On the convergence of Gaussian convex hulls

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Moscow
October 2020

Problem.

The talk is devoted to the review of a number of recent results about asymptotic behavior of convex hulls

$$V_n = \frac{1}{b_n} \mathrm{conv}\{X_1, \dots, X_n\},$$

where (X_k) is a sequence of Gaussian random elements of some linear space $\mathbb{B}$, and b_k are normalizing constants. The main attention is paid to results about the existence of a limit shape and properties of its border.

Motivation.

Let

$$W_n = \mathrm{conv}\{X_1, \ldots, X_n\}. \quad (1)$$

The first point of motivation: if $\mathbb{B} = \mathbb{R}^1$, then W_n is the segment

$$[\min\{ X_1, \ldots, X_n, \}, \max\{ X_1, \ldots, X_n, \}].$$

It means that in some sense our problem is closely connected with the classical theory of extrema.

It is well known that for orthonormal Gaussian r.v. with probability 1

$$V_n = \frac{1}{\sqrt{2 \ln n}} \max\{X_1, \ldots, X_n\} \to 1,$$

which implies that in dimension 1 a.s.

$$V_n \to [-1, 1]$$

The second point is related to the study of convex hulls of paths of random processes.
Let $X_i = \{X_i(t),\ t \in T\}$ be $\mathbb{R}^d$-valued processes on a parametric set T. The random process X_j can be considered as a random element of some functional space $\mathbb{B}$. Consider the convex hulls

$$U_n = \mathrm{conv}\{\, X_1(t), \ldots, X_n(t), \quad t \in T\}. \tag{2}$$

Then in many cases from the convergence of V_n (in $\mathbb{B}$) we can easily deduce important information on asymptotic behavior of $\frac{1}{D_n} U_n$.

Motivation.

My personal interest to the question was motivated by the paper by J. Randon-Furling, Satya N. Majumdar and A. Comptet (2009) inspired by an interesting implication in ecological context in estimating the home range of a herd of animals with population size n. Mathematical results of these articles consist in exact computation of a mean perimeter L_n and area A_n of W_n in the case when $d = 2$ and X is a standard Brownian motion on $T = [0, 1]$. It was shown that

$$L_n \sim 2\pi\sqrt{2\ln n}, \quad A_n \sim 2\pi \ln n, \quad n \to \infty. \tag{3}$$

The relation between L_n and A_n being the same as the relation between the perimeter and area of a circle of the radius $\sqrt{2\ln n}$, it seems credible to suppose that W_n rounds up with the growth of n.

Notation.

$\mathcal{K}(\mathbb{B})$ is the space of compact convex subsets of a Banach space $\mathbb{B}$ provided with Hausdorff distance $\rho_{\mathbb{B}}$:

$$\rho_{\mathbb{B}}(A,B) = \max\{\inf\{\,\epsilon \mid A \subset B^{\epsilon}\},\ \ \inf\{\,\epsilon \mid B \ \subset A^{\epsilon}\}\},$$

A^{ϵ} is the open ϵ-neighbourhood of A.
We set $\mathcal{K}^d = \mathcal{K}(\mathbb{R}^d)$ and $\rho = \rho_{\mathbb{R}^d}$.

X is a centered Gaussian random element of $\mathbb{B}$.
$\mathcal{E}$ is the ellipsoid of concentration of X, that is the closed unit ball in the reproducing kernel Hilbert space of the distribution of X.

[In the finite dimentional case $\mathcal{E} = \{x \in \mathbb{R}^d \mid \langle R^{-1}x, x\rangle \leq 1\}$.]

Goodman's theorem.

Goodman, 1988

Suppose that X_k are i.i.d. $\mathbb{B}$-valued centered Gaussian random variables, the space $\mathbb{B}$ is supposed to be separable. Then with probability 1,

$$\max_{i \le n} d(X_i, \ \sqrt{2 \ln n}\, \mathcal{E}) \to 0,$$

$$\max_{y \in \sqrt{2 \ln n}\, \mathcal{E}} d(y, \ \{X_1, \ldots, X_n\}) \to 0$$

as $n \to \infty$. Here $d(\, , \,)$ denotes the Banach norme distance from a point to a set.

Orthonormal sequences. Limit shape.

Theorem 1

Let X be a centered Gaussian random element of a separable Banach space $\mathbb{B}$. Let (X_i) be a sequence of i.i.d. copies of X and W_n be the convex hull defined by (1). Then with probability 1

$$\rho\left(\frac{1}{\sqrt{2\ln n}}W_n,\ \mathcal{E}\right) = o\left(\frac{1}{\sqrt{\ln n}}\right), \tag{4}$$

where $\mathcal{E}$ is the ellipsoid of concentration of X.

Proof of Theorem 1 follows directly from Goodman's result.

Convex hulls of Gaussian processes.

T is a separable metric space.

$X = \{X(t),\ t \in T\}$ is a bounded centered Gaussian process with values in $\mathbb{R}^d$.

(X_i) is a sequence of i.i.d. copies of X.
$U_n = \mathrm{conv}\{X_1(t), \dots, X_n(t),\ t \in T\}$ (in $\mathbb{R}^d$).
R_t is the covariance matrix of $X(t)$.
$\mathcal{E}_t$ is the ellipsoid of concentration of $X(t)$:

$$\mathcal{E}_t = \{x \in \mathbb{R}^d \mid \langle R_t^{-1} x, x \rangle \leq 1\}.$$

Convex hulls of Gaussian processes.

Theorem 2 (D'2011)

1) Let $X = \{X(t),\ t \in T\}$ be a bounded centered Gaussian process with values in $\mathbb{R}^d$. Let (X_i) be a sequence of i.i.d. copies of X and U_n be defined by (1).
Then with probability 1

$$\frac{1}{\sqrt{2 \ln n}} U_n \xrightarrow{\mathcal{K}^d} U, \tag{5}$$

where $U = \mathrm{conv}\{\mathcal{E}_t,\ t \in T\}$.

2) If T is compact and X is continuous, then a.s.

$$\rho\left(\frac{1}{\sqrt{2 \ln n}} U_n,\ U\right) = o\left(\frac{1}{\sqrt{\ln n}}\right). \tag{6}$$

Examples.

Brownian motion. Let X be a standard d-dimensional Brownian motion on $T = [0, 1]$. Then $\mathcal{E}_t = \sqrt{t} B_d(0, 1)$ and the limit shape is $U = \mathcal{E}_1 = B_d(0, 1)$.

Self-similar processes. $U = \mathcal{E}_1$.

Fractional Brownian motion (FBM). $U = \mathcal{E}_1$.
Fractional Brownian Bridge. It is a fractional Brownian motion Y under the condition $Y(1) = 0$. It coincides in distribution with the process

$$X(t) = \{X_1(t), \ldots, X_d(t)\}, \quad X_i(t) = Y_i(t) - r(t, 1) Y_i(1), \quad t \in [0, 1].$$

It is clear that $\mathcal{E}_t = \sigma(t) B_d(0, 1)$, where
$\sigma^2(t) = t^{2H} - \frac{1}{4}(t^{2H} + 1 - |1 - t|^{2H})^2$. The function σ^2 reaches its maximum at $t = \frac{1}{2}$ and $\sigma^2_{\max} = \frac{1}{2^{2H}} - \frac{1}{4}$. Finally we see that $U = \sigma_{\max} B_d(0, 1)$.

Weak dependent sequences.

Let (X_n) be a sequence of Gaussian $\mathbb{B}$-valued random elements satisfying the following condition:

They have the same distribution $\mathcal{P}$ and for all $x^\star \in \mathbb{B}^\star$

$$\mathbf{E}\langle X_k, x^\star\rangle\langle X_l, x^\star\rangle \to 0 \ \text{ as } \ |k - l| \to \infty. \tag{7}$$

Theorem 3 (D&V. Paulauskas'2014)

Under condition (7) with probability 1

$$\frac{1}{\sqrt{2\ln n}} W_n \to \mathcal{E},$$

where $\mathcal{E}$ is the concentration ellipsoid of $\mathcal{P}$.

Stationary fields.

Let now $X = \{X_t, t \in \mathbb{R}^m\}$ be a Gaussian stationary continuous m-parametric random field and (X_j) are i.i.d. copies of X.
Let (T_n) be an increasing sequence of subsets of $\mathbb{R}^m$ such that $\nu_n = \lambda^m(T_n) \to \infty$ and for all $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{\lambda^m\{(\partial T^n)^\varepsilon\}}{\lambda^m(T_n)} = 0. \tag{8}$$

Theorem 4 (D&V. Paulauskas'2014)

Suppose that conditions of weak dependence (7) and (8) are fulfilled. Then a.s.

$$\frac{1}{\sqrt{2\ln(\nu_n)}} U_n \to \mathcal{E}.$$

Weak convergent sequences.

In condition (7):

R.v. X_n have the same distribution $\mathcal{P}$ and for all $x^\star \in \mathbb{B}^\star$

$$\mathbf{E}\langle X_k, x^\star\rangle\langle X_l, x^\star\rangle \to 0 \quad \text{as} \quad |k-l| \to \infty$$

the equality of marginal distributions can be essentially relaxed substituting it by the weak convergence of the sequence $\{X_n\}$.

Theorem 5 (D&V. Paulauskas'2020)

Suppose that a centered Gaussian sequence of $\mathbb{B}$-valued random elements $\{X_k, k \in \mathbb{N}\}$ satisfies (7) and the following condition:

$$X_n \Rightarrow X. \tag{9}$$

Then a.s.

$$\frac{1}{\sqrt{2\ln n}} W_n \overset{\mathcal{K}_{\mathbb{B}}}{\to} \mathcal{E}, \quad \text{as } n \to \infty. \tag{10}$$

Examples

It is clear, that if the dependence between elements X_k of the sequence is stronger, the sequence of their convex hulls is more concentrated.

1st example

Consider the extreme case, when $X_k \equiv X_0$ for all $k \geq 1$.
Then $W_n = \{X_0\}$ is one point and

$$\lim_{n\to\infty} \frac{1}{g(n)} W_n = \{0\}$$

for any sequence $g(n) \to \infty$.

Examples

Consider the case $\mathbb{B} = \mathbb{R}$.
Let $\{X_k\},\ k \in \mathbb{Z}_+$, be a sequence of $N(0, \sigma_k^2)$ random variables. We can assume that X in (9) is $N(0, 1)$ and $\sigma_k^2 \to 1$.
We have

$$W_n = [\min\{X_1, X_2, \ldots, X_n\}, \max\{X_1, X_2, \ldots, X_n\}] \tag{11}$$

and $\mathcal{E} = [-1, 1]$.
If the covariance function $\rho(m, n) := EX_mX_n \to 0$ as $n, m, |n - m| \to \infty$, then the relation (10) holds.

Examples

2nd example

Let $X_n = p\xi_n + q\xi_0,\ p, q > 0,\ p^2 + q^2 = 1, n = 1, 2, \dots,$
where $\{\xi_k, k \geq 0\}$, are orthogaussian r.v.
Then for $m \neq n$ $EX_nX_m = q^2 \nrightarrow 0$ when $|m - n| \to \infty$.
Now

$$W_n = \mathrm{conv}\{X_k, k = 1, \dots, n\} = \mathrm{conv}\{p\xi_k + q\xi_0, k = 1, \dots, n\}$$
$$= p\,\mathrm{conv}\{\xi_1, \dots, \xi_n\} + q\xi_0.$$

Hence, a.s.,

$$\frac{1}{\sqrt{2\ln n}} W_n \to p[-1, 1].$$

Example 3

Let $\{\xi_j\},\ j \geq 1$, be orthogaussian variables and let $X_k = k^{-1/2} \sum_{j=1}^{k} \xi_j$.
We are in the setting of Theorem 5, but the condition (7) is not satisfied, since, if $n = m + k, k > 0$, then

$$EX_m X_n = \frac{ES_m S_{m+k}}{(m(m+k))^{1/2}} = \frac{m}{(m(m+k))^{1/2}} = \left(1 + \frac{k}{m}\right)^{-1/2}. \qquad (12)$$

On the other hand, we have:

Example 3

Proposition

With probability one

$$\frac{1}{c(n)} W_n \overset{\mathcal{K}_{\mathbb{R}}}{\to} [-1, 1], \quad \text{as } n \to \infty, \tag{13}$$

where $W_n = \mathrm{conv}\{X_1, \ldots, X_n\}$ *and* $c(n) = \sqrt{2 \ln \ln n}$.

Open questions

In connection with these examples it is possible to formulate the following problem.

Suppose that a sequence $\{X_n\}$ has standard normal marginal distributions and covariance function $\rho(m, n)$.

For which functions $g(n)$ and under what conditions for covariance function ρ we can get the relation (13) with function g instead of c?

This Proposition and Theorem 5 give us two examples of such functions g. What other normalizing functions are possible in relation (13)?

Limit shape

In previous results for Gaussian sequences the limit set of convex hulls was the dispersion ellipsoid of limit Gaussian measure. If we dismiss the condition of weak convergence of Gaussian sequence $\{X_k\}$, the limit set may exist, but will not necessarily be an ellipsoid.

Theorem 6 (D'2020)

Let $\mathbb{B}$ be a separable Banach space. Let $V \subset \mathbb{B}$ be a central symmetric compact convex set. Then there exists a sequence of independent Gaussian vectors $\{X_k\}$ such that a.s.

$$\frac{1}{b(n)} W_n \ \rightarrow \ V.$$

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