Extremes of Gaussian non-stationary processes and honest confidence bands for densities

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Asymptotics of $\mathbb{P}\{\sup_{t\in M} X(t) \geq u\}$

Trivial inequality. For a Gaussian process $X = \big(X(t)\big)_{t\geq 0}$ and a set $M \subset \mathbb{R}$

$$\mathbb{P}\{\sup_{t\in M} X(t) \geq u\} \geq \sup_{t\in M} \mathbb{P}\{X(t) \geq u\} = \Psi\big(u/\sigma_M\big)$$

with $\sigma_M^2 = \sup_{t\in M} \operatorname{Var} X(t)$.

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First aim

To show that for some function $\mathscr{P}(u)$ and some $\theta > 0$

$$\mathbb{P}\Big\{\sup_{t\in M} X(t) \geq u\Big\} = \mathscr{P}\big(u\big)\big(1 + O(e^{-u^2\theta})\big), \qquad u \to \infty.$$

Technical side: Rice's method of moments, estimation of the second factorial moment of the number of up-crossings

Honest confidence sets

Given: $X_1, ...X_n \sim f$,
f is an infinitely dimensional object (CDF, PDF,...).

Definition. (*Li, 1989*) Given any $\alpha \in (0, 1)$, we aim to construct $(1 - \alpha)$-confidence set $\mathcal{C}_n(x)$ for f that are *honest* to a given class $\mathcal{F}$ of functions in the sense

$$\sup_{f \in \mathcal{F}} \mathbb{P}\Big\{ f(x) \in \mathcal{C}_n(x),\ \forall x \in \mathbb{R} \Big\} \geq 1 - \alpha + e_n,$$

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Honest confidence sets for CDF: Dvoretsky-Kiefer-Wolfowitz inequality (in strong form - *Massart, 1990*)

$$\mathbb{P}\Big\{ \sqrt{n} \sup_{u \in \mathbb{R}} \Big| \hat{F}_n(u) - F(u) \Big| > x \Big\} \leq 2e^{-2x^2}, \qquad \forall x > 0.$$

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Second aim

To construct honest confidence sets for PDF, which are

- based on the projection estimates;
- honest at polynomial rate e_n with $|\mathcal{C}_n(x)| \sim n^{-1/2}$

Confidence bands for densities

Maximal deviation: for an estimate $\hat{p}_n$ of the density function p, denote

$$\mathcal{D}_n := \sup_{u \in \mathbb{R}} \frac{\left|\hat{p}_n(u) - p(u)\right|}{\sqrt{p(u)}}.$$

SBR-type ("Smirnov-Bickel-Rosenblatt") limit theorems

$$\sup_{p \in \mathcal{F}} \left| \mathbb{P}\left\{ \mathcal{D}_n \le \frac{x}{a_n} + b_n \right\} - e^{-e^{-x}} \right| \to 0$$

for some sequences a_n and b_n tending to infinity as $n \to \infty$.

Smirnov (1950), Bickel and Rosenblatt (1973), Konakov and Piterbarg (1984), Giné, Koltchinskii, Sakhanenko (2004), Giné and Nickl (2010), Bull (2012).

Giné and Nickl (2016). Mathematical foundations of of infinitely-dimensional statistical models.

Challenges

1. SBR-type theorems are known only for kernel density estimate and *certain* wavelet projection density estimates like Haar wavelets or Battle-Lemarie wavelets.

   Chernozhukov, Chetverikov and Kato (Annals of Statistics, 2014):
   ... the SBR condition has not been obtained for other density estimators such as nonwavelet projection kernel estimators based, for example, on Legendre polynomials or Fourier series.

2. The rates of convergence.

   Giné and Nickl (Annals of Statistics, 2010):
   ... we finally remark that the results in this article are clearly of an asymptotic (and hence "theoretical" nature).

Key ingredient

Komlós - Major -Tusnady construction

Most estimates can be represented as

$$\hat{p}_n(x) = \int_{\mathbb{R}} K(x,y) d\mathbb{P}_n(y).$$

The analysis of $\mathcal{D}_n$ leads to the study of asymptotic behaviour of the Gaussian process

$$\Upsilon(x) = \int_{\mathbb{R}} K(x,y) dW(y).$$

Examples:

1. Kernel density estimates: $K(x,y) = \mathcal{K}((x-y)h^{-1})/h^{-1}$.
   $\Upsilon(x)$ is a stationary process.
2. Wavelets: $K(x,y) = 2^j \sum_k \phi(2^j x - k)\phi(2^j y - k)$, $j \in \mathbb{N}$.
   Then $\Upsilon(x)$ is a nonstationary sprocess of some special type:
   $r(x, x+u)$ is periodic in x with the same period for any u.
   Cyclostationary processes: *Konstant and Piterbarg (1993), Hüsler, Piterbarg, and Seleznjev (2003)*

Two types of projection estimates

Let $\Psi := \{\psi_0, \psi_1, \psi_2, \dots\}$ be an orthonormal basis of $L^2([A, B])$.

1. Consider with $J \to \infty$

$$\hat{p}_n(x) = \sum_{j=0}^{J} \left[\int \psi_j(y) d\mathbb{P}_n(y) \right] \psi_j(x).$$

2. Let us divide $I := [A, B]$ on M subintervals, and on each subinterval $I_m = [a_m, b_m] := [A + \delta(m-1), A + \delta m]$, $m = 1..M$, we reproduce Ψ:

$$\psi_j^{(m)}(x) = \sqrt{M} \cdot \psi_j \left(M(x - a_m) + A \right), \quad m = 1..M \;\; j = 0, 1, \dots.$$

Consider with $M \to \infty$

$$\hat{p}_n(x) = \sum_{m=1}^{M} \sum_{j=0}^{J} \left[\int \psi_j^{(m)}(y) d\mathbb{P}_n(y) \right] \psi_j^{(m)}(x).$$

In both cases

$$\Upsilon(x) = \int_I \left(\sum_{j=0}^{J} \psi_j(x)\psi_j(y) \right) dW(y).$$

Building bridge to the Gaussian process $\Upsilon(x)$

$$\mathcal{P}_{q,H,\beta} := \Big\{ p - \text{ p.d.f. }, \quad p \in L^2([A,B]), \quad \inf_{x\in[A,B]} p(x) \ge q, \\ |p(x) - p(y)| \le H|x-y|^{\beta} \Big\}.$$

Theorem

There exists a positive constant κ such that for any $p \in \mathcal{P}_{q,H,\beta}$ and any $u \in \mathbb{R}$ it holds

$$\mathbb{P}\left\{ \sqrt{\frac{n}{M}}\mathcal{R}_n \le u \right\} \le \left[\mathbb{P}\left\{ \sup_{x\in\mathbb{R}} |\Upsilon(x)| \le u + \gamma_{n,M} \right\} \right]^M + \mathcal{C}_1 n^{-\kappa},$$

$$\mathbb{P}\left\{ \sqrt{\frac{n}{M}}\mathcal{R}_n \le u \right\} \ge \left[\mathbb{P}\left\{ \sup_{x\in\mathbb{R}} |\Upsilon(x)| \le u - \gamma_{n,M} \right\} \right]^M - \mathcal{C}_1 n^{-\kappa},$$

where $\mathcal{R}_n := \sup_{u\in\mathbb{R}} \frac{\left|\hat{p}_n(u) - \mathbb{E}\hat{p}_n(u)\right|}{\sqrt{p(u)}}$, $\gamma_{n,M} = \mathcal{C}_2 \frac{\log(n)}{\sqrt{n/M}} + \mathcal{C}_3 \frac{\sqrt{\log(n)}}{\sqrt{M}}$,
and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 > 0$ depend on q, H, β.

Main result

Theorem

Let $X(t)$ be a centered Gaussian process with a.s. twice differentiable trajectories. Assume that the variance $\sigma^2(t) = \mathrm{Var}(X(t))$ reaches its maximum σ_M^2 at only one point $t_\circ \in [A, B]$. For $\delta > 0$, introduce *the informative set*

$$\mathcal{M}(\delta) := \left\{ t \in [A, B] : \sigma^2(t) \geq \frac{\sigma_M^2}{1+\delta} \right\}.$$

Then there exists some $\chi > 0$ (depending on δ and the process X_t) such that

$$\mathbb{P}\Big\{ \max_{t \in [A,B]} |X(t)| \geq u \Big\} = \mathscr{P}(u)\big(1 + O\big(e^{-u^2\chi/(2\sigma_M^2)}\big)\big), \qquad u \to \infty,$$

where

$$\mathscr{P}(u) = \begin{cases} 2\mathbb{E}\left[N_u^+(\mathcal{M}(\delta))\right], & \text{if } t_0 \in (A, B], \sigma'(t_0) = 0, \\ 2\Psi\big(u/\sigma_M\big) + 2\mathbb{E}\left[N_u^+(\mathcal{M}(\delta))\right], & \text{if } t_0 = A, \sigma'(t_0) = 0, \\ 2\Psi\big(u/\sigma_M\big), & \text{if } t_0 = A \text{ or } t_0 = B \\ & \quad \text{and } \sigma'(t_0) \neq 0. \end{cases}$$

Corollaries from the main result

Standard setup: $\operatorname{argmax} \sigma^2(t) = \{t_\circ^{(1)}, \ldots, t_\circ^{(K)}\}$. Let us choose disjoint intervals $\mathcal{M}_i,\ t_\circ^{(i)} \in \mathcal{M}_i,\ i = 1, \ldots, K$:

$$\max_{(s,t)\in\mathcal{M}_i\times\mathcal{M}_j} \rho(s,t) < 1, \qquad \forall i, j = 1..k,\ i \neq j.$$

Then there exists some $\chi > 0$ such that

$$\mathbb{P}\left\{\max_{t\in[A,B]} |X(t)| \geq u\right\} = \sum_{i=1}^{k} \mathscr{P}_i(u)\left(1 + O\left(e^{-u^2\chi/(2\sigma_M^2)}\right)\right),$$

where $\mathscr{P}_i(u)$ are defined above for $\mathbb{P}\left\{\max_{t\in\mathcal{M}_i} |X(t)| \geq u\right\}$.

Outcome for the process: $\Upsilon(x)$

There exists some $\chi = \chi(\delta) > 0$ such that

$$\mathbb{P}\left\{\max_{t\in[-1,1]} |\Upsilon(t)| \geq u\right\} = 4\Psi\left(u/\sigma_M\right)\left(1 + O\left(e^{-u^2\chi/(2\sigma_M^2)}\right)\right).$$

Dependence between χ and δ

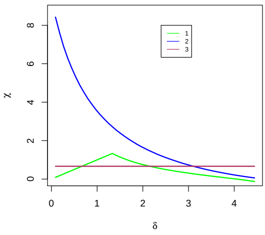
Let δ be a small number such that $\mathcal{M}(\delta) \cap [-1, 0] = [-1, b]$.
Then $\chi < \min\left(\chi_1(\delta), \chi_2(\delta), \chi_3(\delta)\right)$, where

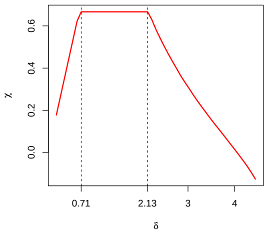
$$\chi_1(\delta) := \min\left\{\delta, \frac{4}{(b-A)J(J+2)} - 1\right\},$$

$$\chi_2(\delta) := \min_{t\in[-1,b]} \left(r_{10}^2(t,t)/r_{11}(t,t)\right),$$

$$\chi_3(\delta) := \begin{cases} J/(J+2), & J \text{ is even}, \\ (J+2)/J, & J \text{ is odd}. \end{cases}$$

Empirical result for $J = 4$: $\chi_{max} = 2/3$, for any $\delta \in (0.71, 2.13)$.





Return to statistical problem

Theorem

Assume that $p \in \mathcal{P}_{q,H,\beta}$ with some $q, H > 0$, $\beta \in (0,1]$. Denote the sequence of distribution functions

$$A_M(x) := \begin{cases} \exp\Big\{-M\sum_{i=1}^{k} \mathscr{P}_i(x)\Big\}, & \text{if } x \geq c_M, \\ 0, & \text{if } x < c_M, \end{cases}$$

where $c_M = (2S\log M)^{1/2} - S$. If $M = \lfloor n^{\lambda} \rfloor$ with $\lambda \in ((2\beta+1)^{-1}, 1)$, then

$$\boxed{\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left\{ \sqrt{\frac{n}{M}}\mathcal{D}_n \leq x \right\} - A_M(x) \right| \leq \bar{c}n^{-\gamma}.}$$

for some positive constants $\bar{c}$ and γ.

Honest confidence bands

Denote

$$k_{\alpha,M} := \sqrt{M_n/n} \cdot q_{\alpha,M},$$

where $q_{\alpha,M}$ is the $(1-\alpha)$- quantile of the distribution function $A_M(\cdot)$.

Then

$$\mathbb{P}\left\{ \frac{|\hat{p}_n(x) - p(x)|}{\sqrt{p(x)}} \le k_{\alpha,M}, \quad \forall x \in I \right\} = 1 - \alpha + e_{n,M},$$

where $e_{n,M}$ converges to zero at polynomial level in both n and M.

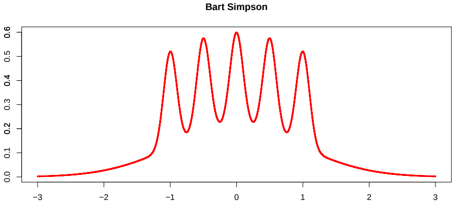
$$\mathcal{C}_n(x) := \Big(\hat{p}_n(x) + (k^2_{\alpha,M}/2) - \left[\hat{p}_n(x) k^2_{\alpha,M} + (k^4_{\alpha,M}/4)\right]^{1/2},$$
$$\hat{p}_n(x) + (k^2_{\alpha,M}/2) + \left[\hat{p}_n(x) k^2_{\alpha,M} + (k^4_{\alpha,M}/4)\right]^{1/2} \Big)$$

is $(1-\alpha)-$confidence set, which is honest to a class $\mathcal{P}_{q,H,\beta}$ at polynomial rate.

Numerical example

Consider the density (Bart Simpson density; the claw)

$$p(x) = \frac{1}{2}\phi_{(0,1)}(x) + \frac{1}{10}\sum_{j=0}^{4}\phi_{((j/2)-1,1/100)}(x),$$



Approximate the distribution of $\mathcal{D}_n = \sup_{u\in\mathbb{R}} |\hat{p}_n(u) - p(u)|/\sqrt{p(u)}$ via

$$A_M(x) := \exp\left\{-4M\left(1 - \Phi\left(\sqrt{6}x/(J+1)\right)\right)\right\} \cdot I\{x \geq c_M\}$$

where $c_M = \frac{(J+1)}{\sqrt{3}}\sqrt{\log(M)} - \frac{(J+1)^2}{6}$.

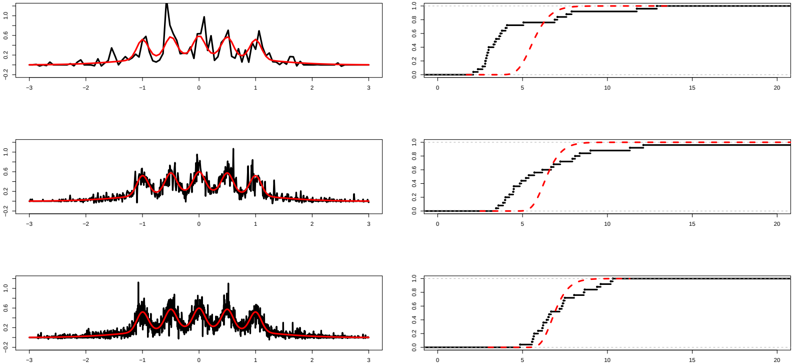


Figure: First raw: projection density estimates (black solid lines) in comparison with the true densities (red lines) based on $n = 500, 3000, 10000$ simulations. In this example, we take $M = \lfloor n^{2/3} \rfloor$. Second raw: empirical distribution functions of $\sqrt{n/M} \cdot \mathcal{D}_n$ (black solid lines) based on 25 simulation runs in comparison with the distribution function $A_M(x)$.

Summary

1. Extremes of Gaussian non-stationary processes: a new theoretical result revealing the asymptotic behaviour of Gaussian processes, **which are neither stationary nor cyclostationary**.
2. Sequence of accompanying laws, which approximates the distribution of maximal deviation at polynomial rate. Confidence sets, which are honest to some classes of densities **at polynomial rate**.
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Thank you for your attention.