Introduction to stochastic differential equations – 5b Markov property

Alexander Veretennikov[1]

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[1]National Research University HSE, Moscow State University, Russia

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Definitions

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \ t \geq 0 \qquad (1)$$

Definition (1)

The process $X_t, t \geq 0$ is called Markov iff for any finite sequence of non-random times $0 \leq t_1 < \ldots < t_{n+1}$ the equality holds a.s. $E(g(X_{t_{n+1}})|X_{t_n}, \ldots X_{t_1})$ = $E(g(X_{t_{n+1}})|X_{t_n})$, $\forall g \in B$. The process adapted to the filtration $(\mathcal{F}_t)$ is called Markov *with respect to this filtration* iff for any $s \leq t$, the equality holds a.s. $E(g(X_t)|\mathcal{F}_s) = E(g(X_t)|X_s)$.

Definition (2)

The process X is strong Markov iff for any a.s. finite stopping times $\tau_1 \leq \tau_2$, $E(g(X_{\tau_2})|\mathcal{F}^X_{\tau_1}) = E(g(X_{\tau_2})|X_{\tau_1})$, and strong Markov with respect to $(\mathcal{F}_t)$ iff almost surely $E(g(X_{\tau_2})|\mathcal{F}_{\tau_1}) = E(g(X_{\tau_2})|X_{\tau_1})$ for any Borel bounded g.

Preliminaries: idea of the Markov property

Lipschitz conditions and linear growth is assumed on b, σ

For any $t_n \geq 0$ (there will be $0 \leq t_1 \leq \ldots \leq t_n$) an SDE may be considered

$$\tilde{X}_t = \xi_{t_n} + \int_{t_n}^{t} b(s, \tilde{X}_s)ds + \int_{t_n}^{t} \sigma(s, \tilde{X}_s)dW_s, \quad t \geq t_n, \quad (2)$$

where the r.v. ξ_{t_n} is $\mathcal{F}^W_{t_n}$-adapted. Now, "clearly" solution $\tilde{X}$ after time t_n is determined by ξ_{t_n} and by the increments of W after time t_0, which increments do not depend on $\tilde{X}_{t_n}$. Hence, if we have a solution X of the equation (1) from 0 to t_n, and define $\xi_{t_n} = X_{t_n}$ (X being the unique solution of (1)), then this (strong) solution does not depend on the past trajectory X before t_n *given* X_{t_n}. Due to the uniqueness, solutions $\tilde{X}_t$ and X_t after t_n coincide a.s.

However, even for WP W itself there is something to prove so as to establish its Markov property. Naturally, there is also something to prove rigorously for SDE solutions.

Preliminaries, ctd.

Let us fix y and consider the SDE (2) with a non-random i.c.

$$Y_t = y + \int_{t_n}^{t} b(s, Y_s)ds + \int_{t_n}^{t} \sigma(s, Y_s)dW_s, \quad t \geq t_n. \qquad (3)$$

The theory of such SDEs under our conditions is similar to the one in the case $t_n = 0$. The solution is denoted by Y_t^y.

Lemma (K.6.11.2)

Y_t^y is measurable wrt $\mathcal{F}_{[t_n,t]}^{\Delta W} \equiv \sigma(W_s - W_{t_n}, t_n \leq s \leq t)$.

Follows from the change of time $t - t_n = t'$.

Lemma (K.6.11.3)

The σ-fields $\mathcal{F}_{[t_n,t]}^{\Delta W}$ and $\mathcal{F}_{t_n}^{W}$ are independent.

Preliminaries, ctd.

For the Proof of the Lemma [K.6.11.3] it suffices to show the equality

$$P(B; W_{t_n+s_1} - W_{t_n} \in \Gamma_1, \ldots W_{t_n+s_k} - W_{t_n+s_{k-1}} \in \Gamma_1)$$
$$= P(B)P(W_{t_n+s_1} - W_{t_n} \in \Gamma_1, \ldots W_{t_n+s_k} - W_{t_n+s_{k-1}} \in \Gamma_1)$$

for any $B \in \mathcal{F}^W_{t_n}$, $\forall k$, $\forall \Gamma_i \in \mathcal{B}(R^d)$, and $\forall\, 0 \leq s_1 < \ldots < s_k$. This equality is a consequence of the definition of WP, that is, of the independence of the increments

$$W_{t_n+s_1} - W_{t_n}, \ldots W_{t_n+s_k} - W_{t_n+s_{k-1}}$$

of the sigma-algebra $\mathcal{F}^W_{t_n}$. The Lemma follows.

Preliminaries, ctd.

$$Y_t^y = y + \int_{t_n}^t b(s, Y_s^y)ds + \int_{t_n}^t \sigma(s, Y_s^y)dW_s, \quad t \geq t_n \qquad (3)$$

Corollary (K.6.11.4)

For any $t \geq t_n$ and any $x \in R^d$, the r.v. Y_t^y and sigma - algebra $\mathcal{F}_{t_n}^W$ are independent.

Follows from the previous two Lemmata.

Lemma (K.6.11.5)

Let $\xi^m = 2^{-m}[2^m X_{t_n}]$, where X_t is a solution of the equation (1). Then

$$Y_t^{\xi^m} \xrightarrow{P} X_t^{t_n, X_{t_n}} = X_t, \quad m \to \infty, \ \forall\, t \geq t_n.$$

Proof. On the set $(\omega \in \Omega : \xi^m = y)$ we have $Y_t^{\xi^m} = Y_t^y$ (by definition of $Y_t^{\xi^m}$), where Y_t^y solves (3) with the i.c. y.

Proof of Lemma K.6.11.5, ctd.

$Y_t^y = y + \int_{t_n}^t b(s, Y_s^y)ds + \int_{t_n}^t \sigma(s, Y_s^y)dW_s, \quad t \geq t_n \qquad (3)$

Recall: on the event $(\omega \in \Omega : \xi^m = y)$ we have $Y_t^{\xi^m} = Y_t^y$. The union of all (no more than countably many) such events is the whole Ω. This means[2] that the process $Y_t^{\xi^m}$ satisfies the equation (3) **with** y **replaced by** ξ^m:

$$Y_t^{\xi^m} = \xi^m + \int_{t_n}^t b(s, Y_s^{\xi^m})ds + \int_{t_n}^t \sigma(s, Y_s^{\xi^m})dW_s, \quad t \geq t_n.$$

Also, X_t for $t \geq t_n$ satisfies (3) **with** y **replaced by** X_{t_n}:

$$X_t = X_{t_n} + \int_{t_n}^t b(s, X_s)ds + \int_{t_n}^t \sigma(s, X_s)dW_s, \quad t \geq t_n.$$

[2]We are using some Lemma about stochastic integrals from one of the previous lectures

Why $X_t^{t_n, X_{t_n}} = X_t$

$$X_t = X_{t_n} + \int_{t_n}^t b(s, X_s)ds + \int_{t_n}^t \sigma(s, X_s)dW_s, \quad t \geq t_n$$

This follows from the equation on X_t if we subtract $X_t - X_{t_n}$. In other words, X_t on $t \geq t_n$ serves as a solution of the equation for Y_t^ξ:

$$Y_t^\xi = \xi + \int_{t_n}^t b(s, Y_s^\xi)ds + \int_{t_n}^t \sigma(s, Y_s^\xi)dW_s, \quad t \geq t_n,$$

with $\xi \in \mathcal{F}_{t_n}^W$. Compare these two equations: if $\xi = X_{t_n}$ then they coincide. However, the solution of this equation is unique (by Ito's theorem for the equations on $t \geq t_n$). Hence, we have,

$$X_t^{t_n, X_{t_n}} = X_t, \quad t \geq t_n.$$

They both coincide with $Y_t^{X_{t_n}}$ by definition of Y_t^ξ. The Theorem follows from the continuity theorem as $\xi^m \rightrightarrows X_{t_n}$.

Theorem (K.6.11.6)

The solution of equation (1) is a Markov process.

Proof. Let us take $t \geq t_n$ and any bounded *continuous* function $f(x) \geq 0$. We will now show

$$E(f(X_t)|\mathcal{F}_{t_n}^W) = E(f(X_t)|X_{t_n}), \quad \text{a.s.}$$

This will not be the "full Markov property", but some restricted version of it, just for $f \in C_b$; the full Markov property would be this equality a.s. for any Borel bounded function f. We just start with a continuous f, which will be later extended to all Borel bounded ones.

Proof of Theorem K6.11.6, ctd.

$$Y_t^y = y + \int_{t_n}^t b(s, Y_s^y)ds + \int_{t_n}^t \sigma(s, Y_s^y)dW_s, \quad t \geq t_n \qquad (3)$$

Denote $\Phi(y) := Ef(Y_t^y)$. This function is *continuous*. Let us now take any $B \in \mathcal{F}_{t_n}^W$. By Corollary K.6.11.4 and Lemma K.6.11.5, we have a sequence of equalities

$$E1(B)f(X_t) \overset{Le\,K.6.11.5}{=} \lim_{m\to\infty} E1(B)f(Y_t^{\xi^m})$$

$$= \lim_{m\to\infty} \sum_{r=-\infty}^{\infty} E1(B)f(Y_t^{r/2^m})1(\xi^m = r/2^m)$$

$$\overset{Cor\,K.6.11.4}{=} \lim_{m\to\infty} \sum_{r=-\infty}^{\infty} E1(B, \xi^m = r/2^m)E(f(Y_t^{r/2^m})|\mathcal{F}_{t_n}^W)$$

$$= \lim_{m\to\infty} \sum_{r=-\infty}^{\infty} E1(B)1(\xi^m = r/2^m)\Phi(r/2^m)$$

$$= \lim_{m\to\infty} E1(B)\Phi(\xi^m) = E1(B)\Phi(X_{t_n}).$$

Proof of Theorem, ctd.

"Restricted Markov property for $f \in C_b$"

$$E1(B)f(X_t) = E1(B)\Phi(X_{t_n}).$$

By definition of conditional expectations this implies the equality

$$E(f(X_t)|\mathcal{F}^W_{t_n}) = \Phi(X_{t_n}),$$

which, in turn, implies

$$E(f(X_t)|X_{t_n}) = E(E(f(X_t)|\mathcal{F}^W_{t_n})|X_{t_n}) = \Phi(X_{t_n}),$$

and

$$E(f(X_t)|X_{t_n}, \ldots X_{t_1}) = E(E(f(X_t)|\mathcal{F}^W_{t_n})|X_{t_n}, \ldots X_{t_1}) = \Phi(X_{t_n}).$$

So, $\forall f \in C_b$

$$E(f(X_t)|X_{t_n}) = E(f(X_t)|X_{t_n}, \ldots X_{t_1}) \quad (a.s.) \tag{4}$$

Proof of Theorem, ctd. $E(f(X_t)|X_{t_n}) \overset{C_b}{=} E(f(X_t)|X_{t_n}, \ldots X_{t_1})$ (4)

It remains to extend this "restricted Markov property" on all $f \in B(R^d)$

Let us fix any $B \in \mathcal{F}_{t_n}^W$ and consider two measures,

$$\mu(\Gamma) = E1(B)E(1(X_t \in \Gamma)|X_{t_n}, \ldots X_{t_1}), \nu(\Gamma) = E1(B)E(1(X_t \in \Gamma)|X_{t_n}).$$

We will integrate bounded function f over each measure μ and ν. For any step function f we have, $\int f(x)\mu(dx) =$

$$= E1(B)E(f(X_t)|X_{t_n}, \ldots X_{t_1}), \int f(x)\nu(dx) = E1(B)E(f(X_t)|X_{t_n}).$$

So, it is also true for any $f \in C_b$, as such f can be *uniformly* approximated by a sequence of step functions. Thus, (4) implies

$$\int f(x)\mu(dx) = \int f(x)\nu(dx), \ \forall f \in C_b.$$

So, $\mu \overset{K.1.2.4}{=} \nu \Longrightarrow E(f(X_t)|X_{t_n}) = E(f(X_t)|X_{t_n}, \ldots X_{t_1}), \ \forall f \in B.$

Strong Markov processes

Strong Markov property is rather useful for studying ergodic properties.

Recall

Definition (2)

The process X is called strong Markov iff for any a.s. finite stopping times $\tau_1 \le \tau_2$, a.s. $E(g(X_{\tau_2})|\mathcal{F}^X_{\tau_1}) = E(g(X_{\tau_2})|X_{\tau_1})$, and strong Markov with respect to $(\mathcal{F}_t)$ iff almost surely $E(g(X_{\tau_2})|\mathcal{F}_{\tau_1}) = E(g(X_{\tau_2})|X_{\tau_1})$ for any Borel bounded g.

There is a nice simple result (see, e.g., [A.D. Wentzel, Theory of random processes ("Kurs teorii sluchajnyh protsessov")]) for discrete time: **any Markov process in discrete time is strong Markov**. Yet, for SDEs it is not suitable. However, it (or, rather, its proof) prompts that some sufficient conditions for a **strong** Markov property may be established if there is some kind of separability, or, more precisely, if the process may be well approximated by its values, say, at rational times. Feller property is the key.

Feller processes

and continuous trajectories

Definition (3)

The Markov process X is called Feller iff for any $f \in C_b$ the function $E_x f(X_t)$ is continuous wrt x for any $t \geq 0$

Under our standard assumptions of Lipschitz and linear growth, the solution of SDE is Feller (follows from the continuity theorem). The following theorem closes the issue.

Theorem (without proof, see (A.D. Wentzel, §9.2))

Any Markov process X with trajectories continuous from the right which is Feller, is also strong Markov with respect to the filtration $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

In other words, $E(X_{\tau_2}|\mathcal{F}_{\tau_1+}) = E(X_{\tau_2}|X_{\tau_1})$ if $\tau_1 \leq \tau_2$ a.s.
Here $\mathcal{F}_{\tau+} = \sigma(A : A \cap (\tau \leq t) \in \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s)$.

The next result is a bit "abstract" and not very useful since usually if the assumptions of the theorem are satisfied then we may show weak uniqueness easier by other means. Let $b, \sigma \in B$, $a = \sigma\sigma^*$ uniformly non-degenerate.

Theorem (follows from N.V. Krylov & M.V. Safonov)

Let there exist a (classical or in $W^{1,2}_{d+1}$) solution of a PDE

$$u_s(s,x) + Lu(s,x) = 0,\ 0 \le s \le T, \quad u(T,x) = g(x),$$

for any $g \in C_b(R^d)$. Then X_t^x is Feller.

The matter is Hölder continuity of $u(s,x)$. Note that no regularity is assumed explicitly except for the measurability of coefficients, but solution should exist for any x.

Weak uniqueness and strong Markov

The last remark is about one more general result (not to be proved in this course); Lipschitz is not assumed.

Theorem (N.V. Krylov)

Any solution of an SDE which possesses weak uniqueness is Markov and strong Markov with respect to the filtration $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$.

Krylov, N. V. (1973). On the selection of a Markov process from a system of processes and the construction of quasi-diffusion processes . Mathematics of the USSR - Izvestija , 7 (3), 691-709;
Nisio, M., On the Existence of Solutions of Stochastic Differential Equations, Osaka J. Math., 1973, 10(1), 185-208.

Weak uniqueness

Sufficient conditions for weak uniqueness for an SDE are:

1. Strong uniqueness.
2. Uniform non-degeneracy of the diffusion coefficient and its continuity (uniform wrt time). In 1D the requirement of continuity of σ may be dropped.
3. In some cases the continuity assumption may be relaxed. [Krylov, N. V. (2004). On weak uniqueness for some diffusions with discontinuous coefficients. Stochastic Processes and their Applications, 113(1), 37-64. https://doi.org/10.1016/j.spa.2004.03.012]
4. Some more assumptions are available at [A.Yu. Veretennikov, On Weak Solutions of Highly Degenerate SDEs, Automation and Remote Control, 2020, 81(3), 398-410. DOI 10.1134/S0005117920030029].