Introduction to stochastic differential equations – 10
Strong solutions; Krylov's estimates

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Why extensions are needed

Abstract

In the previous lectures various aspects of SDEs, including existence and uniqueness in strong and weak sense, Markov property, as well as some ergodic properties of solutions were studied. Yet, this theory would remain incomplete if we do not touch Krylov's estimates and related issues of some extensions of Ito's formula and weak and strong solutions under relaxed regularity conditions. Some brief introduction to these topics will be presented here in this last additional lecture to the course. Not many full proofs will be provided for this advanced material; however, after this lecture the reader should be able to **use** these methods if required.

Principle (Yamada & Watanabe theorem 1971)

Let SDE have a (weak) solution on some probability space, and let there be a pathwise (=strong) uniqueness, that is, on any probability space with a Wiener process, two solutions coincide with probability one. Then every solution is strong, and it exists on any probability space with a WP.

For the proof, see [Yamada, Watanabe 1971], [Ikeda and Watanabe, the monograph], [Zvonkin and Krylov, the paper in the Proceedings of the 1974 Druniskininkai Stochastic workshop (1975, vol. II)], [H.J. Engelbert, On Yamada and Watanabe principle..., 2007], [T.G. Kurtz, The Yamada - Watanabe - Engelbert theorem for general stochastic equations and inequalities, Elec. J. Probab. 12, (2007), 951-965.].

Example 10.1: strong uniqueness $\Longrightarrow$ strong existence

dim = 1; this is a very simple version of Zvonkin's (and mine in d>1) result

Consider a 1D SDE

$$dX_t = b(X_t)dt + dW_t, \quad X_0 = x. \tag{1}$$

Example (10.1)

Let $b \in C_b$. Then the equation (1) is pathwise unique and, hence, has a strong solution.

Recall that it has a (possibly weak) solution by virtue of Girsanov's theorem. Now it follows that it is, actually, strong. The condition $b \in C_b$ may be relaxed just to $b \in B$, but it requires Krylov's estimates and Ito-Krylov's formula, which topic will be presented on the following pages. For more general d=1 result for a *non-homogeneous* SDE see [Zvonkin 1974, Mat sb.]; for d>1 [AYV 1979, Mat sb. & Mat zametki 1982]; later see [Krylov & Röckner 2005], etc.

We will prove strong uniqueness. Let $u(x)$ be a (any) solution of an ODE

$$(Lu(x) =) \qquad \frac{1}{2}u''(x) + b(x)u'(x) = 0.$$

Let us solve this equation. Denote $u' = v$, then assuming $v \neq 0$ and, moreover, $v > 0$, we get

$$\frac{1}{2}v'(x) + b(x)v(x) = 0 \ \sim \ \frac{v'}{v}(x) = (\ln v)'(x) = -2b(x);$$

hence,

$$\ln v(x) = -2\int_0^x b(y)dy + C, \ \text{but we take } C = 0;$$

then,

$$v(x) = \exp(-2\int_0^x b(y)dy) = u'(x).$$

Example 10.1, proof, ctd $v'(x) = -2b(x)v(x)$

$v(x) = \exp(-2\int_0^x b(y)dy)$; $X_t = x + \int_0^t b(s, X_s)ds + W_t$, $t \geq 0$

So, one of solutions (which suffices!) has a form

$$u(x) = \int_0^x \exp(-2\int_0^y b(z)dz)dy + C,$$

and again we take $C = 0$ here. Note that $u \in C^2$. Denote now

$$Y_t = u(X_t).$$

By Ito's formula,

$$dY_t = du(X_t) = u'(X_t)dW_t = v(X_t)dW_t.$$

Since $u' = v > 0$, the mapping $x \mapsto u(x)$ is 1-1. Denote by u^{-1} its inverse (also in C^2), so that $X_t = u^{-1}(Y_t)$. Then

$$dY_t = v(u^{-1}(Y_t))dW_t.$$

Example 10.1, proof, ctd $dY_t = v(u^{-1}(Y_t))dW_t$

$v(x) = \exp(-2\int_0^x b(y)dy)$; $X_t = x + \int_0^t b(s, X_s)ds + W_t$, $t \geq 0$

Note that with $x = u^{-1}(y)$ we have,

$$\frac{d}{dy}v(u^{-1}(y)) = \frac{v'(u^{-1}(y))}{u'(u^{-1}(y))} = \frac{v'(x)}{u'(x)} = \frac{-2b(x)v(x)}{v(x)} = -2b(x).$$

In other words, the new diffusion coefficient for Y_t is C_b^1; in particular, it is Lipschitz. So, Y_t is a pathwise unique solution of the SDE for Y. Equivalently, solution X_t of the initial SDE is also pathwise unique. QED.

In particular, solution X_t is strong, and it exists on any probability space with a WP. The requirement $b \in C$ may be dropped and this example is valid for any $b \in B$, but to show this we need to introduce Krylov's bounds.

[Krylov, Controlled diffusion processes, ch.2]

Simplified versions of K-bounds are presented below

Let ξ_t be an Ito process in R^d, that is,

$$\xi_t = \xi_0 + \int_0^t \sigma_s dW_s + \int_0^t b_s ds,$$

with adapted stochastic processes σ and b which are both bounded and such that $\sigma\sigma^*$ is uniformly nondegenerate. Let $D \subset B_R$ (domain) and $\tau := \inf(t \ge 0 : X_t \notin D)$.

Theorem (Krylov's bounds 1 – in the domain in R^d)

For any $R > 0$ and for any $p \ge d$ there exists N such that

$$\mathbb{E} \int_0^\tau g(\xi_t) dt \le N_R \|g\|_{L_p(D)}, \qquad (2)$$

and

$$\mathbb{E} \int_0^\tau f(t, \xi_t) dt \le N_R \|f\|_{L_{p+1}([0,T]\times D)}. \qquad (3)$$

Krylov's bounds, ctd

Q: if $g = 0$ a.s., is it true that $E\int_0^T g(\xi_t)dt = 0$? We know little about the law of ξ, and it is unclear whether or not it may have a component singular to Λ

The first K-bounds were for a bounded domain, with N depending on R (the diameter of D). Here are K-bounds for the whole space.

Theorem (Krylov's bounds 2 – in R^d)

For any $T > 0$ and for any $p \geq d$ there exists N such that

$$\mathbb{E}\int_0^T g(\xi_t)dt \leq N\|g\|_{L_p(R^d)}, \tag{4}$$

and

$$\mathbb{E}\int_0^T f(t,\xi_t)dt \leq N\|f\|_{L_{p+1}([0,T]\times R^d)}. \tag{5}$$

In both theorems constants N depend also on the constants of ellipticity of $\sigma\sigma^*$ and the B-norm of the drift; in the first theorem in both bounds they also depend on R, and in the second on T.

Application of Krylov's bounds

The law of ξ on $[0, T] \times R^d$ (**not** on R^d $\forall t$) is $\ll$ Lebesgue's measure in t, x

Corollary

Let ξ_t be an Ito process in R^d, that is,

$$\xi_t = \xi_0 + \int_0^t \sigma_s dW_s + \int_0^t b_s ds,$$

with adapted stochastic processes σ and b which are both bounded and such that $\sigma\sigma^*$ is uniformly nondegenerate. Let $g(x) \geq 0$ and $f(t, x) \geq 0$ be Borel measurable functions such that $g(x) = 0$ a.e., and $f(t, x) = 0$ a.e. Then $\forall\, T > 0$,

$$\int_0^T g(\xi_t) dt = 0 \quad \int_0^T f(t, \xi_t) dt = 0 \quad a.s.$$

$E \int_0^T g(\xi_t) dt \leq N \|g\|_{L_d(R^d)} = 0 = N \|f\|_{L_{d+1}} \geq E \int_0^T f(t, \xi_t) dt.$

Sobolev space W_p^2

Recall that the Sobolev space W_p^2 consists of real-valued functions on R^d satisfying the following conditions:

- $f \in L_p(R^d)$;
- There exist vector and matrix functions f_1, f_2 and sequences $f^n, f_1^n, f_2^n \in L_p(R^d) \bigcap C^\infty(R^d)$ (vector and matrix functions) such that (the 2nd term is a **matrix**, not Δf!)

$$f_1^n = \nabla f^n, \; f_2^n = \nabla^2 f^n,$$

and

$$\|f^n - f\|_{L_p(R^d)} + \|f_1^n - f_1\|_{L_p(R^d)} + \|f_2^n - f_2\|_{L_p(R^d)} \underset{n\to\infty}{\to} 0.$$

Here f_1 and f_2 are called, respectively, the first and the second Sobolev derivatives of f.

Sobolev space $W_p^{1,2}$

The Sobolev space $W_p^{1,2}$ consists of real-valued functions $f(t,x)$ on $[0,\infty)\times R^d$ satisfying the following conditions:

- $f \in L_p([0,\infty)\times R^d)$;
- There exist vector and matrix functions f_0, f_1, f_2 and sequences $f_0^n, f^n, f_1^n, f_2^n \in L_p(R^d)\bigcap C^\infty(R^d)$ (scalar, vector and matrix functions) such that

$$f_0^n = \frac{df^n}{dt},\ f_1^n = \nabla_x f^n,\ f_2^n = \nabla_x^2 f^n,$$

and $\|f^n - f\|_{L_p(R^d)}+$

$$+\|f_0^n - \frac{df}{dt}\|_{L_p(R^d)} + \|f_1^n - f_1\|_{L_p(R^d)} + \|f_2^n - f_2\|_{L_p(R^d)} \underset{n\to\infty}{\to} 0.$$

Here f_0 is called the Sobolev derivative of f wrt t.

Ito–Krylov's formula

$\xi_t = \xi_0 + \int_0^t \sigma_s dW_s + \int_0^t b_s ds, \quad L_t = \frac{1}{2}\sum_{ij}(\sigma_t\sigma_t^*)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i (b_t)^i \frac{\partial}{\partial x_i}$

Theorem (Ito–Krylov's formula)

Let $u \in W_p^{1,2}([0,\infty) \times R^d)$ and $\nabla_x u \in L_{2p}([0,\infty) \times R^d)$ with $p \geq d+1$. Then

$$du(s,\xi_s) = \sigma^* \nabla u(s,\xi_s) dW_s$$
$$+[u_s(s,\xi_s) + L_s u(s,\xi_s)]ds,$$

or, equivalently, in the integral form,

$$u(T,\xi_T) = u(0,x) + \int_0^T \sigma^* \nabla u(s,\xi_s) dW_s$$
$$+ \int_0^T [u_s(s,\xi_s) + L_s u(s,\xi_s)]ds.$$

NB: here all derivatives of u are regarded as Sobolev ones.

Idea of proof is straightforward: let us approximate our function u by smooth functions u^n in the norm ($\nabla^2 u$ stands for the (Sobolev) Hessian of u)

$$\|u\|_W := \|u\|_{L_p} + \|\nabla u\|_{L_{2p}} + \|\nabla^2 u\|_{L_p},$$

apply Ito's formula to $u^n(t, X_t)$, and pass to the limit by using Krylov's bounds. Naturally, this idea allows a localization.

Example 10.1 with $b \in B$

via Ito–Krylov's formula, idea

In the setting of the Example 10.1 under the condition $b \in B$, it is possible to approximate b in the norm $\| \cdot \|_{L_1([-N,N])}$ for any N by smooth C^∞ uniformly bounded functions: say,

$$\|b^n - b\|_{L_1([-N,N])} \to 0, \quad n \to \infty.$$

Let

$$v^n(x) = \exp(-2\int_0^x b^n(y)dy) = (u^n)'(x),$$

$$u^n(x) = \int_0^x \exp(-2\int_0^y b^n(z)dz)dy.$$

It is possible to show that locally $\|u^n - u\|_W \to 0$, $n \to \infty$. In the limit we obtain the equation, as earlier,

$$dY_t = \tilde\sigma(Y_t)dW_t, \quad \text{with } \sigma(y) = v(u^{-1}(y)).$$

The diffusion coefficient $\tilde\sigma(y)$ is Lipschitz. Hence, the solution of the SDE is pathwise unique, as required.

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SDEs introduction

Yamada and Watanabe

Krylov's bounds

PDE equation solutions in Sobolev spaces

Krylov's bounds were extended to serve for SDEs with nondegenerate diffusion and jumps over Poisson random measures in the paper [S. Anulova and H. Pragarauskas, On strong Markov weak solutions of stochastic equations, Liet. Mat. Rinkinys XVII (2) (1977) 5-26] and under more rigorous conditions and with just brief proofs in [Lepeltier, J.-P. & Marchal, B. Problème des martingales et équations différentielles stochastiques associées à un opérateur intégro-différentiel. Ann. Inst. H. Poincaré Sect. B (N.S.) 12 (1976), no. 1, 43-103. MR413288]

Parabolic equations in Sobolev spaces

The basic monographs

[Ladyzenskaya, Solonnikov, Ural'tseva]; [Krylov]

Elliptic equations in Sobolev spaces

The basic monographs and articles

[Ladyzenskaya, Ural'tseva]; [Solonnikov]; [Gilbarg, Trudinger]; [Krylov]

It is suggested after reading about Sobolev PDE solutions to return to the lecture 7. All examples there can be reformulated for PDE solutions in Sobolev spaces. A special attention should be paid to the boundary conditions, which now may include some discontinuous indicator functions.

THE END OF THE COURSE