SDEs introduction

Introduction to stochastic differential equations – 3
Links to PDEs

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Heat equation

Example 1

We start with a (d-dimensional) Wiener process W_t and with a few simplest examples. The first one is about heat equation and its relation to WP.

Example (1)

Let $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$ be a solution of the heat equation

$$u_t(t,x) + \frac{1}{2}\Delta u(t,x) = 0, \quad 0 \le t \le T,$$
$$u(T,x) = g(x),$$

with $g \in C_b^2(R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = Eg(x + W_{T-t}).$$

Proof

For the proof, let us apply Ito's formula to $u(t_0+s, x+W_s)$ for $0 \le t_0 < T$ (since $u(T,x) = g(x) \equiv Eg(x+W_0)$ without any calculus):

$$du(t_0+s, x+W_s) = \nabla u(t_0+s, x+W_s)dW_s$$
$$+[u_s(t_0+s, x+W_s) + \frac{1}{2}\Delta u(t_0+s, x+W_s)]ds.$$

In the integral form with $t_0+s=T$,

$$u(T, x+W_{T-t_0}) = u(t_0, x) + \int_0^{T-t_0} \nabla u(t_0+s, x+W_s)dW_s$$
$$+\int_0^{T-t_0}[u_s(t_0+s, x) + \frac{1}{2}\Delta u(t_0+s, x+W_s)]ds.$$

Example 1, Proof, ctd.

Let us now take expectations from both sides of this equality:

$$Eu(T, x + W_{T-t_0}) = u(t_0, x),$$

because

$$E \int_0^{T-t_0} \nabla u(t_0 + s, x + W_s) dW_s = 0,$$

$$\& \quad [u_s(t_0, x) + \frac{1}{2} \Delta u(t_0 + s, x + W_s)] = 0.$$

Remark

The condition $g \in C_b^2(R^d)$ follows automatically from $u(t, x) \in C_b^{1,2}([0, T] \times R^d)$. Both of them can be relaxed.

Relaxed Example 1

Example (2)

Let $u(t,x) \in C_b^{1,2}((0,T) \times R^d) \bigcap C_b([0,T] \times R^d)$ be a solution of the heat equation

$$\begin{aligned} u_t(t,x) + \frac{1}{2}\Delta u(t,x) = 0, \quad 0 \le t \le T, \\ u(T,x) = g(x), \end{aligned}$$

with $g \in C_b(R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = Eg(x + W_{T-t}).$$

The conditions of boundedness of g and u with its derivatives may be further considerably relaxed, too.

Proof of Example 2

Note that the differential form of Ito's equation remains valid,

$$du(t_0 + s, x + W_s) = \nabla u(t_0 + s, x + W_s) dW_s$$
$$+[u_s(t_0 + s, x + W_s) + \frac{1}{2}\Delta u(t_0 + s, x + W_s)]ds.$$

Yet, now we cannot simply integrate it to T, because the derivatives are assumed only on the open interval $(0, T)$. Firstly let us consider $t_0 > 0$. Denote $T_n := T - \frac{1}{n}$. Then, for n such that $t_0 < T_n$ we have,

$$u(T_n, x + W_{T_n - t_0}) = u(t_0, x) + \int_0^{T_n - t_0} \nabla u(t_0 + s, x + W_s) dW_s$$
$$+ \int_0^{T_n - t_0} [u_s(t_0 + s, x) + \frac{1}{2}\Delta u(t_0 + s, x + W_s)]ds.$$

Proof of Example 2, ctd.

$$u(T_n, x + W_{T_n - t_0}) = u(t_0, x) + \int_0^{T_n - t_0} \nabla u(t_0 + s, x + W_s) dW_s$$
$$+ \int_0^{T_n - t_0} [u_s(t_0 + s, x) + \frac{1}{2} \Delta u(t_0 + s, x + W_s)] ds.$$

Let us take expectations here: since

$$[u_s(t_0 + s, x) + \frac{1}{2} \Delta u(t_0 + s, x + W_s)] = 0$$

and because

$$E \int_0^{T_n - t_0} \nabla u(t_0 + s, x + W_s) dW_s = 0,$$

we get

$$Eu(T_n, x + W_{T_n - t_0}) = u(t_0, x).$$

$$u(t_0, x) = Eu(T_n, x + W_{T_n - t_0}).$$

Here we can pass to the limit as $T_n \uparrow T$ in the r.h.s.: since the function u is continuous and bounded up to T, and because W is continuous in time, we get by Lebesgue's bounded convergence theorem that again

$$u(t_0, x) = Eu(T, x + W_{T-t_0}) \equiv Eg(x + W_{T-t_0}),$$

as required. Moreover, in the latter equation we can again pass to the limit as $t_0 \downarrow 0$, to get by the same reasoning

$$u(0, x) = Eg(x + W_T).$$

Example 3

Non-zero right-hand side (rhs)

Now let us consider the equation with a non-zero r.h.s.

Example (3)

Let $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$ be a solution of the heat equation

$$u_t(t,x) + \frac{1}{2}\Delta u(t,x) = -f(t,x), \quad 0 \le t \le T,$$
$$u(T,x) = g(x),$$

with $g \in C_b^2(R^d)$, $f(t,x) \in C_b([0,T] \times R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = E\left[\int_0^{T-t} f(t+s, x+W_s)ds + g(x+W_{T-t})\right].$$

Proof of Example 3

Recall Ito's formula,

$$du(t_0+s,x+W_s) = \nabla u(t_0+s,x+W_s)dW_s$$
$$+[u_s(t_0+s,x+W_s)+\frac{1}{2}\Delta u(t_0+s,x+W_s)]ds.$$

Now it can be rewritten as follows,

$$du(t_0+s,x+W_s) = \nabla u(t_0+s,x+W_s)dW_s$$
$$-f(t_0+s,x+W_s)ds,$$

or, in the integral form,

$$u(T,x+W_{T-t_0}) = u(t_0,x) + \int_0^{T-t_0} \nabla u(t_0+s,x+W_s)dW_s$$
$$-\int_0^{T-t_0} f(t_0+s,x+W_s)ds.$$

Proof of Example 3, ctd.

Taking expectations from both sides we get,

$$
\begin{aligned}
u(t_0, x) &= Eu(T, x + W_{T-t_0}) + E \int_0^{T-t_0} f(t_0 + s, x + W_s) ds \\
&= Eg(x + W_{T-t_0}) + E \int_0^{T-t_0} f(t_0 + s, x + W_s) ds,
\end{aligned}
$$

as required.

Remark

Conditions of the Example may also be relaxed, as earlier, assuming derivatives only in the open cylinder $((0, T) \times R^d)$ *along with continuity of u only in the closed cylinder* $([0, T] \times R^d)$. *Yet, it is not all that may be relaxed here.*

The issue is that for heat equations with a non-zero r.h.s. it is not often that solutions are classical, that is, from $C_b^{1,2}$.

How to verify that solution $u \in C_b^{1,2}$?

In PDE theory often solutions are only with Sobolev derivatives!

However, in the case of the Laplacian there is a simple approach to check it. Let us take the function

$$v(t,x) = E\left[\int_0^{T-t} f(t+s, x+W_s)ds + g(x+W_{T-t})\right].$$

We do not know whether it is a classical solution of the heat equation with the r.h.s. f and a terminal condition g; but after the previous examples we guess that this is likely to be a solution. Can we check that this function belongs to one of the classes we considered in the previous Examples? In other words, the task is as follows: how can we differentiate the function v w.r.t. x and t? From the first sight this looks doubtful because the trajectories of W are only Hölder continuous but not differentiable.

Differentiate v^1

Nevertheless, if we write the expectation (for each summand separately) in the form (change of variables $z = y + x$)

$$v^1(t,x) = Eg(x + W_{T-t})$$
$$= \int g(x+y)\frac{1}{(2\pi(T-t))^{d/2}}\exp(-\frac{1}{2(T-t)}y^2)dy$$
$$= \int g(z)\frac{1}{(2\pi(T-t))^{d/2}}\exp(-\frac{1}{2(T-t)}(x-z)^2)dz,$$

it becomes clear that this expression, indeed, is differentiable both in x and in t. It is a good exercise to check that $v^1 \in C_b^{1,2}$ here and that the equation holds true without using WP,

$$v_t^1 + \frac{1}{2}\Delta v^1 = 0, \quad \& \quad v^1(T,x) = g(x).$$

Differentiate v^2

For the other term (we use Fubini theorem & change of variables $t + s = r$, so that $s = r - t$)

$$v^2(t,x) = \int_0^{T-t} Ef(t+s, x+W_s)ds,$$

we can also use the density of WP, hence, rewriting it as

$$\begin{aligned} v^2(t,x) &= \int_t^T Ef(r, x+W_{r-t})dr \\ &= \int_t^T dr \int_{R^d} \frac{f(r,x+y)}{(2\pi(r-t))^{d/2}} \exp(-\frac{y^2}{2(r-t)})dy \\ &= \int_t^T dr \int_{R^d} \frac{f(r,z)}{(2\pi(r-t))^{d/2}} \exp(-\frac{1}{2(r-t)}(x-z)^2)dz. \end{aligned}$$

It is also a good exercise to differentiate this expression in t and (twice) in x and to check the corresponding equation.

Example 4

Homework! Here c is a constant, but it may be made variable.

Similarly a PDE "with a potential" can be considered.

Example (4)

Let $u(t,x) \in C_b^{1,2}([0,T] \times R^d)$ be a solution of the heat equation with a potential

$$u_t(t,x) + \frac{1}{2}\Delta u(t,x) - cu(t,x) = -f(t,x), \quad 0 \le t \le T,$$
$$u(T,x) = g(x),$$

with $g \in C_b^2(R^d)$, $f(t,x) \in C_b([0,T] \times R^d)$. Then for any $0 \le t \le T$ the value $u(t,x)$ can be represented in the form

$$u(t,x) = E\int_0^{T-t} e^{-cs} f(t+s, x+W_s)ds$$
$$+Ee^{-c(T-t)} g(x+W_{T-t}).$$

Example 5: Laplace equation

Probabilists like the multiplier $1/2$; of course, in the equation it is redundant.

Let D be a bounded domain (by definition open one and connected; condition to be connected can be dropped, it is just for simplicity) in R^d. Consider the Laplace equation

$$\frac{1}{2}\Delta u(x) = 0, \ x \in D, \quad \& \quad u|_\Gamma = \phi(x),$$

where $\Gamma = \partial D$ is the boundary of D. Denote $D^c := R^d \setminus D$. Let

$$\tau := \inf(t \geq 0 : x + W_t \in D^c).$$

Example (5)

Let $u(x) \in C_b^2(\bar{D})$ be a solution of the Laplace with $\phi \in C(\bar{D})$. Then $u(x)$ can be represented as

$$u(x) = E\phi(x + W_\tau), \quad x \in D.$$

Proof of Example 5

$\frac{1}{2}\Delta u(x) = 0,\ x \in D, \quad \& \quad u|_\Gamma = \phi(x)$

Let us apply Ito's formula to $u(x + W_t)$:

$$du(x + W_t) = \nabla u(x + W_t) dW_t + \frac{1}{2}\Delta u(x + W_t) dt.$$

In the integral form we have (assuming $u \in C_b^2(R^d)$),

$$u(x + W_t) - u(x) = \int_0^t \nabla u(x + W_s) dW_s + \frac{1}{2}\int_0^t \Delta u(x + W_s) ds,$$

but it is not what we need, because, remember, we do not know anything about u outside $\bar{D}$, or, at most, outside some its neighbourhood. So, we are to use stopping time τ.

Proof of Example 5, ctd.

It is also true that left hand side (lhs) here equals right hand side (rhs) if we integrate from 0 to $t \wedge \tau$ (for which $u \in C_b^2(\bar{D})$ suffices):

$$u(x + W_{t\wedge\tau}) - u(x) = \int_0^{t\wedge\tau} \nabla u(x + W_s) dW_s.$$

Take expectations:

$$Eu(x + W_{t\wedge\tau}) - u(x) = E\int_0^{t\wedge\tau} \nabla u(x + W_s) dW_s = 0.$$

So,

$$u(x) = Eu(x + W_{t\wedge\tau}).$$

This is true for any $t > 0$ and we want now to let $t \to \infty$.

Proof of Example 5, ctd.

An issue: why $P(\tau < \infty) = 1$, or even better, $E \int_0^\infty 1(s < \tau) |\nabla u(x + W_s)|^2 ds < \infty$?

If we knew that $\tau < \infty$ a.s., then in the limit we would have, clearly,

$$u(x) = Eu(x + W_\tau).$$

And since $x + W_\tau \in \Gamma$, and due to continuity of u, and because of the boundary condition (called Dirichlet's b.c.), we would conclude that, indeed,

$$u(x) = E\phi(x + W_\tau),$$

as required. But why this property – or, equivalently, why $E\tau < \infty$ – holds? To resolve this difficulty, recall that $x + W_t$ is a Markov and strong Markov process (the material of the autumn semester: a strong advice is to repeat this stuff). Note that $\inf_x P_x(x + W_1 \not\in D) > 0$.

Lemma for Markov process (MP)

For any MP, $\inf_x P_x(\text{exit from } D \text{ on } [0,1]) > 0 \Longrightarrow \sup_x E_x\tau < \infty$.

Lemma

Let for an MP X_t and a domain D exist $q > 0$ such that

$$P_x(X_t \text{ exits from } D \text{ on } [0,1]) \geq q.$$

Then $\sup_x E_x\tau < \infty$, where $\tau = \inf(t \geq 0 : X_t \notin D)$; in particular, $P_x(\tau < \infty) = 1$. More than that, $\exists\, \alpha > 0$ such that

$$\sup_x E_x \exp(\alpha\tau) < \infty.$$

Proof. All these claims follow from the inductive estimate

$$P_x(\tau > n) = E_x 1(\tau > n-1) E_{X_{n-1}} 1(\tau > 1)$$
$$\leq E_x(1-q)1(\tau > n-1) \leq \ldots \leq (1-q)^n.$$

Proof of Example 5, ctd.

Using the Lemma

Now we can complete the proof of this Example. We have,

$$P_x(x + W_1 \notin D) = \int_{R^d \setminus D} \frac{1}{(2\pi)^{d/2}} \exp(-\frac{1}{2}(y-x)^2) dy$$
$$= 1 - \int_D \frac{1}{(2\pi)^{d/2}} \exp(-\frac{1}{2}(y-x)^2) dy \geq q > 0.$$

[Consider the last inequality as a homework!]

Hence, by Lemma, $P(\tau > n) \leq q^n$, and $\sup_x E_x \tau < \infty$, as required. Therefore, the claim of the Example 5 follows,

$$u(x) = E\phi(x + W_\tau).$$

Averaging properties of WP

$|B(r)| = d$-volume of $B(r)$, $|\Gamma(r)| = d-1$-area of the $(d-1)$-surface $\Gamma(r)$

Before we start a new topic, Poisson equations, let us discuss some interesting and useful corollary. Assume that $x \in D$, and let $B_x(r)$ denote the open ball of radius r with x the center. Suppose $r > 0$ is small enough, so that $B_x(r) \subset\subset D$. Denote also $\tau^r := \inf(t \geq 0 : x + W_t \notin B_x(r))$.

Corollary

We have,

$$u(x) = Eu(x + W_{\tau^r}) = \frac{1}{|\Gamma(r)|} \int_{\Gamma_x(r)} u(y)dy,$$

and also

$$u(x) = Eu(x + W_{\tau^r}) = \frac{1}{|B(r)|} \int_{B_x(r)} u(z)dz.$$

Proof of averaging properties

We show the first one, because the second one follows by one more integration from it. Due to the strong Markov property, we have, denoting $X_t = x + W_t$,

$$
\begin{aligned}
u(x) = E\phi(x + W_\tau) &= E(E\phi(x + W_\tau)|\mathcal{F}_{\tau^r}) \\
= E(E\phi(x + W_\tau)|x + W_{\tau^r}) &= E(E\phi(X_\tau)|X_{\tau^r}) \\
= Eu(X_{\tau^r}) &= \frac{1}{|\Gamma(r)|}\int_{\Gamma_x(r)} u(y)dy,
\end{aligned}
$$

the last equality *by the symmetry of W:* for the WP starting from x, to hit any area on $\Gamma_x(r)$ at stopping time τ^r is proportional to the $d-1$-dimensional volume of this area. *[Homework: Show that any Borel measurable bounded function satisfying the two averaging properties above must be continuous in x. (In fact, this property even implies the Laplace equation in D for such a function.)]*

Example 6, Poisson equation

It is a "Laplace equation with a non-trivial rhs".

Let D be a bounded domain in R^d. Consider the Poisson equation

$$\frac{1}{2}\Delta u(x) = -\psi(x), \; x \in D, \quad \& \quad u(x)|_\Gamma = \phi(x),$$

where $\Gamma = \partial D$ is the boundary of D. Recall that $D^c := R^d \setminus D$, $\tau := \inf(t \geq 0 : x + W_t \in D^c)$.

Example (6)

Let $u(x) \in C_b^2(\bar{D})$ be a solution of the Poisson equation with $\phi \in C(\Gamma)$, $\psi \in C(\bar{D})$. Then $u(x)$ in D can be represented as

$$u(x) = E[\int_0^\tau \psi(x + W_s)ds + \phi(x + W_\tau)].$$

Proof

By Ito's formula,

$$du(x+W_t) = \nabla u(x+W_t)dW_t + \frac{1}{2}\Delta u(x+W_t)dt$$
$$= \nabla u(x+W_t)dW_t - \psi(x+W_t)dt.$$

So, in the integral form with a stopping time,

$$u(x+W_{t\wedge\tau}) - u(x) = \int_0^{t\wedge\tau} \nabla u(x+W_s)dW_s$$
$$- \int_0^{t\wedge\tau} \psi(x+W_s)ds.$$

Taking expectations, we get

$$Eu(x+W_{t\wedge\tau}) - u(x) = -E\int_0^{t\wedge\tau} \psi(x+W_s)ds.$$

Proof of Example 6, ctd.

Recalling that $\sup_x E_x\tau < \infty$ and letting $t \to \infty$, we have due to continuity of u, W and the integral wrt t and by virtue of Lebesgue's dominated convergence theorem,

$$Eu(x + W_\tau) - u(x) = -E\int_0^\tau \psi(x + W_s)ds,$$

or, equivalently,

$$u(x) = E\psi(x + W_\tau) + E\int_0^\tau \psi(x + W_s)ds,$$

as required.

Example 7

Poisson equation with a potential $c(\cdot)$

Let D be a bounded domain in R^d. Consider the Poisson equation with a (variable) potential $0 \leq c(x) \in C(\bar{D})$

$$\frac{1}{2}\Delta u(x) - c(x)u(x) = -\psi(x), \; x \in D, \quad \& \quad u(x)|_\Gamma = \phi(x).$$

Denote $\kappa(t) := \int_0^t c(x + W_s)ds$. Recall that $D^c := R^d \setminus D$, $\tau := \inf(t \geq 0 : x + W_t \in D^c)$.

Example (7)

Let $u(x) \in C_b^2(\bar{D})$ be a solution of the Poisson equation with $\phi \in C(\Gamma)$, $\psi \in C(\bar{D})$. Then $u(x)$ in D can be represented as

$$u(x) = E[\int_0^\tau e^{-\kappa(s)}\psi(x + W_s)ds + e^{-\kappa(\tau)}\phi(x + W_\tau)].$$

Proof of Example 7

$\kappa(t) := \int_0^t c(x + W_s)ds$

By Ito's formula,

$$de^{-\kappa(t)}u(x+W_t) = e^{-\kappa(t)}\nabla u(x+W_t)dW_t$$

$$+e^{-\kappa(t)}[\frac{1}{2}\Delta u(x+W_t) - c(x+W_t)u(x+W_t)]dt$$

$$= e^{-\kappa(t)}\nabla u(x+W_t)dW_t - e^{-\kappa(t)}\psi(x+W_t)dt.$$

So, in the integral form with a stopping time,

$$e^{-\kappa(t\wedge\tau)}u(x+W_{t\wedge\tau}) - u(x) = \int_0^{t\wedge\tau} e^{-\kappa(s)}\nabla u(x+W_s)dW_s$$

$$-\int_0^{t\wedge\tau} e^{-\kappa(s)}\psi(x+W_s)ds.$$

Taking expectations, we get

$$Ee^{-\kappa(t\wedge\tau)}u(x+W_{t\wedge\tau}) - u(x) = -E\int_0^{t\wedge\tau} e^{-\kappa(s)}\psi(x+W_s)ds.$$

Proof of Example 7, ctd.

$\kappa(t) := \int_0^t c(x + W_s) ds$

From the equation

$$Ee^{-\kappa(t\wedge\tau)}u(x + W_{t\wedge\tau}) - u(x) = -E\int_0^{t\wedge\tau} e^{-\kappa(s)}\psi(x + W_s)ds,$$

by letting $t \to \infty$, we obtain due to continuity of all terms in t, because of $\sup_x E\tau < \infty$, and by virtue of the Lebesgue dominated convergence theorem,

$$Ee^{-\kappa(\tau)}u(x + W_\tau) - u(x) = -E\int_0^{\tau} e^{-\kappa(s)}\psi(x + W_s)ds,$$

or, equivalently,

$$u(x) = Ee^{-\kappa(\tau)}u(x + W_\tau) + E\int_0^{\tau} e^{-\kappa(s)}\psi(x + W_s)ds,$$

as required. Note that the condition $c \geq 0$ was essential.

Remark on the case $d = 1$

From the above it follows why it is useful to solve explicitly one-dimensional second order ODEs (also known as 1D "partial" elliptic differential equations of order 2, although, there is no really partial derivatives in this case; however, for 1D parabolic equations the name "partial" is genuine):

$$u''(x) + b(x)u'(x) = 0,$$

$$u''(x) + b(x)u'(x) = -f(x),$$

$$u''(x) + b(x)u'(x) - c(x)u(x) = 0,$$

with various boundary conditions. We will use some explicit solutions for such 1D equations later in this course. Explicit formulae (for the "elliptic" 1D case) can be found in many sourses, e.g., in [I.I. Gikhman, A.V. Skorokhod, Stochastic differential equations, Kiev, Naukova Dumka, 1968].

Extensions?

From WP to SDE solutions; from Δ to elliptic operators of the 2nd order.

After we establish Markov and strong Markov property of solutions of SDEs, we will be able to extend this analysis to more general operators of the second order, parabolic and elliptic.

It is not the goal of this short course, but a similar analysis may be extended also to (strong) Markov "diffusions with jumps", which is the name used for solutions of SDEs driven by WP and Lévy processes. They correspond to integro-differential equations instead of PDEs; also there is a link to fractional Laplacians.

Also there are differential operators with other boundary conditions; this is a more difficult topic and not in the scope of this course.