Introduction to stochastic differential equations

Alexander Veretennikov[1]

Spring 2020

April 10, 2020

[1] National Research University HSE, Moscow State University, Russia

online mini-course

Abstract

strong solutions are wanted under Lipschitz condition

A stochastic differential equation in $\mathbb{R}^d$ is considered

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \;\; t \geq 0, \qquad X_0 = x_0, \tag{1}$$

Or, equivalently in the integral form,

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{2}$$

Here $(W_t, \mathcal{F}_t)$ is a standard d-dimensional Wiener process, b and σ are vector and matrix Borel functions of corresponding dimensions d and $d \times d$. The initial value x_0 may be non-random, or random but $\mathcal{F}_0$-measurable.

Ito isometry, 2D version

Reminder in 1D: $E(\int_0^T f_t dW_t)^2 = E\int_0^T f_t^2 dt$

Equivalently in 1D,

$$E\left((\int_0^T f_t dW_t) \times (\int_0^T g_t dW_t) \right) = E\int_0^T f_t g_t dt.$$

For two independent WP, W^1 and W^2,

$$E\left((\int_0^T f_t dW_t^1) \times (\int_0^T g_t dW_t^2) \right) = 0.$$

Both identities can be written in a unified formula,

$$E\left((\int_0^T f_t dW_t^i) \times (\int_0^T g_t dW_t^j) \right) = \delta_{ij} E\int_0^T f_t g_t dt.$$

Ito isometry, multidimensional version

Reminder: $E\left(\left(\int_0^T f_t dW_t^i\right) \times \left(\int_0^T g_t dW_t^j\right)\right) = \delta_{ij} E \int_0^T f_t g_t dt.$

For a $d \times d$ matrix-valued process $g_t = (g_t^{ij})$ and a d-dimensional WP let us consider a SI $Ig = \int_0^T g_t dW_t$.

$$E\left|\int_0^T g_t dW_t\right|^2 = \sum_i E\left|\sum_j \int_0^T g_t^{ij} dW_t^j\right|^2$$

$$= \sum_i \sum_j E \int_0^T |g_t^{ij}|^2 dt = E \int_0^T \sum_i \sum_j |g_t^{ij}|^2 dt$$

$$= E \int_0^T \|g_t\|^2 dt \overset{also}{=} E \int_0^T \mathrm{Tr}(g g_t^*) dt.$$

Here g^* is a transposed matrix (adjoint operator) and $\|g\|$ is its "Euclidean norm",

$$\|g\| = \sqrt{\sum_{ij} |g^{ij}|^2}.$$

Lipschitz coefficients, uniqueness

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,\ t \geq 0, \quad X_0 = x_0 \quad (1)$

Theorem (Ito)

Assume that there exists $c > 0$ such that for any t, x, x',

$$|b(t,x) - b(t,x')| + \|\sigma(t,x) - \sigma(t,x')\| \leq C|x - x'|,$$

and

$$|b(t,x)| + \|\sigma(t,x)\| \leq C(1 + |x|).$$

*Then there is no more than one solution of the equation (**??**).*

Proof. Suppose there are two solutions X_t and Y_t. We have,

$$|X_t - Y_t|^2 \leq 2\left|\int_0^t (b(s,X_s) - b(s,Y_s))ds\right|^2 + 2\left|\int_0^t (\sigma(s,X_s) - \sigma(s,Y_s))dW_s\right|^2.$$

$((a+b)^2 \leq 2a^2 + 2b^2)$

Lipschitz uniqueness, proof

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \; t \geq 0, \quad X_0 = x_0 \quad (1).$ Hence,

$$E|X_t - Y_t|^2 \leq 2E|\int_0^t |b(s, X_s) - b(s, Y_s)|ds|^2$$

$$+2E\int_0^t Tr(\sigma(s, X_s) - \sigma(s, Y_s))(\sigma(s, X_s) - \sigma(s, Y_s))^* ds$$

$$\leq CE|\int_0^t |X_s - Y_s|ds|^2$$

$$+CE\int_0^t Tr(\sigma(s, X_s) - \sigma(s, Y_s))(\sigma(s, X_s) - \sigma(s, Y_s))^* ds$$

$$\leq (C + Ct)\int_0^t E|X_s - Y_s|^2 ds.$$

Since $\sup_{s\leq t} E|X_s - Y_s|^2 \leq 2(\sup_{s\leq t} E(|X_s|^2 + |Y_s|^2)) < \infty$, now Gronwall's (Grönwall's) inequality implies $E|X_t - Y_t|^2 = 0$, and $P(X_t = Y_t, \forall t \geq 0) = 1$, as required.

Lipschitz coefficients, existence

$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \; t \geq 0, \quad X_0 = x_0 \quad (1)$

Theorem (Ito)

Assume that there exists $c > 0$ such that for any t, x, x',

$$|b(t,x) - b(t,x')| + \|\sigma(t,x) - \sigma(t,x')\| \leq C|x - x'|,$$

and

$$|b(t,x)| + \|\sigma(t,x)\| \leq C(1 + |x|).$$

*Then there is a solution of the equation (**??**).*

Proof. The proof will use successive approximations. Let $X_t^0 := x_0, \; t \geq 0$. Given $X_t^n, t \geq 0$, let

$$X_t^{n+1} := x_0 + \int_0^t b(s, X_s^n)ds + \int_0^t \sigma(s, X_s^n)dW_s.$$

Let us consider the difference $X_t^{n+1} - X_t^n$. We have, similarly to the calculus in the uniqueness theorem,

Proofs

$(a+b)^2 \leq 2a^2 + 2b^2$

$$|X_t^{n+1} - X_t^n|^2 \leq 2\left|\int_0^t (b(s, X_s^n) - b(s, X_s^{n-1}))ds\right|^2 + 2\left|\int_0^t (\sigma(s, X_s^n) - \sigma(s, X_s^{n-1}))dW_s\right|^2.$$

So, with some $C_t = C + Ct$,

$$E|X_t^{n+1} - X_t^n|^2 \leq 2E\left|\int_0^t (b(s, X_s^n) - b(s, X_s^{n-1}))ds\right|^2 + 2E\left|\int_0^t (\sigma(s, X_s^n) - \sigma(s, X_s^{n-1}))dW_s\right|^2 \leq C_t E\int_0^t |X_s^n - X_s^{n-1}|^2 ds.$$

Moreover, by using Doob's inequality for stochastic integrals we also have (with different constants $C_T > 0$ on different lines)

$$E\sup_{t\le T}|X_t^{n+1}-X_t^n|^2 \le 2E\sup_{t\le T}\Big|\int_0^t (b(s,X_s^n)-b(s,X_s^{n-1}))ds\Big|^2$$

$$+2E\sup_{t\le T}\Big|\int_0^t (\sigma(s,X_s^n)-\sigma(s,X_s^{n-1}))dW_s\Big|^2$$

$$\le C_T E\int_0^T |X_s^n - X_s^{n-1}|^2 ds \overset{(<\infty)}{\le} C_T\int_0^T E\sup_{t\le s}|X_t^n - X_t^{n-1}|^2 ds.$$

Denoting $a_T^n := E\sup_{t\le T}|X_t^{n+1}-X_t^n|^2$, by induction we get,

$$0 \le a_T^n \le \frac{C_T^n a_T^0}{n!}.$$

Clearly, the series $\sum_n a_T^n$ converges. Let us estimate the probability

$$P(\sup_{t\le T}|X_t^{n+1}-X_t^n|\ge 2^{-n})\le 4^n a_T^n \le \frac{(4C_T)^n a_T^0}{n!}.$$

So, by the Borel – Cantelli lemma, with probability one for all n starting with some $n_0(\omega)$,

$$\sup_{t\le T}|X_t^{n+1}-X_t^n|<2^{-n}.$$

This implies that the following object is well-defined in $L_2(\Omega\times[0,T],P\times\Lambda)$ and simultaneously in $C([0,T]$ a.s. in ω:

$$X_t := X_t^0+\sum_{n\ge 0}(X_t^{n+1}-X_t^n).$$

Moreover, there is a uniform in $t \in [0, T]$ convergence

$$X_t^{n+1} = X_t^0 + \sum_{k=0}^{n} (X_t^{k+1} - X_t^k) \rightrightarrows X_t, \quad n \to \infty.$$

We are now able to pass to the limit in the equation

$$X_t^{n+1} := x_0 + \int_0^t b(s, X_s^n) ds + \int_0^t \sigma(s, X_s^n) dW_s,$$

which, of course, results in the limiting version of it,

$$X_t := x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s,$$

as required.

To justify this limiting equation, we firstly note that the process X_t is $(\mathcal{F}_t)$-adapted, along with all X^n. Now, in the right hand side we clearly have $X_t^{n+1} \to X_t$ a.s. (and even uniformly in $t \le T$).
If the drift b is, actually, bounded, then we immediately obtain convergence a.s.

$$\int_0^t b(s, X_s^n)ds \to \int_0^t b(s, X_s)ds, \quad n \to \infty,$$

from the Lebesgue's bounded convergence theorem. Under the more relaxed linear growth condition on b the same result for this Lebesgue integral follows from the a priori bounds (uniform in n, by induction, and for X_t, too)

$$\sup_{t \le T} E|X_t^n|^2 < \infty.$$

This exercise is left as a homework to the readers.

Finally, similarly, but now with the help of the Ito isometry we obtain convergence

$$\int_0^t \sigma(s, X_s^n) dW_s - \int_0^t \sigma(s, X_s) dW_s \to 0, \quad n \to \infty.$$

straightforwardly in the case of bounded σ. Indeed,

$$E|\int_0^t (\sigma(s, X_s^n) - \sigma(s, X_s)) dW_s|^2$$

$$= E \int_0^t \|\sigma(s, X_s^n) - \sigma(s, X_s))\|^2 ds \to 0, \quad n \to \infty.$$

For unbounded σ the same convergence of stochastic terms follows from similar a priori bounds for the fourth moment,

$$\sup_{t \le T} E|X_t^n|^4 < \infty$$

(also left as a homework). The existence theorem is proved.

A priori moment bounds

Second moment

The estimate $\sup_{t\leq T} E|X_t|^2 < \exp(CT)Ex_0^2$ with some C would follow easily from Gronwall's inequality, if we only knew already that $E|X_t|^2 < \infty$. Indeed, we have,

$$\begin{aligned} E|X_T|^2 &\leq 3Ex_0^2 + 3E\sup_{t\leq T}|\int_0^t b(s,X_s)ds|^2 \\ &+3E\sup_{t\leq T}|\int_0^t \sigma(s,X_s)dW_s|^2 \\ &\leq 3Ex_0^2 + C_T\int_0^T (1+E|X_t|^2)dt. \end{aligned}$$

It remains to apply Gronwall's inequality. However, while applying it, we also need to know that both sides of the inequality are finite. This can be tackled by stopping times. Similar a priori bounds for $3E(X_0^n)^2$ can be obtained by induction (homework).

Gronwall's inequality

If for a nonrandom function $a(t)$ the following holds true,

$$0 \leq a(t) \leq C_1 + C_2 \int_0^t a(s)ds, \quad \forall 0 \leq t \leq T,$$

and either $\sup_{s \leq t} a(s) < \infty$ for all $t > 0$, or, $\int_0^t a(s)ds < \infty$ for all t, then,

$$a(t) \leq C_1 \exp(C_2 t), \quad t \leq T.$$

$$\begin{aligned} a(t) &\leq C_1 + C_2 \int_0^t a(s)ds \\ &\leq C_1 + C_2 \int_0^t (C_1 + C_2 \int_0^s a(s_2)ds_2)ds \\ &\leq C_1 + C_1 C_2 t + C_1 c_2^2 \frac{t^2}{2!} + \ldots + C_2^n \|a\| \frac{t^n}{n!} \end{aligned}$$

Solution X_t can be written as

$$X_t^{0,x_0};\ X_t^{s,x}\ (t \geq s)$$