathbb{R}$ such that

1. For any $(i, y) \in \mathbb{X} \times \mathbb{R}$ and $n \geq 1$,

$$\mathbb{E}_i \left(V\left(X_n, y + S_n\right) ; \tau_y > n \right) = V(i, y).$$

2. For any $i \in \mathbb{X}$, the function $V(i, \cdot)$ is non-decreasing and for any $(i, y) \in \mathbb{X} \times \mathbb{R}$,

$$V(i, y) \leq c\left(1 + \max(y, 0)\right).$$

3. For any $i \in \mathbb{X}$, $y > 0$ and $\delta \in (0, 1)$,

$$(1 - \delta)\, y - c_\delta \leq V(i, y) \leq (1 + \delta)\, y + c_\delta.$$

Conditioned limit theorem

1. We need the asymptotic behaviour of the conditioned limit law of the Markov walk $(y+S_n)_{n\geq 0}$ jointly with the Markov chain $(X_n)_{n\geq 0}$. (Extends previous results of the authors where the asymptotic of $\frac{y+S_n}{\sigma\sqrt{n}}$ given the event $\{\tau_y > n\}$ has been considered.)

Theorem 5

Assume that $k'(0) = 0$. 1. For any $(i, y) \in \mathrm{supp}(V)$ and $t \geq 0$,

$$\mathbb{P}_i\left(\frac{y+S_n}{\sigma\sqrt{n}} \leq t, X_n = j \,\middle|\, \tau_y > n \right) \underset{n\to+\infty}{\longrightarrow} \Phi^+(t)\boldsymbol{\nu}(j).$$

2. There exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $n \geq 1$, $t_0 > 0$, $t \in [0, t_0]$ and $(i, y) \in \mathbb{X} \times \mathbb{R}$,

$$\left| \mathbb{P}_i\left(\frac{y+S_n}{\sqrt{n}\sigma} \leq t, X_n = j, \tau_y > n \right) - \frac{2V(i,y)}{\sqrt{2\pi n}\sigma} \Phi^+(t)\boldsymbol{\nu}(j) \right| \leq c_{\varepsilon,t_0} \frac{\left(1+\max(y,0)^2\right)}{n^{1/2+\varepsilon}}.$$

Bound in the local limit theorem

1. We need a bound in the local limit theorem for the Markov walk $(y + S_n)_{n \geq 0}$ jointly with the Markov chain $(X_n)_{n \geq 0}$.

Theorem 6

Assume that $k'(0) = 0$. Then there exists $c > 0$ such that for any $a > 0$, non-negative function $\psi \in \mathscr{C}(\mathbb{X})$, $y \in \mathbb{R}$, $t \geq 0$ and $n \geq 1$,

$$\sup_{i \in \mathbb{X}} \mathbb{E}_i \left(\psi\left(X_n\right) ; \, y + S_n \in [t, t+a] \, , \, \tau_y > n \right)$$
$$\leq \frac{c \left\| \psi \right\|_\infty}{n^{3/2}} \left(1 + a^3\right) (1 + t) \left(1 + \max(y, 0)\right).$$

The dual Markov chain

1. Note that the invariant measure ν is positive on $\mathbb{X}$. Therefore the dual Markov kernel

$$\mathbf{P}^*(i,j) = \frac{\nu(j)}{\nu(i)} \mathbf{P}(j,i), \quad i,j \in \mathbb{X} \tag{10}$$

   is well defined.

2. On an extension of the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we consider the dual Markov chain $(X_n^*)_{n\geq 0}$ with values in $\mathbb{X}$ and with transition probability $\mathbf{P}^*$. The dual chain $(X_n^*)_{n\geq 0}$ can be chosen to be independent of the chain $(X_n)_{n\geq 0}$.

The dual Markov walk

1. Accordingly, the dual Markov walk $(S_n^*)_{n\geq 0}$ is defined by setting

$$S_0^* = 0 \qquad \text{and} \qquad S_n^* = -\sum_{k=1}^{n} \rho\left(X_k^*\right), \quad n \geq 1, \tag{11}$$

where we recall that $\rho(i) = \ln f_i'(1), \quad i \in \mathbb{X}$.

2. For any $y \in \mathbb{R}$ define the first time when the Markov walk $(y + S_n^*)_{n\geq 0}$ becomes non-positive:

$$\tau_y^* := \inf\left\{k \geq 1 : y + S_k^* \leq 0\right\}. \tag{12}$$

For any $i \in \mathbb{X}$, denote by $\mathbb{P}_i^*$ and $\mathbb{E}_i^*$ the probability and the associated expectation generated by the finite dimensional distributions of the Markov chain $(X_n^*)_{n\geq 0}$ starting at $X_0^* = i$.

Duality

1. It can be verified that ν is also $\mathbf{P}^*$ invariant and that the transition operator $\mathbf{P}^*$ satisfies the same condition as the transition operator $\mathbf{P}$ of the forward Markov chain. This implies that the conditioned limit theorems hold also for the dual Markov chain $(X_n^*)_{n\geq 0}$ and the Markov walk $(y+S_n^*)_{n\geq 0}$.
2. In particular there is a harmonic function V^* such that, for any $(i,y)\in\mathbb{X}\times\mathbb{R}$ and $n\geq 1$,

$$\mathbb{E}_i\left(V^*\left(X_n, y+S_n^*\right) ; \tau_y^*>n\right)=V^*(i,y).$$

3. The following duality property is obvious:

Lemma 7

[Duality] For any $n\geq 1$ and any function $g\colon \mathbb{X}^n\to\mathbb{C}$,

$$\mathbb{E}_i\left(g\left(X_1,\ldots,X_n\right) ; X_{n+1}=j\right)=\mathbb{E}_j^*\left(g\left(X_n^*,\ldots,X_1^*\right) ; X_{n+1}^*=i\right)\frac{\nu(j)}{\nu(i)}.$$

The change of probability measure

1. Fix any $(i, y) \in \mathrm{supp}(V)$ and $z \in \mathbb{N}$. The harmonic function V from Proposition (Harmonic function), allows us to introduce the probability measure $\mathbb{P}^+_{i,y,z}$ on $(\mathbb{X} \times \mathbb{N})^{\mathbb{N}}$ and the corresponding expectation $\mathbb{E}^+_{i,y,z}$, by the following relation: for any $n \geq 1$ and any bounded measurable g: $(\mathbb{X} \times \mathbb{N})^n \mapsto \mathbb{R}$,

$$\begin{aligned} &\mathbb{E}^+_{i,y,z}\left(g\left(X_1, Z_1, \dots, X_n, Z_n\right)\right) \\ &:= \frac{1}{V(i,y)} \mathbb{E}_{i,z}\Big(g\left(X_1, Z_1, \dots, X_n, Z_n\right) \\ &\qquad \times V\left(X_n, y + S_n\right) ; \tau_y > n\Big). \end{aligned} \tag{13}$$

Elements of the proof - 1

1. For any $n \geq 1$, $z \geq 1$ and $s \in [0,1]$, the following quantity will play an important role in our study:

$$q_{n,z}(s) = 1 - \left(f_{X_1} \circ \cdots \circ f_{X_n}(s)\right)^z \in [0,1].$$

For any $n \geq 1$ and $z \geq 1$, the function $s \mapsto q_{n,z}(s)$ is convex on $[0,1]$. Since the sequence $(\xi_i^{n,j})_{j,n \geq 1}$ is independent of the Markov chain $(X_n)_{n \geq 0}$, with $s = 0$, we have, $\mathbb{P}^+_{i,y,z}$-a.s.,

$$q_{n,z}(0) = \mathbb{P}^+_{i,y,z}\left(Z_n > 0 \middle| (X_k)_{k \geq 0}\right). \tag{14}$$

Elements of the proof - 2

1. The following formula is similar to the well-known statements from the papers by Agresti 1974 and Geiger and Kersting 2001: for any $s \in [0,1)$ and $n \geq 1$,

$$\frac{1}{q_{n,z}(s)} = \frac{1}{z f'_{X_1}(1) \cdots f'_{X_n}(1)(1-s)} + \frac{1}{z} \sum_{k=1}^{n} \frac{\varphi_{X_k} \circ f_{X_{k+1}} \circ \cdots \circ f_{X_n}(s)}{f'_{X_1}(1) \cdots f'_{X_{k-1}}(1)} + \psi_z \circ f_{X_1} \circ \cdots \circ f_{X_n}(s). \tag{15}$$

Elements of the proof - 3

1. We can rewrite (15) in the following more convenient form: for any $s \in [0,1)$ and $n \geq 1$,

$$q_{n,z}(s)^{-1} = \frac{1}{z}\left(\frac{e^{-S_n}}{1-s} + \sum_{k=0}^{n-1} e^{-S_k}\, \eta_{k+1,n}(s)\right) + \psi_z \circ f_{X_1} \circ \cdots \circ f_{X_n}(s), \tag{16}$$

   where
$$\eta_{k,n}(s) = \varphi_{X_k} \circ f_{X_{k+1}} \circ \cdots \circ f_{X_n}(s).$$

2. Since $\frac{1}{2}\varphi(0) \leq \varphi(s) \leq 2\varphi(1)$, for any $k \in \{1,\ldots,m\}$,

$$0 \leq \eta_{k,m}(s) \leq \frac{f''_{X_k}(1)}{f'_{X_k}(1)^2} \leq \eta := \max_{i\in\mathbb{X}} \frac{f''_i(1)}{f'_i(1)^2}. \tag{17}$$

Non-degeneracy of martingale limit

We deal first with the non-degeneracy of the limit of the associated martingale $\frac{Z_n}{e^{S_n}}$. The next lemma establish the convergence of the Laplace transform.

Lemma 8

Let $s \geq 0$. For any $(i, 0) \in \operatorname{supp} V$ and $z \in \mathbb{N}$, $z \neq 0$, there exists a positive random variable $W_{i,z}$ such that

$$\lim_{n \to \infty} \sqrt{n} \mathbb{E}_{i,z}(e^{-e^{-s} \frac{Z_n}{e^{S_n}}}; Z_n > 0, \tau_0 > n) = 2 \frac{V(i, 0)}{\sqrt{2\pi}\sigma} P_\infty(i, s, z),$$

where

$$P_\infty(i, s, z) := \mathbb{E}^+_{i,0,z}\left(e^{-W_{i,z} e^{-s}}; \cap_{p \geq 1} \{Z_p > 0\}\right) \leq 1.$$

Moreover, for any $(i, 0) \in \operatorname{supp} V$ and $z \in \mathbb{N}$, $z \neq 0$, it holds $\mathbb{P}^+_{i,0,z}$-a.s.

$$\cap_{p \geq 1} \{Z_p > 0\} = \{W_{i,z} > 0\}.$$

Proof of lemma 8

1. For the proof we use a result from Kersting 2017 stated in a more general setting of branching processes with varying environment. To apply it we condition with respect to the environment $(X_n)_{n\geq 0}$, so that one can consider that the environment is fixed.
2. The condition (A) in Kersting 2017 is verified because we assumed that:

$$\inf_{i\in\mathbb{X}} \mathbb{P}(\xi_i \geq 2) > 0.$$

The key point of the proof

The next result plays the key role in the proof.

Theorem 9

For any $i \in \mathbb{X}$, $s \in \mathbb{R}$ and $z \in \mathbb{N}$, $z \neq 0$,

$$\lim_{n\to\infty} \sqrt{n}\mathbb{E}_{i,z}\left(e^{-e^{-s}\frac{Z_n}{e^{S_n}}}; Z_n > 0\right)$$
$$= \frac{2}{\sqrt{2\pi}\sigma} \sum_{k=1}^{\infty} \mathbb{E}_{i,z}\Big(V(X_k,0)\mathbf{1}_{\operatorname{supp} V}(X_k,0)P_\infty(X_k, s+S_k, Z_k);$$
$$Z_k > 0,\, T_k = k\Big) =: u(i,z,e^{-s}).$$

With $s = +\infty$,

$$\lim_{n\to\infty} \sqrt{n}\mathbb{P}_{i,z}(Z_n > 0) = u(i,z),$$

where

$$u(i,z) = \frac{2}{\sqrt{2\pi}\sigma} \sum_{k=1}^{\infty} \mathbb{E}_{i,z}\Big(V(X_k,0)\mathbf{1}_{\operatorname{supp} V}(X_k,0)\mathbb{P}^{+}_{X_k,0,Z_k}(W_{i,z} > 0);$$
$$Z_k > 0,\, T_k = k\Big) > 0$$

is from Theorem 1.

Proof of Theorem 2

$$\lim_{n\to\infty}\sqrt{n}\,\mathbb{P}_{i,z}\left(\frac{S_n}{\sigma\sqrt{n}}\le t, X_n=j, Z_n>0\right)=\Phi^+(t)\boldsymbol{\nu}(j)u(i,z)$$

1. Let $i,j\in\mathbb{X}$, $y\ge 0$, $z\in\mathbb{N}$, $z\neq 0$, $t\in\mathbb{R}$. Then, for any $n\ge 1$ and $y\ge 0$,

$$\begin{aligned}&\mathbb{P}_{i,z}\left(\frac{S_n}{\sqrt{n}\sigma}\le t, X_n=j, Z_n>0\right)\\&=\mathbb{P}_{i,z}\left(\frac{S_n}{\sqrt{n}\sigma}\le t, X_n=j, Z_n>0, \tau_y>n\right)\\&+\mathbb{P}_{i,z}\left(\frac{S_n}{\sqrt{n}\sigma}\le t, X_n=j, Z_n>0, \tau_y\le n\right)\\&=I_1(n,y)+I_2(n,y).\end{aligned}\tag{18}$$

2. Using the bound in the conditioned limit theorem,

$$\sqrt{n}I_2(n,y)\le\sqrt{n}P_{i,z}(Z_n>0,\tau_y\le n)\le cze^{-y}(1+y)\underset{y\to\infty}{\to}0.\tag{19}$$

Proof of Theorem 2

1. To bound the firs term $I_1(n, y)$ we use the following lemma:

Lemma

Suppose that $i, j \in \mathbb{X}$, $z \in \mathbb{N}$, $z \neq 0$ *and* $t \in \mathbb{R}$. *Then, for any* $y \geq 0$ *sufficiently large,*

$$\lim_{m\to\infty} \lim_{n\to\infty} \sqrt{n}\mathbb{P}_{i,z}\left(\frac{S_n}{\sqrt{n}\sigma} \leq t, X_n = j, Z_m > 0, \tau_y > n\right)$$
$$= \frac{2\Phi^+(t)}{\sqrt{2\pi}\sigma} V(i,y)U(i,y,z)\boldsymbol{\nu}(j). \quad (20)$$

Proof of Yaglom theorem (Theorem 3)

1. Using the fact that $\mu_{i,z}$ is a probability mesure of mass 0 in 0 we sho that

$$\lim_{n\to+\infty} \sqrt{n}\mathbb{P}_{i,z}\left(\left| \frac{\log Z_n}{\sigma\sqrt{n}} - \frac{S_n}{\sqrt{n}\sigma} \right| \geq \varepsilon, X_n = j, Z_n > 0 \right) = 0.$$

As ε is arbitrary, using Theorem 2 we conclude the proof.

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Thank you !!!

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