Introduction to stochastic differential equations – 4
Links to PDEs – 2

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Example 8

Particular case of Example 5

Let D be a bounded domain (by definition open one and connected; condition to be connected can be dropped, it is just for simplicity) in R^d. Consider the Laplace equation

$$\frac{1}{2}\Delta u(x) = 0,\ x \in D, \quad \& \quad u|_\Gamma = \phi(x), \tag{1}$$

where $\Gamma = \partial D$ and $\phi(x) = 1(x \in A)$, $A \subset \Gamma$. Recall that $D^c := R^d \setminus D$. Let $\tau := \inf(t \geq 0 : x + W_t \in D^c)$.

Example (8 (already proved!))

1. Let $u(x) \in C_b^2(\bar{D})$ be a solution of the Laplace equation (1) with $\phi \in C(\bar{D})$. Then $u(x)$ can be represented as

$$u(x) = E\phi(x + W_\tau) = P(x + W_\tau \in A), \quad x \in D.$$

2. **Vice versa**: the probability $P(x + W_\tau \in A)$ as a function of x satisfies the Laplace equation (1).

Example 8, Comments

Some obstacle may arise form the fact that often an indicator of a part of the boundary turns out to be discontinuous. Indeed, is it possible that the function $\phi(x) = 1(x \in A) \in C$? And, if it is not, how can we apply the earlier results in such a case?

One option is to consider domains with a disconnected boundary as in the next example 9 of a ring on the plane: in R^2 let $D := \{x : 0 < a < |x| < b\}$, and, say, we are interested in computing the probability $P(|x + W_\tau| = a)$. Here $\phi(x) = 1(|x| = a)$, while the complement of the boundary is the $(x : |x| = b)$. In this example the function ϕ is smooth on $\Gamma = \partial D$, not at all discontinuous.

Another option is to study solutions in Sobolev classes which allow discontinuities in boundary conditions. Not in this lecture.

General Theorem $\frac{1}{2}\Delta u(x) = 0, \; x \in D \quad (1)$

Homework: to prove! (Hint: use stopping times as in Example 2)

Theorem (variation of K.6.6.6)

Let $u(x) \in C_b^2(D)$ be a solution of the Laplace equation (1), and let $u \in C(\bar{D})$. Assume that $E \int_0^\tau |\Delta u(x + W_s)| ds < \infty$. Then $u(x)$ can be represented as

$$u(x) = Eu(x + W_\tau) - \frac{1}{2} E \int_0^\tau \Delta u(x + W_s) ds, \quad x \in D.$$

Note, we already know that for a bounded domain D, the stopping time τ is a.s. finite, and even $E\tau < \infty$, due to the Lemma from the previous lecture.

NB: In his lectures N.V. Krylov states Theorem 6.6.6 for a more general diffusion. Hence, he has to assume that $P(\tau < \infty) = 1$.

Poisson equation and expected hitting time

Example 9, K.6.7.1 $\frac{1}{2}\Delta u(x) = -\psi(x),\; x \in D;$ choose $\psi \equiv 1$

Here $\phi \equiv 0$ and $\psi \equiv 1$; both functions are very-very smooth!

Example (9)

In R^d consider an open ball $B_R = \{x : |x| < R\}$ with some $R > 0$, and let

$$\tau := \inf(t \geq 0 : W_t \notin B_R).$$

Consider the function

$$u(x) := \frac{1}{d}(R^2 - x^2).$$

Then

$$E\tau = \frac{R^2}{d}.$$

We do not explain how function u was found: an educated guess.

Example 9, Proof

By Ito's formula,

$$u(W_t) = -t - \int_0^t \frac{2}{d} W_s dW_s + \frac{R^2}{d}.$$

Replace t by $t \wedge \tau$ and take expectations:

$$Eu(W_{t\wedge\tau}) = -Et \wedge \tau - E \int_0^{t\wedge\tau} \frac{2}{d} W_s dW_s + \frac{R^2}{d}.$$

We have, $u(W_{t\wedge\tau}) \leq \frac{R^2}{d}$. Since we already know (from the Lemma of the previous lecture) that $E\tau < \infty$, and as $E \int_0^t W_s^2 ds < \infty$, then we can pass to the limit here as $t \to \infty$:

$$E\tau = \frac{R^2}{d} - Eu(W_\tau) = \frac{R^2}{d}.$$

Again, $\phi \equiv 0$ and $\psi \equiv 1$.

Example (10)

Again $B_R = \{x : |x| < R\}$ in R^d, and let

$$\tau_x := \inf(t \geq 0 : x + W_t \notin B_R).$$

Consider the same function from Example 9,

$$u(x) := \frac{1}{d}(R^2 - x^2).$$

Then

$$E\tau_x = u(x), \quad |x| \leq R.$$

This is a homework!

Example (11)

$D_{\epsilon,R} = \{x : \epsilon < |x| < R\}, \tau_{\epsilon,R} = \inf(t \ge 0 : x_0 + W_t \notin D_{\epsilon,R}),$

$$\phi(x) := \begin{cases} A(|x|^{-(d-2)} - R^{-(d-2)}), & \textit{if} \quad d \ge 3, \\ A(\ln|x| - \ln R), & \textit{if} \quad d = 2, \\ A(|x| - R), & \textit{if} \quad d = 1, \end{cases}$$

$$A := \begin{cases} (\epsilon^{-(d-2)} - R^{-(d-2)})^{-1}, & \textit{if} \quad d \ge 3, \\ (\ln\epsilon - \ln R)^{-1}, & \textit{if} \quad d = 2, \\ (\epsilon - R)^{-1}, & \textit{if} \quad d = 1, \end{cases}$$

Then $\Delta\phi(x) = 0$ in $D_{\epsilon,R}$, $P_{\epsilon,R} = E\phi$, and

$$P_\epsilon := \lim_{R\to\infty} P_{\epsilon,R} = \begin{cases} (\epsilon/|x_0|)^{(d-2)}, & d \ge 3, \\ 1, & d \le 2. \end{cases}$$

Proof

Homework: check yourself that $\Delta\phi(x) = 0$ in $D_{\epsilon,R}$, & the gradient $\nabla\phi$ bounded.

Now apply Ito's formula to $\phi(x_0 + W_t)$ for $t < \tau_{\epsilon,R}$:

$$
\begin{aligned}
d\phi(x_0 + W_t) = \nabla\phi(x_0 + W_t)dW_t + \frac{1}{2}\Delta\phi(x_0 + W_t)dt \\
= \nabla\phi(x_0 + W_t)dW_t.
\end{aligned}
$$

Hence,

$$
\phi(x_0 + W_{t\wedge\tau_{\epsilon,R}}) = \phi(x_0) + \int_0^{t\wedge\tau_{\epsilon,R}} \nabla\phi(x_0 + W_s)dW_s.
$$

Since $\nabla\phi(x_0 + W_s)$ is bounded on $s < \tau_{\epsilon,R}$, we obtain,

$$
E\phi(x_0 + W_{t\wedge\tau_{\epsilon,R}}) = \phi(x_0).
$$

Proof, ctd.

(Recall: we have obtained)

$$E\phi(x_0 + W_{t\wedge\tau_{\epsilon,R}}) = \phi(x_0).$$

Taking $t \to \infty$, and using that $\tau_{\epsilon,R} < \infty$ a.s. (by Lemma), we get due to the Lebesgue bounded convergence theorem,

$$\phi(x_0) = E\phi(x_0 + W_{\tau_{\epsilon,R}}).$$

But at $\tau_{\epsilon,R}$ by the choice of ϕ, the value of $E\phi(x_0 + W_{\tau_{\epsilon,R}})$ coincides with

$$E\phi(x_0 + W_{\tau_{\epsilon,R}}) = P(|x_0 + W_{\tau_{\epsilon,R}}| = \epsilon).$$

Therefore,

$$P(|x_0+W_{\tau_{\epsilon,R}}| = \epsilon) = \phi(x_0) = \begin{cases} A(|x_0|^{-(d-2)} - R^{-(d-2)}),\ d \geq 3, \\ A(\ln|x_0| - \ln R),\ d = 2, \\ A(|x_0| - R),\ d = 1. \end{cases}$$

Proof, ctd.

$P_\epsilon = \lim_{R\to\infty} P(|x_0 + W_{\tau_{\epsilon,R}}| = \epsilon)$

Recall that

$$A := \begin{cases} (\epsilon^{-(d-2)} - R^{-(d-2)})^{-1}, & \text{if} \quad d \geq 3, \\ (\ln \epsilon - \ln R)^{-1}, & \text{if} \quad d = 2, \\ (\epsilon - R)^{-1}, & \text{if} \quad d = 1, \end{cases}$$

Hence, as $R \to \infty$, we obtain from

$$P(|x_0 + W_{\tau_{\epsilon,R}}| = \epsilon) = \begin{cases} A(|x_0|^{-(d-2)} - R^{-(d-2)}), \ d \geq 3, \\ A(\ln |x_0| - \ln R), \ d = 2, \\ A(|x_0| - R), \ d = 1, \end{cases}$$

that

$$P_\epsilon = \begin{cases} \epsilon^{(d-2)} |x_0|^{-(d-2)}, \ d \geq 3, \\ 1, \ d = 2, \\ 1, \ d = 1. \end{cases}$$

Proof, ctd.

Indeed,

$$\lim_{R\to\infty} \frac{(|x_0|^{-(d-2)} - R^{-(d-2)})}{(\epsilon^{-(d-2)} - R^{-(d-2)})} = \frac{|x_0|^{-(d-2)}}{\epsilon^{-(d-2)}} = \frac{\epsilon^{(d-2)}}{|x_0|^{(d-2)}};$$

$$\lim_{R\to\infty} \frac{(\ln|x_0| - \ln R)}{(\ln\epsilon - \ln R)} = 1;$$

and

$$\lim_{R\to\infty} \frac{(|x_0| - R)}{(\epsilon - R)} = 1.$$

By words, a 1D WP a.s. returns to any neighborhood – and, hence, to the origin as well – of $0 \in R^1$; a 2D WP a.s. returns to any neighborhood – but, actually, not to the origin! – of the origin in R^2; and any WP in dimension $d \geq 3$ returns to any of the origin in R^d with a probability strictly less than one (it is, hence, probably transient).

Example 13

K.6.7.4

As we have seen, in dimension $d \geq 3$ a WP is transient. Does it go to $+\infty$ by absolute value then?

Example (13)

In R^d with $d \geq 3$,

$$\lim_{t\to\infty} |W_t| = +\infty \quad \text{a.s.}$$

Consider the function which slightly differs from ϕ from the previous Example,

$$f(x) = |x|^{2-d}, \quad |x| > 0.$$

Let $|x_0| > \epsilon > 0$, and define the stopping time

$$\tau_\epsilon := \inf(t \geq 0 : |x_0 + W_t| \leq \epsilon).$$

Example 13, Proof

$f(x) = |x|^{2-d}, \quad |x_0| > \epsilon > 0, \quad \tau_\epsilon := \inf(t \geq 0 : |x_0 + W_t| \leq \epsilon)$

Let

$$X_t := |x_0 + W_{t \wedge \tau_\epsilon}|^{2-d}.$$

We have seen earlier that $\Delta f(x) = 0$ at any $|x| > 0$. Hence, the process X_t is a *bounded* martingale. (It is bounded because before τ_ϵ the modulus $|x_0 + W_{t \wedge \tau_\epsilon}|$ is greater than ϵ, and this value to the negative $2 - d$ is bounded by $1/\epsilon^{d-2}$.) By one of the limit theorems for non-negative (super)martingales, such a process has an a.s. limit as $t \to \infty$. Hence, we write,

$$|x_0|^{2-d} = EX_0 = EX_t = \lim_{t\to\infty} EX_t = E \lim_{t\to\infty} X_t,$$

the last equality by Lebesgue's bounded convergence.

Example 13, Proof, ctd.

$f(x) = |x|^{2-d}, \quad |x_0| > \epsilon > 0, \quad \tau_\epsilon := \inf(t \geq 0 : |x_0 + W_t| \leq \epsilon)$

$$
\begin{aligned}
|x_0|^{2-d} = E \lim_{t\to\infty} X_t = E \lim_{t\to\infty} |x_0 + W_{t\wedge\tau_\epsilon}|^{2-d} \\
= E \lim_{t\to\infty} \frac{1}{|x_0 + W_{t\wedge\tau_\epsilon}|^{d-2}} \\
= E \frac{1}{|x_0 + \lim_{t\to\infty} W_{t\wedge\tau_\epsilon}|^{d-2}} 1(\tau_\epsilon = \infty) + \epsilon^{2-d} E 1(\tau_\epsilon < \infty).
\end{aligned}
$$

We highlight that from the existence of the limit for X_t it follows that there exists a limit a.s. for the term

$$
\exists \quad |x_0 + \lim_{t\to\infty} W_{t\wedge\tau_\epsilon}|^{2-d}
$$

on the set $(\tau_\epsilon = \infty)$, which set has a positive P-measure (see the previous Example): $P(\tau_\epsilon = \infty) = 1 - (\epsilon/|x_0|)^{d-2}$.

Example 13, Proof, ctd.

$f(x) = |x|^{2-d}, \quad |x_0| > \epsilon > 0, \quad \tau_\epsilon := \inf(t \geq 0 : |x_0 + W_t| \leq \epsilon)$

Repeat:

$$|x_0|^{2-d} = E\frac{1}{|x_0 + \lim_{t\to\infty} W_{t\wedge\tau_\epsilon}|^{2-d}} 1(\tau_\epsilon = \infty)$$

$$+\epsilon^{2-d} E1(\tau_\epsilon < \infty).$$

The last term here equals (see the previous Example)

$$\epsilon^{2-d} E1(\tau_\epsilon < \infty) = \epsilon^{2-d} P(\tau_\epsilon < \infty)$$
$$= \epsilon^{2-d} \epsilon^{(d-2)} |x_0|^{-(d-2)} = |x_0|^{2-d}.$$

So (we drop "$\wedge\tau_\epsilon$" on $\tau_\epsilon = \infty$),

$$E\frac{1}{|x_0 + \lim_{t\to\infty} W_t|^{2-d}} 1(\tau_\epsilon = \infty) = 0.$$

Proof, ctd

$E \frac{1}{|x_0 + \lim_{t\to\infty} W_t|^{2-d}} 1(\tau_\epsilon = \infty) = 0$ for any $\epsilon > 0$

This means that for any $\epsilon > 0$

$$\lim_{t\to\infty} |W_t| = +\infty \quad \text{a.s. on the set } (\tau_\epsilon = \infty).$$

But the union of all these sets $(\tau_\epsilon = \infty)$ (which increase as ϵ decreases) equals $\bigcup_{\epsilon>0}(\tau_\epsilon = \infty) = \bigcup_{m\geq 1}(\tau_{1/m} = \infty)$, which probability equals

$$P(\bigcup_{m\geq 1}(\tau_{1/m} = \infty)) = \lim_{m\to\infty} P(\tau_{1/m} = \infty)$$

$$= 1 - \lim_{m\to\infty} \frac{1}{(m|x_0|)^{d-2}} = 1.$$

Therefore,

$$\lim_{t\to\infty} |W_t| = +\infty \quad \text{a.s.},$$

as required.