Branching Diffusions in Inhomogeneous Media

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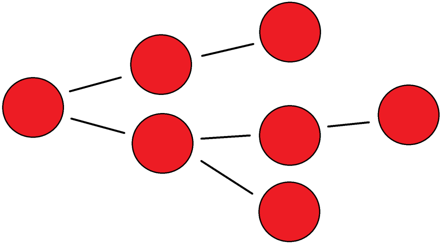
Parts of the talk are based on joint work with:

(a) D. Dolgopyat, P. Hebbar, M. Perlman (on multi-type branching processes)

(b) S. Molchanov (on branching with constant and with compactly supported potentials)

(c) P. Hebbar, J. Nolen (on branching in periodic media)

Single-type branching process



Example: $Z_0 = 1$, $Z_1 = 2$, $Z_2 = 3$, $Z_3 = 1$, $Z_4 = 0$.

Time-homogeneous - distribution of the number of children is prescribed: $\mathrm{P}(Z_1 = m | Z_0 = 1)$ is given for each m.

Under mild assumptions, everything depends on the average number of children $\lambda = \mathrm{E}(Z_1 | Z_0 = 1)$. $\lambda < 1$ - process is sub-critical, $\lambda = 1$ - critical, or $\lambda > 1$ - super-critical.

More on the types of behavior

1) Subcritical ($\lambda < 1$) and critical ($\lambda = 1$).

$$\sum_{k=1}^{\infty} \frac{1}{\lambda^k} = \infty.$$

1a) Sub-critical: Distribution of Z_n, conditioned on survival ($Z_n \neq 0$) tends to a limit. The probability of survival, $\mathrm{P}(Z_n \neq 0)$ decays exponentially.

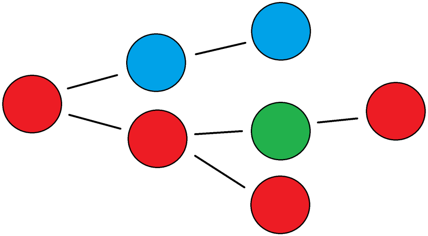
1b) Critical: The probability of survival decays as $1/n$. Distribution of Z_n, conditioned on survival and normalized, tends to the exponential distribution.

$$\lambda^n \sum_{k=1}^{n} \frac{1}{\lambda^k} \to \infty \quad \text{as} \quad n \to \infty.$$

2) Super-critical: Asymptotically, the population is a non-random multiple of a random variable that doesn't depend on n. The population grows exponentially.

$$\sum_{k=1}^{\infty} \frac{1}{\lambda^k} < \infty.$$

Multi-type branching process



Example: $Z_0 = (1, 0, 0)$, $Z_1 = (1, 1, 0)$, $Z_2 = (1, 1, 1)$, $Z_3 = (1, 0, 0)$, $Z_4 = (0, 0, 0)$.

Distribution of the number of children of each type is prescribed:
$\mathrm{P}(Z_1 = (m_1, ..., m_d)|Z_0 = e_j)$ is given for each $(m_1, ..., m_d)$ and each coordinate vector e_j.

Everything depends on $A(j, i) = \mathrm{E}(Z_1(i)|Z_0 = e_j)$ or, rather, the top eigenvalue λ.

Time-inhomogeneous process

Now we assume that $\mathrm{P}(Z_{n+1} = (m_1, ..., m_d)|Z_n = e_j)$ is given for each n, $(m_1, ..., m_d)$ and each coordinate vector e_j.

$A_n(j, i) = \mathrm{E}(Z_{n+1}(i)|Z_n = e_j)$.
$A_0 A_1 ... A_{n-1}(j, i) = \mathrm{E}(Z_n(i)|Z_0 = e_j)$. Λ_n is the top eigenvalue of this matrix.
In the single-type time-homogeneous case, $\Lambda_n = \lambda^n$.

Mild non-degeneracy assumptions:
(A1) $\mathrm{P}(Z_{n+1}(i) \geq 2|Z_n = e_j) \geq \varepsilon$;
(A2) $\mathrm{P}(Z_{n+1} = (0, ..., 0)|Z_n = e_j) \geq \varepsilon$;
(A3) $\mathrm{E}(\|Z_{n+1}\|^2|Z_n = e_j) \leq K$ (plus uniform integrability for our second theorem).

Theorem 1. (Criterion for sub-critical or critical behavior vs the super-critical behavior)

(a) Extinction for some initial population $\Rightarrow$ $\sum_{k=1}^{\infty}(1/\Lambda_k) = \infty$.

(b) $\sum_{k=1}^{\infty}(1/\Lambda_k) = \infty \Rightarrow$ Extinction for every initial population.

Theorem 2. (Criterion for sub-critical vs critical behavior) Assume that extinction occurs.
(a) For some initial population, the distribution of $|Z_n|$, conditioned on survival and normalized, tends to the exponential distribution $\Rightarrow$ $\Lambda_n \sum_{k=1}^{n}(1/\Lambda_k) \to \infty$.

(b) $\Lambda_n \sum_{k=1}^{n}(1/\Lambda_k) \to \infty \Rightarrow$ for every initial population, the distribution of $|Z_n|$, conditioned on survival and normalized, tends to the exponential distribution

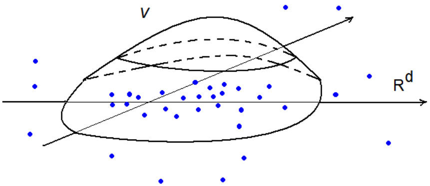
Earlier results (single-type processes): Jagers (1974), Agresti (1975).

An ingredient in the proof

Irrespective of the initial population, if the population at time n is large, then $(Z_n(1), Z_n(2), ..., Z_n(d))$ (which is random) is nearly a random scalar multiple of the top eigenvector of $A_0 A_1 ... A_{n-1}$ (with eigenvalue Λ_n).

So, the problem is almost one-dimensional: just track the magnitude of the population, since you almost know the (changing) distribution between the types.

v - intensity of branching.



Initially - one particle is located at $x \in \mathrm{R}^d$. Goal is to describe the distribution of particles when t is large.

One can treat particle location as its type; instead of $A_n(j,i) = \mathrm{E}(Z_{n+1}(i)|Z_n = e_j)$, look at

$$L_t u(x) = \frac{1}{2}\Delta u(x) + v(t,x)u(x).$$

Open problem

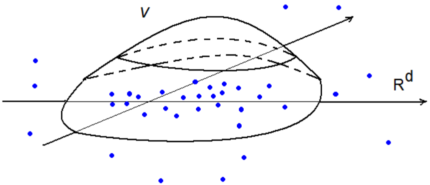
For time-dependent branching potential, can one distinguish between different types of behavior (sub-critical, critical, super-critical) based on the spectral properties of the operators L_t (or operators giving time t solutions to the time-dependent heat equation)?

If the phase space is compact, we can still look at the top eigenvalue of the evolution operator (analogue of Λ_n). If the space is not compact, there might be no eigenvalues, and the edge of the continuous spectrum may play a role.

Time-independent compactly supported potential: distribution of the population

v - time-independent. Sufficiently large for the existence of a positive eigenvalue of $Lu = \frac{1}{2}\Delta u + vu$ (super-critical case).

$$L\psi = \lambda_0 \psi$$



Question 1: Growth of the front with time (boundary of the set where particles can be found with sufficiently high probability (and how do we interpret that?)).

Let U_y be a domain of fixed size centered at $y \in \mathbb{R}^d$. At time zero, there is one particle at $x \in \mathbb{R}^d$. Average number of particles in U_y satisfies the equation

$$\frac{\partial \rho(t,x,y)}{\partial t} = \frac{1}{2}\Delta_x \rho(t,x,y) + v(x)\rho(t,x,y), \quad \rho(0,x,y) = \chi_{U_y}(x).$$

Equation is in x, while y is just a parameter. The front is the boundary of the region $A_t = \{y : \rho(t,0,y) \geq 1\}$. The average number of particles near $y \in A_t$ is one or more.

Probability of finding at least one particle in U_y satisfies the equation

$$\frac{\partial u(t,x,y)}{\partial t} = \frac{1}{2}\Delta_x u(t,x,y) + v(x)u(t,x,y)(1 - u(t,x,y)), \quad u(0,x,y) = \chi_{U_y}(x).$$

The front is the boundary of the region $B_t = \{y : u(t,0,y) \geq 1/2\}$. The probability of finding a particle near $y \in B_t$ is 1/2 or more.

Theorem. (Watanabe 1968) (total number of particles) When the initial particle is located at x, there is a random variable ξ^x, such that

$$\frac{N_t^x}{e^{\lambda_0 t}} \to \xi^x \quad \text{as} \quad t \to \infty.$$

(we can show that all the moments converge).

Theorem. (shape of the front) For each $\delta > 0$, there is $T > 0$ (random) such that, on the event $\xi^x > 0$, almos surely,

$$B((\sqrt{\lambda_0/2} - \delta)t) \subseteq B_t \subseteq B((\sqrt{\lambda_0/2} + \delta)t), \qquad t \geq T.$$

Front speads linearly, with the speed $\sqrt{\lambda_0/2}$.

$n_t^x(U)$ - number of particles in U.
Population is intermittent along $y(t)$ (say, $y(t) = rt$) if there is $k \geq 2$ such that

$$\frac{\mathrm{E}(n_t^x(U_{y(t)}))^k}{(\mathrm{E}n_t^x(U_{y(t)}))^k} \to \infty \quad \text{as} \quad t \to \infty.$$

Theorem. If $\|r\| < \sqrt{\lambda_0/2}$,

$$\frac{n_t^x(U_{rt})}{\mathrm{E}n_t^x(U_{rt})} \to \xi^x.$$

All the moments converge, so there is no intermittency inside the front.

$\rho_1(t,x,y_1)$ - particle density. For fixed y_1:

$$\partial_t \rho_1(t,x,y_1) = \frac{1}{2}\Delta \rho_1(t,x,y_1) + v(x)\rho_1(t,x,y_1),$$
$$\rho_1(0,x,y_1) = \delta_{y_1}(x).$$

This can be derived by examining the behavior of the original particle on the time interval $[0,\Delta t]$, as $\Delta t \downarrow 0$, with the first term on the right hand side coming from the effect of the diffusion and the second term from the branching.

$\rho_n(t,x,y_1,...,y_n)$ - higher order correlation functions (finding particles near y_1,...,y_n, simultaneously). For fixed $y_1,...,y_n$, they satisfy the equations

$$\partial_t \rho_n(t,x,y_1,...,y_n) = \frac{1}{2}\Delta \rho_n(t,x,y_1,...,y_n) +$$
$$+ v(x)\left(\rho_n(t,x,y_1,...,y_n) + H_n(t,x,y_1,...,y_n)\right),$$
$$\rho_n(0,x,y_1,...,y_n) \equiv 0.$$

Here

$$H_n = \sum_{U \subset Y, U \neq \emptyset} \rho_{|U|}(t,x,U)\rho_{n-|U|}(t,x,Y\setminus U).$$

Lemma 1: Let $K \subset \mathbb{R}^d$ be a compact set. For sufficiently small ε,

$$\rho_1(t,x,y) = \exp(\lambda_0 t)\psi(x)\psi(y) + q(t,x,y),$$

where

$$\sup_{x \in K} |q(t,x,y)| \le A_\varepsilon \exp((\lambda_0 - \varepsilon)t - |y|\sqrt{2(\lambda_0 - \varepsilon)}) \quad \text{for} \quad t \ge 1/2.$$

Lemma 2 Let $K \subset \mathbb{R}^d$ be a compact set. For sufficiently small ε,

$$\rho_n(t,x,y_1,...,y_n) = \exp(n\lambda_0 t) f_n(x)\psi(y_1)\cdot ... \cdot \psi(y_n)$$

$$+q_n(t,x,y_1,...,y_n),$$

where q_n decays sufficiently fast in t, y_1,...,y_n when $x \in K$. Here $f_1 = \psi$ and f_n can be defined explicitly (inductively, with a combinatorial formula).

Proof of the theorem on the asymptotic number of particles

For simplicity, let's look at a time-independent domain U. Want:

$$\frac{n_t^x(U)}{e^{\lambda_0 t}\int_U \psi(y)dy} \to \xi^x \quad \text{as} \quad t \to \infty.$$

Look at the asymptotics of the moments.

$$\mathbb{E}(n_t^x(U))^n = \int_U \dots \int_U \rho_n(t, x, y_1, \dots, y_n)dy_1 \dots dy_n + \text{contribution from } \rho_{n-1}, \dots, \rho_1.$$

Divide by $\exp(n\lambda_0 t)$ and check that the limiting quantities satisfy the Carleman condition.

Sufficient to check that

$$\sum_{n=1}^{\infty}\left(\frac{1}{f_n(x)}\right)^{\frac{1}{2n}} = \infty.$$

Now assume that $v(x) = a > 0$ (constant). No top eigenvalue/eigenfunction to look at, but we have nearly-explicit formulas for ρ_1, ρ_2, etc. In particular,

$$\partial_t \rho_1(t, x, y_1) = \frac{1}{2}\Delta \rho_1(t, x, y_1) + a\rho_1(t, x, y_1),$$

$$\rho_1(0, x, y_1) = \delta_{y_1}(x),$$

so

$$\rho_1(t, x, y_1) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{\|x - y_1\|^2}{2t}} e^{at}.$$

Solving $\rho_1(t, 0, y_1) = 1$ gives

$$\|y_1\| \approx (\sqrt{2a})t - \frac{d}{2\sqrt{2a}} \ln(t).$$

So, A_t grows linearly (up to a logarithmic correction). There is a further logarithmic correction for B_t (Bramson 1978).

Known: there is a random variable ξ, such that

$$\frac{N_t^x}{e^{at}} \to \xi \quad \text{as} \quad t \to \infty.$$

If $\|r\| < \sqrt{2a}$,

$$\frac{n_t^x(U_{rt})}{\mathrm{E} n_t^x(U_{rt})} \to \xi \;\; \text{in distribution} \quad \text{as} \quad t \to \infty.$$

Theorem. If $y(t) = o(t)$, then all the moments for $\frac{n_t^x(U_{y(t)})}{\mathrm{E} n_t^x(U_{y(t)})}$ converge to those of ξ.
If $0 < \|r\| < \sqrt{2a}$, then there is k such that

$$\frac{\mathrm{E}(n_t^x(U_{rt}))^k}{(\mathrm{E} n_t^x(U_{rt}))^k} \to \infty \quad \text{as} \quad t \to \infty.$$

So there is no intermittency at sub-linear distances and there is intermittency at linear, in time, distances from the location of the initial particle.

In the case of constant branching potential, we had

$$\partial_t \rho_1(t,x,y_1) = \frac{1}{2}\Delta \rho_1(t,x,y_1) + v(x)\rho_1(t,x,y_1),$$

$$\rho_1(0,x,y_1) = \delta_{y_1}(x),$$

with $v \equiv a > 0$, so we had

$$\rho_1(t,x,y_1) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{\|x-y_1\|^2}{2t}} e^{at},$$

which was relatively easy to put into the equation for ρ_2, etc. Now, we don't have an explicit expression for ρ_1, but we can use a precise (up to pre-exponential terms) asymptotics for ρ_1 at linear, in t, distances between x and y_1.

The behavior of the branching diffusion process is qualitatively similar to the case of homogeneous branching.

Combined effects ...

Assume that the branching potential is $v(x) + a$, where the top eigenvalue of $Lu = \frac{1}{2}\Delta u + (v + a)u$ is $\lambda_0 > a$. Killing is also allowed (either due to v changing sign or $a < 0$).

Theorem*. Up to certain, linear in t, distances from the origin, there is no intermittency. However, if $\lambda_0 - a \ll a$, then there is intermittency near the front.

Theorem*. One can describe the 'sharpness' of the front (i.e., whether B_t lags behind A_t by a logarithmic, in t, distance) in terms of the relationship between λ_0 and a.