Transforming i.d. random variables into each other with conditional expectations.

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Based on joint work with Stanislav A. Molchanov and Isaac M. Sonin

Stochastic Analysis and its Application in Economics RS at HSE

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Story begins in 2007, when A.Cherny and P.Grigoriev published a noteworthy result for Risk Theory.

Theorem 1

Let (Ω, F, P) be a nonatomic probability space, X ,Y two bounded functions with the same distribution. Then for any $\epsilon > 0$ there is a sequence of σ-subalgebras $F_1, F_2, \dots, F_n \subseteq F$ such that for a sequence of random variables $X_0 = X, X_1 = E(X_0|F_1)$, $X_2 = E(X_1|F_2), \dots, X_n = E(X_{n-1}|F_n)$, the following inequality holds $||X_n - Y||_\infty < \epsilon$.

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This result is highly connected to **Risk Theory**.

- One of the important objects studied in Financial Mathematics and Risk Theory is **risk measures**.
- For example, risk measure is used to determine the amount of an asset (currency) to be kept in reserve.
- A risk measure is defined as a mapping $\rho : \mathcal{L} \to \mathbb{R}$ from a set of random variables to the real numbers:
  1. **Normalized** $\rho(0) = 0$;
  2. **Translative** If $a \in \mathbb{R}, X \in \mathcal{L}$, then $\rho(X + a) = \rho(X) - a$;
  3. **Monotone** If $X, Y \in \mathcal{L}$ and $X \leq Y$, then $\rho(Y) \leq \rho(X)$.

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- For example: $\rho(X) = \mathbb{E}[-X]$.
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- Important family: **Law invariant risk measures**.

- **Coherent risk measures**: $\rho : L^\infty \to \mathbb{R}$ with
  1. **Normalized + Monotonicity + Translation invariance**
  2. **Subadditivity** $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
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Theorem (Föllmer, Schied in '04)

Any law invariant convex risk measure on an atomless probability space has to satisfy

$$\rho(\mathbb{E}(X|G)) \leq \rho(X)$$

for any $X \in L^\infty$ and $\mathcal{G} \subset \mathcal{F}$.

- Property $\rho(\mathbb{E}(X|G)) \leq \rho(X)$ for any $X \in \mathcal{L}^\infty$ and $\mathcal{G} \subset \mathcal{F}$ was introduced by Leitner in '04 and is called **dilatation monotoninicty**.
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Theorem

On an atomless probability space a convex risk measure is law invariant iff it is dilatation monotone.

Theorem 1

Let (Ω, F, P) be a nonatomic probability space, X ,Y two bounded functions with the same distribution. Then for any $\epsilon > 0$ there is a sequence of σ-subalgebras $F_1, F_2, \ldots, F_n \subseteq F$ such that for a sequence of random variables $X_0 = X, X_1 = E(X_0|F_1)$, $X_2 = E(X_1|F_2), \ldots, X_n = E(X_{n-1}|F_n)$, the following inequality holds $||X_n - Y||_\infty < \epsilon$.

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In other words, we can «fully» transform r.v. X into Y with the sequence of «averaging» operations.

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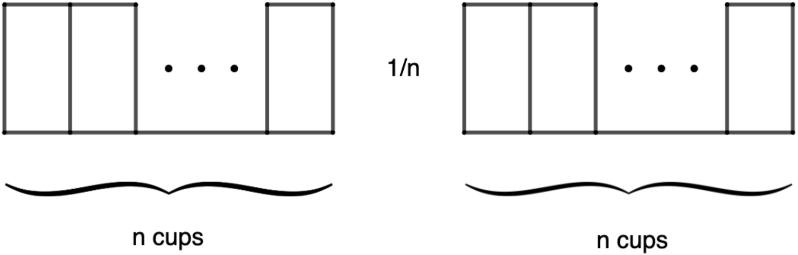
Main results:

- Transparent interpretation using «hydrostatic» setup;
- Explicit and optimal algorithm of transformations;
- Exact first term of the asymptotic behavior.

The structure:

- Transparent interpretation;
- Optimal algorithm for a simple case;
- Asymptotic behaviour of the optimal algorithm (Basic Lemma);
- Implication of the Basic Lemma to the Theorem 1.
- Optimality of the algorithm.

Transparent interpretation



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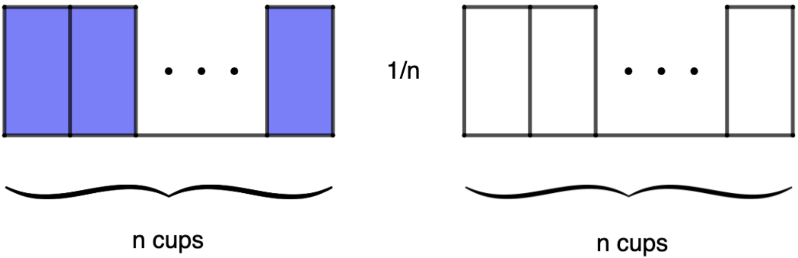

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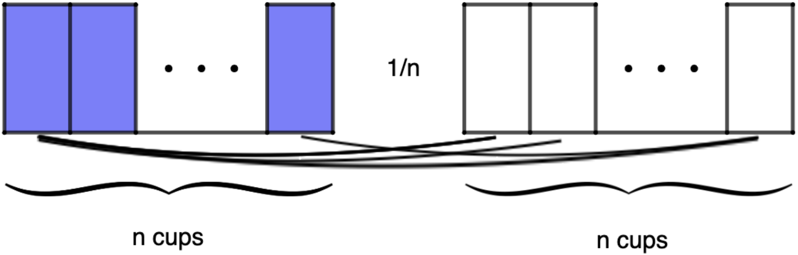

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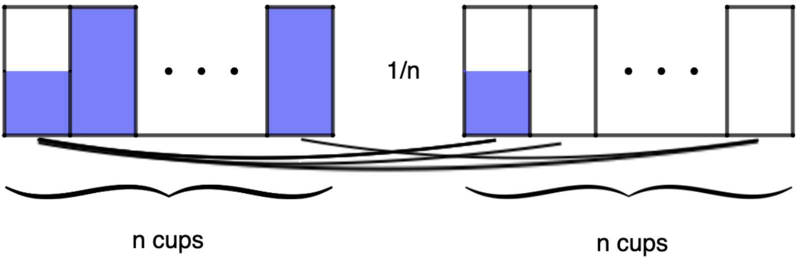

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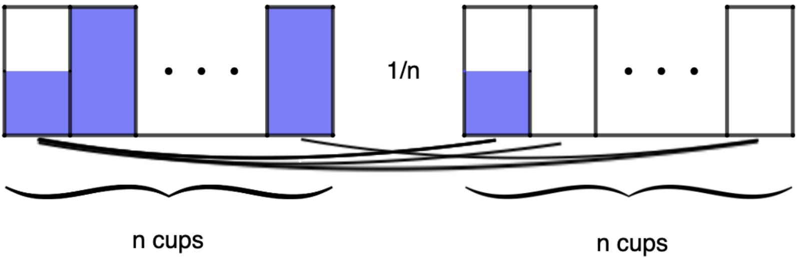
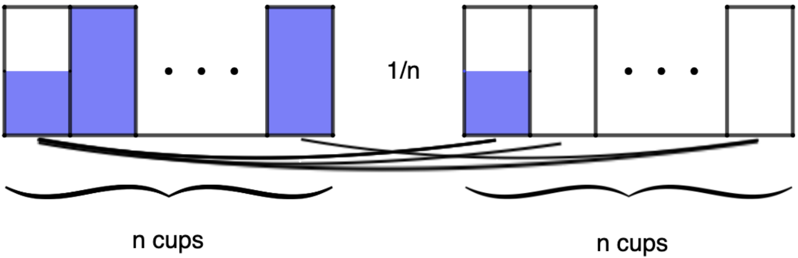


Problem

How much water can be transferred from the left to the right cups with a series of such «mixing operations»?



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How much water can be transferred from the left to the right cups with a series of such «mixing operations»?

Basic Lemma

In the setup with $2n$ cups the first term of the asymptotic behaviour of water amount left under the optimal transfer is $\frac{2}{\sqrt{n\pi}}$.

- Let the cups be numbered as $1, 2, \ldots, n$ from the center to the right, and as $-1, -2, \ldots, -n$ from the center to the left;
- On the first stage, we start with connecting the full cup -1 sequentially with all cups $1, 2, 3, \ldots, n$;

Optimal algorithm

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- As a result, these n cups receive levels $1/2, 1/4, 1/8, \dots, 1/2^n$, the cup -1 gets the same level as the cup n, i.e. $1/2^n$;

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- As a result, these n cups receive levels $1/2, 1/4, 1/8, \ldots, 1/2^n$, the cup -1 gets the same level as the cup n, i.e. $1/2^n$;
- On our second stage, we connect the full cup -2 sequentially again with all cups $1, 2, 3, \ldots, n$, or again with all cups $1, 2, \ldots;$
- As a result, these n cups involved receive levels $(1 + 1/2)/2 = 3/4, (3/4 + 1/4)/2 = 1/2, (1/2 + 1/8) = 5/16, \ldots$

- Let $x_i(j)$ be the relative level in the cup $j, j = 1, 2, \dots, n$ after i stages of our procedure, $i = 1, 2, \dots, n$;
- Denote by d_n (the «deficit») the total amount of water untransferred to the right after n stages of finite transfer;
- The deficit equals

$$d_n = \frac{1}{n} \sum_{i=1}^{n} x_i(n). \tag{1}$$

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Objective

We want to prove that

$$d_n = \frac{1}{n}\sum_{i=1}^{n} x_i(n) \sim \frac{2}{\sqrt{n\pi}}.$$

Lemma 1

Denote S_k as a random variable with Negative Binomial distribution with parameters $k, p = 1/2$ (a number of failures before the k-th success occurs). Then

$$x_k(j) = P(S_1 = j-1) + \cdots + P(S_k = j-1), j = 1, 2, \ldots \qquad (2)$$

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- Introduce generating function $F_n(z) = \sum_{j=1}^{\infty} x_n(j) z^j, n = 1, 2, \ldots$
- Use recursive relation $x_i(j) = \frac{x_{i-1}(j)}{2} + \frac{x_{i-1}(j-1)}{2^2} + \cdots + \frac{x_{i-1}(1)}{2^j} + \frac{1}{2^j}, j = 1, 2, \ldots$

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- Get $F_n(z) = \sum_{k=1}^{n} \frac{z}{(2-z)^k}, n = 1, 2, \ldots$

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Proposition 2

Using (1) and (2) we obtain $nd_n = \sum_{k=1}^{n} P\left(S_k \geq n\right)$.

Using notation $m_n = n^{3/5}$ we can rewrite (3) as

$$\begin{aligned} nd_n = \sum_{k=1}^{n-\lfloor m_n \rfloor} P\left(S_k \geq n\right) + \sum_{k=n-\lfloor m_n \rfloor+1}^{n} P\left(S_k \geq n + m_n\right) + \\ + \sum_{k=n-\lfloor m_n \rfloor+1}^{n} P\left(n \leq S_k < n + m_n\right) = A_1 + A_2 + A_3 \end{aligned} \quad (3)$$

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Proposition 3

If $k \leq n - m_n$, then $P\left(S_k \geq n\right) \leq \exp\left(-\frac{n^{\frac{1}{5}}}{4} + O\left(n^{-\frac{1}{5}}\right)\right)$.

Proposition 4

For any $k \leq n$ $P\left(S_k \geq n + m_n\right) \leq \exp\left(-\frac{n^{\frac{1}{5}}}{4} + O\left(n^{\frac{1}{6}}\right)\right)$.

- If $1 \leq k \leq n - m_n$ and $1 < z < 2$, Markov inequality implies

$$P\left(S_k \geq n\right) = P\left(z^{S_k} \geq z^n\right) \leq \frac{Ez^{S_k}}{z^n} = \frac{1}{(2-z)^k z^n}.$$

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- Put $z = 1 + \varepsilon$. Then for large n and small ε we have
$$P(S_k \geq n) = P\left((1+\varepsilon)^{S_k} \geq (1+\varepsilon)^n\right) \leq (1-\varepsilon)^{-k}(1+\varepsilon)^{-n} \leq$$
$$\leq (1-\varepsilon)^{-n+m_n}(1+\varepsilon)^{-n} =$$
$$= \exp\left[-m_n\varepsilon + n\varepsilon^2 + O\left(m_n\varepsilon^2\right) + O\left(n\varepsilon^3\right)\right].$$

Proof of Propositions 3

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- Put $\varepsilon = \varepsilon_n = \frac{1}{2n^{2/5}}$, as it approximately minimizes of the exponent. It gives for $k \leq n - m_n$

$$P\left(S_k \geq n\right) \leq \exp\left(-\frac{1}{4}n^{\frac{1}{5}} + O\left(n^{-\frac{1}{5}}\right)\right),$$

Therefore, $A_1(n) \to 0, A_2(n) \to 0$ and $nd_n = A_3(n) + o(1)$.

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- Recall $A_3(n) = \sum_{k=n-\lfloor m_n \rfloor+1}^{n} P\left(n \leq S_k < n + m_n\right)$;
- Note that $P\left(S_{k+1} = m\right) = b(k|k+m)$, where $b(k|n)$ is a Binomial distribution with parameters $n, 1/2$;

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- By the De Moivre-Laplace Theorem $b(k|n,p) = \frac{1}{\sqrt{2\pi npq}} \exp\left[-\frac{(k-np)^2}{2npq} + o(1)\right]$;
- Therefore, we obtain

$$A_3(n) = \sum_{k=n-\lfloor m_n \rfloor}^{n-1} \sum_{m=n}^{n+\lfloor m_n \rfloor} \sqrt{\frac{2}{\pi(k+m)}} \exp\left(-\frac{(m-k)^2}{2(k+m)} + o(1)\right);$$

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- Or $A_3(n) = \frac{1}{\sqrt{\pi n}} \sum_{l=1}^{\lfloor n^{3/5} \rfloor} (l+1) \exp\left(-\frac{l^2}{4n}\right) + O\left(n^{\frac{7}{10}} \exp\left(-\frac{n^{\frac{1}{5}}}{4}\right)\right)$

Euler-Maclaurin formula

Suppose that L is a positive integer, $f(x)$ is a non-negative continuous function with the absolute maximum at the point $x_0 \in [0, L]$, and monotonously increasing on $[0, x_0]$, and monotonously decreasing on $[x_0, L]$. Then for $S_L = f(0) + f(1) + \cdots + f(L)$

$$\left| S_L - \int_0^L f(x)dx \right| \leq 3f\left(x_0\right) = 3 \max_{x \in [0,L]} f(x).$$

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- Apply to function $(x+1)\exp\left(-\frac{x^2}{4n}\right)$ with maximum of an order $\sqrt{n}$.
- Get

$$\frac{1}{\sqrt{\pi n}} \sum_{l=1}^{\lfloor n^{3/5} \rfloor} (l+1)\exp\left(-\frac{l^2}{4n}\right) =$$

$$= \frac{1}{\sqrt{\pi n}} \int_1^{\lfloor n^{3/5} \rfloor} (x+1)\exp\left(-\frac{x^2}{4n}\right) dx + O(1) \sim \frac{2}{\sqrt{n\pi}}.$$

- We devided nd_n into sum $nd_n = A_1(n) + A_2(n) + A_3(n)$;
- We showed that $A_1(n) \to 0, A_2(n) \to 0$;
- We showed that $A_3(n) \sim 2\sqrt{\frac{n}{\pi}} + o(1)$;
- Therefore, we proved Basic Lemma.

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Remark

Basic Lemma proves Theorem 1 for the case when $X = c_1 1_A + c_2 1_B, Y = c_2 1_A + c_1 1_B$ for some disjoint subsets $A, B \subseteq \Omega$ such that $\mu(A) = \mu(B)$.

We should show the derivation of the Cherny-Grigoriev Theorem 1 from this basic case.

- It is sufficient to consider only simple bounded function X, Y (functions with finitely many values).
- Indeed, for an arbitrary $\delta > 0$ and equidistributed bounded functions X, Y there exist equidistributed bounded **simple** functions $\tilde{X}, \tilde{Y}$ such that $||X - \tilde{X}||_\infty < \delta, ||Y - \tilde{Y}||_\infty < \delta$.

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- We now will *transform* X into Y step by step.
- At the first step we will make X equal (up to fixed ϵ) to x_N on B_N.
- Then without touching previous results we will make our new X_1 equal (up to fixed ϵ) to x_{N-1} on B_{N-1} and so on.
- As the result, we will obtain X_N such that on every B_N it is equal to Y up to fixed ϵ, i.e. $||X_N - Y||_\infty < \epsilon$.

- Suppose that $A_N \cap B_N = \varnothing$, if not we will not touch their intersection.
- Suppose X takes values $x_1 < x_2 < \cdots < x_N$ on subsets $C_1, \ldots, C_N \subset B_N$ respectively, $\cup_{i=1}^{N} C_i = B_N$.
- Divide A_N into disjoint sets $D_1, \ldots, D_N \subset A_N$ such that $\cup_{i=1}^{N} D_i = A_N$ and $\forall i = 1, \ldots, N : \mu(D_i) = \mu(C_i)$.
- Basic Lemma implies that we can *swap* the values of X on all these subsets C_i and D_i up to a fixed ϵ: first, swap C_1 and D_1, then C_2 and D_2, etc.

- After that, we will obtain new function X_1 such that $\forall i = 1, \dots, N : ||X_1|_{C_i} - Y|_{C_i}||_\infty < \epsilon$ or $||X_1|_{B_N} - Y|_{B_N}||_\infty < \epsilon$.
- Moreover, on $\Omega \setminus B_N$ up to an arbitrary small ϵ X_1 and Y again equidistributed, but take only $N - 1$ values.
- Therefore, we can similarly obtain next X_2 such that $||X_2|_{B_{N-1}} - Y|_{B_{N-1}}||_\infty < \epsilon$ or $||X_2|_{B_N \cup B_{N-1}} - Y|_{B_N \cup B_{N-1}}||_\infty < \epsilon$.
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- What is about **optimality**?

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  2. Possible values of functions are in a fixed grid;
  3. Discretizing discrete time even more, allowing only one unit of a transfer at a time.
- Suppose that levels of the cups are integers $s = 0, 1, \ldots$ or fractions s/M, where s and M are integers (for continuous case let n and M tend independently to infinity).

- Suppose that the i-th cup on the left has level $a(i)$ and the j-th cup on can connect if $a(i) > b(j)$).
- After the connection levels become $a(i)' = a(i) - 1$ and $b(j)' = b(j) + 1$.

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- Say that a cup i on the left is **open** at moment m, if there is at least one cup on the right with the lower the level.
- Denote $A^0_m(i)$, the «*inner set*» of a cup i at the moment m, i.e. all cups available to it, which are not available to any cup with a lower level.

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Optimal strategy

Let π_* be the following strategy: *At each moment a cup with lowest level from the left among all open cups is connected with an available for it cup with the greatest level on the right*. And if at any moment there are some draws, connections can be ordered in some arbitrary way.

- At each moment the state of a system is described by a $2n$ dimensional vector
  $c = (a, b), a = (a(i), i = 1, 2, \ldots, n), b = (b(j), j = 1, 2, \ldots, n)$.
- Define $k(c|\pi)$ as the total number of connections obtained from state c using a strategy π and let $\pi(c)$ be the next state obtained from c, when π is applied.
- Define $k_0(c) = \sup_\pi k(c|\pi)$, i.e. maximum possible number of connections from state c. Let $k_*(c)$ be the number of connections from state c using the strategy π_*.

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Objective

We want to show that for any given state c we have $k_*(c) = k_0(c)$.

- We shall prove Theorem 2 by induction on the maximal number of remaining steps n.
- The induction statement $P_n, n = 0, 1, \ldots$ is: For any state c, if $k_0(c) = n$, then $k_*(c) = n$.

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- Now suppose that P_k holds for all $1 \leq k \leq n$, and $k_0(c) = n + 1$. Let an optimal strategy π_0 asks (s, t) connection and the strategy π_* asks (i, j) connection.
- Let us denote states $c_0 = \pi_0(c)$ and $c_* = \pi_*(c)$. According to the induction assumption the strategy π_0 after the first step can be continued by the strategy π_*, i.e. $k_0(c_0) = k_*(c_0) = n$.

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- And we want to show that $k_*(c_*) = k_0(c_0) = n$ either, or that there exists a strategy π' such that $k(c_*|\pi') = n$.

- If an optimal strategy π_0 is different from strategy π_*, then the definition of π_* implies that $a(s) \geq a(i)$ and if $b(t) < a(i)$, then $b(j) \leq b(j)$.
- Only the following four situations are possible:
  1. $a(s) > a(i)$ and $b(t) \geq a(i)$;
  2. $a(s) > a(i)$ and $b(t) < b(j)$;
  3. $a(s) > a(i)$ and $b(t) = b(j)$;
  4. $a(s) = a(i)$ and $b(t) < b(j)$.

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  3. $a(s) > a(i)$ and $b(t) = b(j)$;
  4. $a(s) = a(i)$ and $b(t) < b(j)$.
- In all cases we shall show that the state c_* is «at least good» as state c_0, i.e. there is a matching strategy π' applied to c_* with at least n steps.

- In case 1) the strategy π_* applied to c_0 asks for (i,j) connection, because in this state, as in state c, the left cup i is still the lowest open and the right cup j is still its greatest available.

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- Hence $c' = \pi_*(c_0) = (a(s) - 1, a(i) - 1, \dots; b(t) + 1, b(j) + 1, \dots)$ and $k_*(c') = k_0(c') = n - 1$.

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- But state c' can be obtained also from state c_* using strategy π' with the first connection (s, t).
- If this connection is continued by π_0 then $k_0(c_*) \geq k(c_*|\pi') = 1 + k_0(c') = 1 + (n - 1) = n$.
- Therefore, we obtain P_{n+1}.

Main results:

- We showed transparent interpretation of Theorem 1 using «hydrostatic» setup;
- We provided explicit and optimal algorithm of transformations for the simplest case;
- We found an exact first term of the asymptotic behavior (Basic Lemma);
- We proved that our simplest case implies Theorem 1.

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