Introduction to stochastic differential equations – 5a
Continuity with respect to initial data

Alexander Veretennikov[1]

Spring 2020

April 20, 2020

[1]National Research University HSE, Moscow State University, Russia

2nd moment a priori bound

Under a linear growth condition

Let $x \in R^d$, and let there exist $K > 0$ such that for all x, x' the following *linear growth condition* holds,

$$|b(t,x)|^2 + \|\sigma(t,x)\|^2 \le K(1 + |x|^2). \qquad (1)$$

Suppose X_t is a solution (weak or strong) of the equation

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x,$$

on the whole half-line $t \ge 0$, where the i.c. x is non-random.

Theorem (5.1)

Under the assumptions above, there exists $C > 0$ such that

$$E \sup_{0 \le t \le T} |X_t|^2 \le C(1 + T + x^2) \exp(CT).$$

Proof

$dX_t^i = b^i(t, X_t)dt + \sigma_{ij}(t, X_t)dW_t^j;\ |b(t,x)|^2 + \|\sigma(t,x)\|^2 \leq K(1+|x|^2)$

Let $\tau_N := \inf(t \geq 0 : |X_t| \geq N)$. We write,

$$X_{t\wedge\tau_N} = x + \int_0^{t\wedge\tau_N} b(s, X_s)ds + \int_0^{t\wedge\tau_N} \sigma(s, X_s)dW_s$$

So, due to Doob's inequality,

$$\frac{1}{3}E \sup_{0\leq s\leq t\wedge\tau_N} |X_s|^2 \leq x^2 + E(\int_0^{t\wedge\tau_N} |b(s, X_s)|ds)^2$$

$$+E \sup_{0\leq s\leq t\wedge\tau_N} (\int_0^{s\wedge\tau_N} \sigma(r, X_r)dW_r)^2$$

$$\leq x^2 + E(\int_0^{t\wedge\tau_N} |b(s, X_s)|ds)^2 + 4E\int_0^{t\wedge\tau_N} \sigma(r, X_r)^2 dr.$$

Proof, ctd.

$0 \le g_t \le A + B\int_0^t g_s ds \le C \stackrel{Gronwall}{\Longrightarrow} g_t \le A\exp(Bt)$. It follows

$$\begin{aligned}\frac{1}{3}E\sup_{0\le s\le t}|X_{s\wedge\tau_N}|^2 &\le x^2 + E\Big(\int_0^t |b(s\wedge\tau_N, X_{s\wedge\tau_N})|ds\Big)^2\\ &+4E\int_0^t \sigma(s\wedge\tau_N, X_{s\wedge\tau_N})^2 ds\\ &\le x^2 + Ct\,E\int_0^t C(1+|X_{s\wedge\tau_N}|^2)ds\\ &+C\,E\int_0^t C(1+|X_{s\wedge\tau_N}|^2)ds\end{aligned}$$

By virtue of Gronwall's inequality we have,

$$\frac{1}{3}E\sup_{0\le s\le t}|X_{s\wedge\tau_N}|^2 \le C(x^2+1+t)\exp(Ct), \quad t\ge 0.$$

Proof, ctd.

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \ X_0 = x$

From

$$\frac{1}{3} E \sup_{0 \le s \le t} |X_{s \wedge \tau_N}|^2 \le C(x^2 + 1 + t) \exp(Ct), \quad t \ge 0,$$

we conclude by Fatou's lemma as $N \to \infty$,

$$\frac{1}{3} E \sup_{0 \le s \le t} |X_s|^2 \le C(x^2 + 1 + t) \exp(Ct), \quad t \ge 0,$$

as required. The Theorem is proved.

Remark

If the i.c. is random (and $\mathcal{F}_0$-measurable), then the bound for the second moment reads,

$$\frac{1}{3} E \sup_{0 \le s \le t} |X_s|^2 \le C(EX_0^2 + 1 + t) \exp(Ct), \quad t \ge 0. \qquad (2)$$

Extensions: higher a priori moment bounds

Assume linear growth conditions; $dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \; X_0 = x$

Let $m \geq 1$. We have,

$$E \sup_{0 \leq s \leq t} |X_s|^{2m} \leq C_{2m} \Big(E|X_0|^{2m} + t^{2m-1} E \int_0^t |b(s, X_s)|^{2m} ds \\ \overset{\text{(Doob)}}{+} \; C_{2m} E \int_0^t \|\sigma(s, X_s)\|^{2m} ds \Big)$$

Using stopping times $\tau_N := \inf(t \geq 0 : |X_t| \geq N)$ and via Gronwall's inequality, we obtain the a priori bound:

Theorem (5.2)

Under the linear growth conditions (1), for any $m \geq 1$

$$E \sup_{0 \leq s \leq t} |X_s|^{2m} \leq C_{2m} E|X_0|^{2m} (1 + E|X_0|^{2m} + t^{2m-1}) \exp(Ct).$$

Continuity wrt i.c. simple version of K.6.9.3

Assume (3): Lipschitz condition in x uniform wrt t and linear growth

$$|b(t,x)-b(t,x')|+\|\sigma(t,x)-\sigma(t,x')\|\le K|x-x'|,$$
$$|b(t,x)|+\|\sigma(t,x)\|\le K(1+|x|). \tag{3}$$

Let (with random $\xi_n, \xi_0 \in \mathcal{F}_0$)

$$X_t=\xi_0+\int_0^t b(s,X_s)ds+\int_0^t \sigma(s,X_s)dW_s,$$
$$X_t^n=\xi_n+\int_0^t b(s,X_s^n)ds+\int_0^t \sigma(s,X_s^n)dW_s,$$

Theorem (version of K.6.9.3: let $E|\xi_0|^2+\sup_n E|\xi_n|^2<\infty$)

Assume also (3) and $E|\xi_n-\xi_0|^2\to 0$. Then $\forall T>0$, $\forall c>0$,

$$P(\sup_{0\le t\le T}|X_t^n-X_t|^2>c)\overset{\exists C}{\le} Cc^{-1}E|\xi_n-\xi_0|^2\to 0,\ n\to\infty. \tag{4}$$

Proof

Theorem (5.1): $E\sup_{0\le t\le T}|X_t|^2 \le C(1+T+x^2)\exp(CT)$

We subtract,

$$X_t - X_t^n = \xi_0 - \xi_n + \int_0^t (b(s,X_s) - b(s,X_s^n))ds + \int_0^t (\sigma(s,X_s) - \sigma(s,X_s^n))dW_s.$$

So, using Doob's inequality and a priori 2nd moment bounds (2) from Remark above both for X^n and X, we get

$$E\sup_{0\le s\le t}|X_s - X_s^n|^2 \le 3tE\int_0^t |b(s,X_s) - b(s,X_s^n)|^2 ds + 3E|\xi_0 - \xi_n|^2 + 12\,E\int_0^t \|\sigma(s,X_s) - \sigma(s,X_s^n)\|^2 ds.$$

Proof, ctd.

goal: $P(\sup_{0\le t\le T}|X_t^n - X_t|^2 > c) \stackrel{\exists C}{\le} Cc^{-1}E|\xi_n - \xi_0|^2$

Repeat the last inequality

$$E\sup_{0\le s\le t}|X_s - X_s^n|^2 \le 3tE\int_0^t |b(s,X_s) - b(s,X_s^n)|^2 ds$$

$$+3E|\xi_0 - \xi_n|^2 + 12\,E\int_0^t \|\sigma(s,X_s) - \sigma(s,X_s^n)\|^2 ds.$$

Hence, from the Lipschitz condition it follows,

$$E\sup_{0\le s\le t}|X_s - X_s^n|^2 \le 3E|\xi_0 - \xi_n|^2$$

$$+C(1+t)E\int_0^t |X_s - X_s^n|^2 ds,$$

and now due to Gronwall's inequality we obtain $\forall\, t > 0$,

$$E\sup_{0\le s\le t}|X_s - X_s^n|^2 \le 3E|\xi_0 - \xi_n|^2 \exp(C(1+t)t).$$

Bienaymé–Chebyshev–Markov's inequality now implies (4).

Let us state without proof the exact version of the Theorem in Krylov's textbook.

Theorem (K.6.9.3)

Assume (3) and let

$$\xi_n - \xi_0 \to 0, \quad n \to \infty, \qquad a.s.$$

Then $\forall T > 0$, $\forall c > 0$,

$$P(\sup_{0 \le t \le T} |X_t^n - X_t|^2 > c) \to 0, \; n \to \infty. \tag{5}$$

Continuity: some extensions

Does classical continuity wrt i.c. under Lipschitz condition hold?

Assume for simplicity b and σ bounded.

Theorem (without proof)

Under the Lipschitz conditions the random field X_t^x has a modification $\tilde{X}_t^x$ (i.e., $P(X_t^x = \tilde{X}_t^x) = 1, \forall t, x$) continuous wrt (t, x) almost surely.

There is some more option in the 1D case with the help of the monotonicity of pathwise unique strong solutions with respect to the initial conditions.

Derivatives wrt i.c.

By e we denote any non-random unit vector in R^d; it determines a direction

Let in addition to (3), $b, \sigma \in C_b^{0,1}([0,\infty) \times R^d)$. Let

$$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s.$$

Theorem (5.4: on directional derivative in L_2)

Assume (3), and $b, \sigma \in C_b^{0,1}([0,T] \times R^d)$. Then $\forall e \in R^d$

$$E \sup_{0\le t\le T} \left| \frac{X_t^{x+he} - X_t^x}{h} - \xi_t^e \right|^2 \to 0, \quad h \to 0 \tag{6}$$

where ξ_t^e is a (strong) vector-valued solution of the SDE

$$\xi_t^e = e + \int_0^t (\nabla b(s, X_s^x))\xi_s^e ds + \int_0^t (\nabla \sigma(s, X_s^x))\xi_s^e dW_s. \tag{7}$$

Solution of (7) $\exists$ and is unique on $[0,T]$, $E\sup_{t\le T}|\xi_t^e|^2 < \infty$.

ξ_t^e exists and is unique; $E \sup_{t \le T} |\xi_t^e|^2 < \infty$

Idea of proof $\quad (\xi_t^e = e + \int_0^t (\nabla b(s, X_s^x))\xi_s^e ds + \int_0^t (\nabla \sigma(s, X_s^x))\xi_s^e dW_s)$

The assertion of existence and uniqueness can be proved similarly to the proof of Ito's theorem but with random adapted coefficients satisfying the same Lipschitz and linear growth conditions. For example, the drift is now $B(s, \xi) = \nabla b(s, X_s^x)\xi$. Note that here the existence of solution X_t^x is already established; so we can treat coefficients as random and *linear in* ξ.

The second moment bound then can be estimated also similarly to the previously treated case, now with random coefficients, which are linear in ξ.

Proof of (6)

$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s$

$$X_t^{x+eh} - X_t^x = (x + eh - x) + \int_0^t (b(s, X_s^{x+eh}) - b(s, X_s^x))ds + \int_0^t (\sigma(s, X_s^{x+eh}) - \sigma(s, X_s^x))dW_s.$$

Denote

$$\xi_t^{e,h} := \frac{X_t^{x+eh} - X_t^x}{h}.$$

Then

$$\xi_t^{e,h} = e + \int \frac{(b(s, X_s^{x+eh}) - b(s, X_s^x))}{h}ds + \int_0^t \frac{(\sigma(s, X_s^{x+eh}) - \sigma(s, X_s^x))}{h}dW_s.$$

Proof, ctd

$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s$

Hadamard's version of the Newton–Leibnitz formula reads,

$$f(y) - f(x) = \int_0^1 \nabla f(\alpha x + (1-\alpha)y))(y-x)d\alpha.$$

So,

$$b(s, X_s^{x+eh}) - b(s, X_s^x)$$

$$= \int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)(X_s^{x+eh} - X_s^x)d\alpha,$$

$$\sigma(s, X_s^{x+eh}) - \sigma(s, X_s^x)$$

$$= \int_0^1 \nabla \sigma(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)(X_s^{x+eh} - X_s^x)d\alpha.$$

Proof, ctd.

$X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s; \quad \xi_s^{e,h} = (X_t^{x+eh} - X_t^x)/h$

Therefore, the process $\xi_t^{e,h}$ may be represented as

$$\xi_t^{e,h} = e + \int_0^t \left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha \right) \xi_s^{e,h} ds$$

$$+ \int_0^t \left(\int_0^1 \nabla \sigma(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha \right) \xi_s^{e,h} dW_s.$$

Let us subtract $\xi_t^{e,h} - \xi_t^e =: \eta_t^{e,h}$: recall that

$$\xi_t^e = e + \int_0^t (\nabla b(s, X_s^x))\xi_s^e ds + \int_0^t (\nabla \sigma(s, X_s^x))\xi_s^e dW_s.$$

Note that by the continuity theorem 5.1, for any $T > 0$,

$$\sup_{0 \le s \le T} |X_s^{x+eh} - X_s^x| \xrightarrow{P} 0, \quad h \to 0.$$

Proof, ctd. $X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s$

$d\xi_t^e = (\nabla b(t, X_t^x))\xi_t^e dt + (\nabla\sigma(t, X_t^x))\xi_t^e dW_t; \xi_s^{e,h} = (X_t^{x+eh} - X_t^x)/h; \eta_t^{e,h} = \xi_t^{e,h} - \xi_t^e$

So, we have as $h \to 0$,

$$\sup \|\nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x) - \nabla b(s, X_s^x)\| \xrightarrow{P} 0,$$

$$\sup \|\nabla \sigma(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x) - \nabla \sigma(s, X_s^x)\| \xrightarrow{P} 0. \tag{8}$$

Also, we write,

$$\left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha\right)\xi_s^{e,h} - \nabla b(s, X_s^x))\xi_s^e$$

$$= \left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha\right)(\xi_s^{e,h} - \xi_s^e)$$

$$+ \left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha - \nabla b(s, X_s^x)\right)\xi_s^e.$$

Proof, ctd. $X_t^x = x + \int_0^t b(s, X_s^x)ds + \int_0^t \sigma(s, X_s^x)dW_s$

$\eta_t^{e,h} = \xi_t^{e,h} - \xi_t^e$; $\xi_t^e = e + \int_0^t (\nabla b(s, X_s^x))\xi_s^e ds + \int_0^t (\nabla \sigma(s, X_s^x))\xi_s^e dW_s$

Hence,

$$\eta_t^{e,h} = \int_0^t \left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha \right) \eta_s^{e,h} ds$$
$$+ \int_0^t \left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha - \nabla b(s, X_s^x) \right) \xi_s^e ds$$
$$+ \int_0^t \left(\int_0^1 \nabla \sigma(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha \right) \eta_s^{e,h} dW_s$$
$$+ \int_0^t \left(\int_0^1 \nabla \sigma(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha - \nabla \sigma(s, X_s^x) \right) \xi_s^e dW_s.$$

Take expectations of the left and right hand sides squared.

The following terms are *small* ($o_h(1)$) in L_2 by virtue of the Lebesgue dominated convergence theorem because of the bound $E\sup_{t\le T}|\xi_t^e|^2 < \infty$, and since (8) holds true, and due to the continuity theorem 5.1: $\zeta_T^{e,h} :=$

$$\sup_{t\le T}\left(\int_0^t\left(\int_0^1 \nabla b(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha - \nabla b(s, X_s^x)\right)\xi_s^e ds\right)^2$$

$$+\sup_{t\le T}\left(\int_0^t\left(\int_0^1 \nabla\sigma(s, \alpha X_s^{x+eh} + (1-\alpha)X_s^x)d\alpha - \nabla\sigma(s, X_s^x)\right)\xi_s^e dW_s\right)^2.$$

Hence, due to Gronwall's inequality we estimate

$$E\sup_{0\le t\le T}|\eta_t^{e,h}|^2 \le E\zeta_T^{e,h}\exp(CT) \to 0,$$

The Theorem 5.4 on directional derivative in L_2 is proved.

Derivatives: of further extensions

In L_p; continuous derivatives; with respect to parameters other than i.c. x

Further extensions can be as follows.

- L_p - derivatives, see [N.V. Krylov, Controlled diffusion processes, chapter 2]; often $p > d$ or $p > d + 1$ is desirable, while $p = 2$ may not suffice;
- Higher L_2 or L_p-derivatives (2nd et al.): for them, apparently, a higher regularity of b and σ is required;
- Classical (continuous) derivatives: normally, one additional L_p- derivative is needed for that [M.I. Freidlin, et al.]; "+one" can be replaces by Hölder [H. Tanaka]
- We may wish to use the gradient $\nabla X_t^x =: Z_t$ which is a matrix-valued process satisfying a matrix-valued SDE

$$dZ_t = I + \int_0^t (\nabla b(s, X_s^x)) Z_s ds + \int_0^t (\nabla \sigma(s, X_s^x)) Z_s dW_s.$$

- Similar can be done for derivatives wrt parameters.