Introduction to stochastic differential equations – 6
Girsanov and weak uniqueness

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Parabolic PDEs and Markov property

$X_t^{0,x} = x + \int_{t_0}^t b(s, X_s^{0,x})ds + \int_{t_0}^t \sigma(s, X_s^{0,x})dW_s,\ t \geq t_0 \quad (1), \quad a = \sigma\sigma^*$

Generally, it is a special part of a non-trivial theorem. The **idea** is as follows. If we have a weak uniqueness, then we have a non-homogeneous semigroup of operators $T_{s,t}f(x) := E_x f(X_t^{s,x})$ with the semigroup property $T_{s,t} = T_{s,r}T_{r,t}$ $(s < r < t)$. Indeed, let us compute $E(u(t, X_t^{0,x})|\mathcal{F}_{t_0})$, $0 \leq t_0 < t$: by Ito's formula for smooth $u(t,x)$ we have with $L = \frac{1}{2}\sum_{ij} a_{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_i b^i(t,x)\frac{\partial}{\partial x^i}$,

$$E(u(t, X_t^{0,x})|\mathcal{F}_{t_0}) = u(t_0, X_{t_0}^{0,x})$$
$$+E(\int_{t_0}^t (u_s + Lu)(s, X_s^{0,x})ds|\mathcal{F}_{t_0}).$$

Suppose u solves the PDE $u_s + Lu = 0$ with the terminal condition $u(t,x) = g(x)$. Then

$$E(g(X_t^{0,x})|\mathcal{F}_{t_0}) = u(t_0, X_{t_0}^{0,x}).$$

Parabolic PDEs and Markov property, ctd

Non-rigorous presentation (no precise assumptions provided)

It follows that

$$E(g(X_t^{0,x})|X_{t_0}^{0,x}) = E(E(g(X_t^{0,x})|\mathcal{F}_{t_0})|X_{t_0}^{0,x})$$
$$= E(u(t_0, X_{t_0}^{0,x})|X_{t_0}^{0,x}) = u(t_0, X_{t_0}^{0,x}).$$

Hence, we should have a.s.

$$E(g(X_t^{0,x})|\mathcal{F}_{t_0}) = E(g(X_t^{0,x})|X_{t_0}^{0,x}).$$

The subtle point here is Ito's formula if the solution of the parabolic PDE is not classical but in Sobolev spaces, and whether or not this PDE has a (Sobolev) solution at all. We will learn about Ito-Krylov's formula for Sobolev derivatives, and there will be some short review about solving parabolic PDEs (classical and in Sobolev classes). Generally, it is easier to show weak uniqueness for particular cases, and for these cases also to show Markov property in this way.

How to extend to bounded Borel g

$E(g(X_t^{0,x})|\mathcal{F}_{t_0}) = E(g(X_t^{0,x})|X_{t_0}^{0,x})$, $\forall\, g \in C_b$, and we want it $\forall g \in B$

Let any $B \in \mathcal{F}_{t_0}$; let us introduce two measures on $\mathcal{B}(R^d)$,

$$\mu(\Gamma) := E1(B)E(1(X_t^{0,x} \in \Gamma)|\mathcal{F}_{t_0}),$$
$$\nu(\Gamma) := E1(B)E(1(X_t^{0,x} \in \Gamma)|X_{t_0}).$$

It can be checked that for any step function g, and, hence, by approximation also for any $g \in C_b$,

$$\int g(x)\mu(dx) = E1(B)E(g(X_t^{0,x} \in \Gamma)|\mathcal{F}_{t_0}),$$
$$\int g(x)\nu(dx) = E1(B)E(g(X_t^{0,x} \in \Gamma)|X_{t_0}).$$

So, $\int g(x)\mu(dx) = \int g(x)\nu(dx)$, $\forall g \in C_b$. So, $\mu = \nu$. Thus, $E(g(X_t^{0,x})|\mathcal{F}_{t_0}) = E(g(X_t^{0,x})|X_{t_0}^{0,x})\,\forall\, g \in B$.

Strong uniqueness implies weak uniqueness

Idea only; Yamada – Watanabe principle will be studied later

Again idea only will be presented. Suppose we have a strongly (pathwise) unique solution. By the **Yamada – Watanabe principle** *any solution is then strong*, and, hence, can be constructed on any probability space with any WP as a measurable mapping from the trajectory $(W_s, 0 \leq s \leq t) \mapsto X_t = F_t(W_s, 0 \leq s \leq t)$. The mapping F_t is then uniquely determined for each t, at least, a.s. Then, the law of $(X_t, 0 \leq t \leq T)$ is the law of $(F_t(W_s, 0 \leq s \leq t), 0 \leq t \leq T)$. Every WP on any probability space has the same distribution; therefore, the distribution of $(F_t(W_s, 0 \leq s \leq t), 0 \leq t \leq T)$ is also uniquely determined, which means exactly weak uniqueness.

Note that if σ is $d \times d$ and non-degenerate, then, in turn, $dW_t = \sigma^{-1}(t, X_t)(dX_t - b(t, X_t)dt$. So, (X_t, W_t) is weakly unique. See [R.Bass, Diffusions and Elliptic Operators]

Weak uniqueness via Girsanov; NB: the drift is now σb

$$X_t^{t_0,x} = x + \int_{t_0}^t \sigma b(s, X_s^{t_0,x})ds + \int_{t_0}^t \sigma(s, X_s^{t_0,x})dW_s,\ t \geq t_0 \qquad (2)$$

Assume that the solution (X, W) of the equation *without b* has a unique law. Let X there exist a weak solution of the equation (2) on some $(\Omega, \mathcal{F}, \mathbb{P})$ with a WP W. Using the "inverse" Girsanov transformation, we find that X is a solution on $[0, T]$ of the equation without drift with a new WP

$$\tilde{W}_t = W_t + \int_0^t b(s, X_s)ds, \quad 0 \leq t \leq T,$$

under the new probability measure $\tilde{\mathbb{P}} = \mathbb{P}^\rho$ with

$$\rho_T = \exp(-\int_0^T b(s, X_s)dW_s - \frac{1}{2}\int_0^T b(s, X_s)^2 ds).$$

The distribution of the pair $(X, \tilde{W})$ on $[0, T]$ with respect to the measure $\mathbb{P}^\rho$ is uniquely determined by the assumption. The law of X under $\mathbb{P}$ is obtained from the law of the couple $(X, \tilde{W})$ under $\tilde{\mathbb{P}}$ by Girsanov's transformation.

Weak uniqueness via Girsanov, ctd

Indeed, for any bounded Borel function g on $C[0, T; \mathbb{R}^d]$ we have,

$$\mathbb{E} g(X) = \mathbb{E}^\rho \rho^{-1} g(X) = \mathbb{E}^\rho \rho^{-1} g(X) =$$

$$= \mathbb{E}^\rho g(X) \exp\left(+ \int_0^T b(t, X_t)\, dW_t + \frac{1}{2} \int_0^T |b(t, X_t)|^2\, dt \right) =$$

$$= \mathbb{E}^\rho g(X) \exp\left(+ \int_0^T b(t, X_t)\, d\tilde{W}_t - \frac{1}{2} \int_0^T |b(t, X_t)|^2\, dt \right).$$

Since thel law of $(X, \tilde{W})$ under $\mathbb{P}^\rho$ is unique, it follows from here that the value $\mathbb{E} g(X)$ is also unique, as required. The method was used by Gikhman and Skorokhod, among others.

Recall from the first pages,

$$E(u(t, X_t^{0,x})|\mathcal{F}_{t_0}) = u(t_0, X_{t_0}^{0,x}),$$

or, equivalently,

$$E(g(X_t^{0,x})|\mathcal{F}_{t_0}) = u(t_0, X_{t_0}^{0,x}).$$

For $t_0 = 0$ this implies

$$Eg(X_t^{0,x}) = u(0,x).$$

Here $g \in C_b$, and for such g expectations are uniquely determined if solution of the PDE is unique. Further, any indicator of a *closed* set may be monotonically approximated by such functions. It follows that $\mathcal{L}(X_t)$ is unique for any t.

Let $0 < t_1 < t_2$. We will show that the law $\mathcal{L}(X_{t_1}, X_{t_2})$ is also unique.

We are to establish uniqueness of the expressions

$$E_x g_1(X_{t_1}) g_2(X_{t_2}), \quad \forall g_1, g_2 \in C_b,$$

which then will be extended to $\forall g_1, g_2 \in B$. We already know that the process X is Markov. Let $u(s,x)$ be a (classical) solution of the equation $u_s(s,x) + Lu(s,x) = 0,\ t_1 \le s \le t_2,\ u(t_2, x) = g_2(x)$; we assume that solution u is unique. Then we have,

$$\begin{aligned} E_x g_1(X_{t_1}) g_2(X_{t_2}) &= E_x(E_x(g_1(X_{t_1}) g_2(X_{t_2}) | \mathcal{F}_{t_1}) \\ = E_x g_1(X_{t_1})(E_x(g_2(X_{t_2}) | \mathcal{F}_{t_1}) &= E_x g_1(X_{t_1})(E_x(g_2(X_{t_2}) | X_{t_1}) \\ &= E_x g_1(X_{t_1}) E_{X_{t_1}} g_2(X_{t_2}) = E_x g_1(X_{t_1}) u(t_1, X_{t_1}). \end{aligned}$$

Weak uniqueness via PDEs, ctd.

Note that $u(t_1, \cdot) \in C_b$ [Krylov and Safonov 1979]

Now let $v(s,x)$ be a (classical) solution of the equation $v_s(s,x) + Lv(s,x) = 0$, $0 \le s \le t_1$, $v(t_1,x) = g_1(x)u(t_1,x)$; we assume that solution v is unique. Then we have,

$$E_x g_1(X_{t_1}) g_2(X_{t_2}) = E_x g_1(X_{t_1}) u(t_1, X_{t_1})$$
$$= E_x v(0,x) = v(0,x).$$

Due to the uniqueness of solution v (and u), we conclude that the expectation $E_x g_1(X_{t_1}) g_2(X_{t_2})$ is uniquely determined $\forall g_1, g_2 \in C_b$. By induction we extend this to any finite product

$$E_x \prod_{k=1}^{n} g_k(X_{t_k}),$$

for any n, any $0 < t_1 < \ldots < t_n$, and any $g_1, \ldots g_n \in C_b$.

Weak uniqueness via PDEs, ctd

$E_x \prod_{k=1}^n g_k(X_{t_k})$, is unique $\forall g_1, \dots g_n \in C_b$; want to extend it to $\forall g_i \in B$

By uniform approximation this equality is extended to any indicators of closed sets $g_i(x) = 1(x \in G_i)$. Therefore, it is also true for any indicators of open sets (as complementary ones) $g_i(x) = 1(x \in O_i)$, and due to the regularity of measures on R^d also for any indicators of arbitrary Borel sets $g_i(x) = 1(x \in A_i)$. This extends to any Borel indicators, which means that the measure on the cylinders $(X_{t_1} \in A_1, \dots X_{t_n} \in A_n)$ $(\forall A_i \in \mathcal{B}(R^d))$ and further on $\sigma(X_{t_1}, \dots X_{t_n})$ is uniquely determined. Hence, it shows that the distribution of X is unique, as required.

Reference on I.V. Girsanov's paper where he used this method based on parabolic PDEs to show weak uniqueness: I.V. Girsanov [1962], Stochastic equations and their certain generalisations, Proceedings of the VI All-Soviet workshop ("Vsesoyuznoe sovesh'anie") on probability theory, Vilnius 1960 (in Russian).