# Introduction to stochastic differential equations – 9
# Multidimensional ergodic SDEs

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SDEs introduction

Markov–Dobrushin's condition via Girsanov


If in the 1D case we have used **intersections** of two independent solutions of the same SDE with different initial values, but how could we arrange it in dimension $d > 1$? A simple version of the *method of coupling* will be presented below for this aim. In a more general situation such a method can be based on a *parabolic version of Harnack's inequality* shown very briefly in the end of the lecture, but it is a little beyond the scope of this course.

For simplicity, we restrict our study to the case $\sigma \equiv I$, a unit diffusion matrix. In this case (as well as in the nondegenerate case!) it turns out that we may use Girsanov's transformation instead of Harnack. An alternative simplified way would be to use PDE results on the transition densities satisfying Gaussian type lower and upper bounds [Solonnikov, Eidelman, Friedman]; it will be also mentioned briefly in the end.

# Stochastic exponential; Markov – Dobrushin

$X_t = x + \int_0^t b(X_s)ds + W_t,\ t \geq 0$; $b$ bounded (could be linear growing in $x$)



Let $$\rho_T := \exp\left(-\int_0^T b(t, X_t)\, dW_t - \frac{1}{2}\int_0^T |b(t, X_t)|^2\ dt\right).$$

Recall that $\rho_T$ is a probability density for any $T > 0$. Denote by $\mu_t$ the marginal distribution of $X_t$.

**Theorem (Markov – Dobrushin's condition)**

*For any $T > 0$ and $R > 0$*

$$\kappa(R, T) := \inf_{x_0, x_1 \in B_R} \int_{B_R} \left(\frac{\mu_T^{x_0}(dy)}{\mu_T^{x_1}(dy)} \wedge 1\right) \mu_T^{x_1}(dy) > 0. \quad (1)$$

This inequality suffices for applications to coupling and convergence rates (given suitable recurrence estimates). If desirable, a localised version of it may be established.

# Proof

$X_t = x + \int_0^t b(X_s)ds + W_t, \ t \geq 0$

Note that $\mu_T^x(dy) \ll dy$, and

$$\frac{\mu_T^x(dy)}{dy} > 0 \quad \text{a.e.} \qquad (2)$$

Indeed, let us denote $\mu_T^{x,\rho}(dy) := \mathbb{E}_x^\rho 1(X_T \in dy)$; then $\mu_T^x(A) = \mathbb{E}_x^\rho \rho^{-1} 1(X_T \in A) = \mathbb{E}_x^\rho 1(X_T \in A) \mathbb{E}_x^\rho(\rho^{-1}|X_T)$; so,

$$\frac{\mu_T^x(dy)}{dy} \stackrel{!}{=} \frac{\mathbb{E}_x^\rho 1(X_T \in dy) \mathbb{E}_x^\rho(\rho^{-1}|X_T)}{dy} \stackrel{!}{=} \frac{\mu_T^{x,\rho}(dy)}{dy} \mathbb{E}_x(\rho_T^{-1}|X_T)|_{X_T=y}$$

Here both $\mu_T^\rho(x;dy)/dy$ and $\mathbb{E}(\rho_T^{-1} \mid X_T)|_{X_T=y}$ are positive: the second one a.s. since $0 < \rho^{-1} < \infty$, while the first one because it is a nondegenerate Gaussian density. So, (1) may be rewritten as

$$\inf_{x_0,x_1 \in B_R} \int_{B_R} \left( \frac{\mu_T^{x_0}(dy)}{dy} \wedge \frac{\mu_T^{x_1}(dy)}{dy} \right) dy > 0. \qquad (3)$$

# Proof, ctd



Let $L > 0$, and consider the (sub-probability) density

$$\frac{\mu_T^{x_0,L}(dy)}{dy} := \frac{\mathbb{E}_{x_0} 1(X_T \in dy)\, 1(\rho_T > L)}{dy}.$$

Here $\mu_T^{x_0,L}(dy)$ is absolutely continuous wrt Lebesgue's measure $dy$ since it is dominated by $\mu_T^{x_0}(dy)$. The r.v. $\rho_T$ is a probability density. So, we can denote

$$\frac{\mu_T^{x_0}(dy)}{dy} \equiv \frac{\mathbb{E}_{x_0}^{\rho} \rho^{-1} 1(X_T \in dy)}{dy} =: p_{x_0}(y; T),$$

$$\frac{\mu_{T;x_0}^{\rho}(dy)}{dy} \equiv \frac{\mathbb{E}_{x_0}^{\rho} 1(X_T \in dy)}{dy} =: p_{x_0}^{\rho}(y; T),$$

$$\frac{\mu_{T;x_0}^{L}(dy)}{dy} = \frac{\mathbb{E}_{x_0}^{\rho} 1(X_T \in dy)\, 1(\rho_T > L)}{dy} =: p_{x_0}^{L}(y; T).$$

# Proof, ctd



Since $\rho_T^{-1} \geq L^{-1}$ on the set $(\rho_T \leq L)$, we get

$$\frac{\mu_T^{x_0}(dy)}{dy} = \frac{\mathbb{E}^{\rho}_{x_0}\rho_T^{-1} 1(X_T \in dy)\, 1(\rho_T \leq L)}{dy} + \frac{\mathbb{E}^{\rho}_{x_0}\rho_T^{-1} 1(X_T \in dy)\, 1(\rho_T > L)}{dy}$$

$$\geq \frac{\mathbb{E}^{\rho}_{x_0}\rho_T^{-1} 1(X_T \in dy)\, 1(\rho_T \leq L)}{dy}$$

$$= \frac{\mathbb{E}^{\rho}_{x_0}\rho_T^{-1} 1(X_T \in dy)\, (1 - 1(\rho_T > L))}{dy} \geq$$

$$\geq L^{-1}\, \frac{\mathbb{E}^{\rho}_{x_0} 1(X_T \in dy)\, (1 - 1(\rho_T > L))}{dy} \geq$$

$$\geq L^{-1} \left( \frac{\mathbb{E}^{\rho}_{x_0} 1(X_T \in dy)}{dy} - \frac{\mathbb{E}^{\rho}_{x_0} 1(X_T \in dy)\, 1(\rho_T > L)}{dy} \right).$$

# Proof, ctd

$p^{\rho}_{x_0}(y;T) = \frac{1}{(2\pi T)^{d/2}} \exp(-\frac{(x_0-y)^2}{2T})$



We estimate,

$$\frac{\mu_T^{x_0}(dy)}{dy} \ge L^{-1}\left(\frac{\mathbb{E}^{\rho}_{x_0} 1(X_T \in dy)}{dy} - \frac{\mathbb{E}^{\rho}_{x_0} 1(X_T \in dy)\, 1(\rho_T > L)}{dy}\right)$$
$$= L^{-1}\left(\frac{1}{(2\pi T)^{d/2}} \exp(-\frac{(x_0-y)^2}{2T}) - p^{L}_{x_0}(y;T)\right).$$

Using the elementary inequality with $b, d \ge 0$

$$(a-b)\wedge(c-d) \ge (a\wedge c) - b - d,$$

$((a-b)\wedge(c-d) \ge (a-b-d)\wedge(c-b-d) = (a\wedge c) - b - d)$ we obtain (on the next page),

# Proof, ctd

$$p^{\rho}_{x_0}(y;T) = \frac{1}{(2\pi T)^{d/2}} \exp(-\frac{(x_0-y)^2}{2T})$$



$$\begin{aligned}
\kappa(R,T) &:= \inf_{x_0,x_1 \in B_R} \int_{B_R} \left( \frac{\mu_T^{x_0}(dy)}{\mu_T^{x_1}(dy)} \wedge 1 \right) \mu_T^{x_1}(dy) \\
&= \inf_{x_0,x_1 \in B_R} \int_{B_R} \left( \frac{\mu_T^{x_0}(dy)}{dy} \wedge \frac{\mu_T^{x_1}(dy)}{dy} \right) dy \\
&\equiv \inf_{x_0,x_1 \in B_R} \int_{B_R} (p_{x_0}(y;T) \wedge p_{x_1}(y;T))\, dy \\
&\geq \inf_{x_0,x_1 \in B_R} \int_{B_R} L^{-1} \Big( p^{\rho}_{x_0}(y;T) \wedge p^{\rho}_{x_1}(y;T) \\
&\qquad - p^{L}_{x_0}(y;T) - p^{L}_{x_1}(y;T) \Big)\, dy \\
&\geq L^{-1} \left( \inf_{x,x' \in B_R} p^{\rho}_{x}(x';T)\, |B_R| - 2 \sup_{x \in B_R} \mathbb{P}^{\rho}_{x}(\rho_T > L) \right).
\end{aligned}$$

*NB: clearly,* $\inf \int_{B_R} p \wedge p \geq \inf p\, |B_R|$ *is not a very precise bound.*

The latter bound $\inf\int_{B_R} p\wedge p \ge \inf p\,|B_R|$ was not very precise. In fact, a bit more general statement holds true. Let

$$\kappa^0(R,T) := \inf_{x_0,x_1\in B_R}\int_{B_R}\left(\frac{\mu^\rho_{T,x_0}(dy)}{\mu^\rho_{T,x_1}(dy)}\wedge 1\right)\mu^\rho_{T,x_1}(dy).$$

**Remark ($dX_t = b(X_t)dt + \sigma(X_t)dW_t$)**

*Under the nondegeneracy of $\sigma\sigma^*$, if $\kappa^0(R,T)>0$, then $\kappa(R,T)>0$.*

Indeed, from the last calculus repeated with $\sigma$ and with the reference measure $\mu^{x_0}_T(dy)+\mu^{x_1}_T(dy)$ instead of Lebesgue's one, it follows,

$$\kappa(R,T)\ge L^{-1}\left(\kappa^0(R,T) - 2\sup_{x\in B_R}\mathbb{P}^\rho_x(\rho_T > L)\right),$$

which may be made $>0$ by choosing $L$ large enough.

The second term $\sup P(\ldots)$ admits the upper bound

$$\sup_{x_0\in B_R}\mathbb{P}^\rho_{x_0}(\rho_T \ge L) \le \frac{\sup_{x_0\in B_R}\mathbb{E}^\rho_{x_0}\rho_T}{L},$$

where the numerator in the right hand side is bounded and the denominator can be made arbitrarily large. The value $\inf\limits_{x,x'\in B_R} p^\rho_x(x';T)\,|B_R|$ (in the last line on the previous slide) does not depend on $L$. Hence, (1) holds, as required. QED

Note that a popular condition for an MC $X_n$ coupling is a +recurrence towards a so called "petite set" $D$ with a probability measure $\nu$ on it such that on this set $D$ all transition kernels satisfy $P(x,dy) \ge c\,\nu(dy)$ with $c > 0$. For SDEs this condition may only be checked under the condition (4) on the next page, or under a similar condition wrt some other reference measure $\Lambda$, but except for Lebesgue's one, sufficient conditions are not known to me.

- The simplest way to verify the condition (1)

$$\kappa(R,T) := \inf_{x_0,x_1\in B_R} \int_{B_R} \left(\frac{\mu_T^{x_0}(dy)}{\mu_T^{x_1}(dy)} \wedge 1\right) \mu_T^{x_1}(dy) > 0,$$

is if there is a property of the transition density of $X$: $\exists R > 0$ such that

$$0 < a_1 \le \inf_{x,x'\in B_R} p_t(x,x') \le \sup_{x,x'\in B_R} p_t(x,x') \le a_2 < \infty, \quad (4)$$

for some $t$. It holds, e.g., if $b$ and $\sigma$ are bounded and Hölder's continuous in $x$ uniformly wrt $t$ and $\sigma\sigma^*$ is nondegenerate, as this implies Gaussian type lower and upper bounds on $p_t$. Similar bounds hold true also for some degenerate $\sigma$ under so called Hörmander's type non-degeneracy (hypo-ellipticity) conditions. Refs: [Solonnikov; Eidelman; Friedman] (for $\sigma\sigma^*$ non-denegerate).

# Parabolic Harnack inequality

implies some version of Markov – Dobrushin's condition (simplified)



- A more advanced tool is a parabolic Harnack inequality (under the non-degeneracy of $\sigma\sigma^*$ condition), which in the probabilistic terms may be written as follows (with some slight changes from [Krylov, Safonov, 1980]):

$$\sup_{|x_1|\le 1/4, |x_2|\le 1/2} \frac{P(X_\tau^{0,x_1}\in d\gamma)}{P(X_\tau^{\epsilon,x_2}\in d\gamma)}|_{\Gamma_\epsilon} \le N < \infty, \qquad (5)$$

where $\Gamma_\epsilon$ is the parabolic boundary of the cylinder $((t,x) : |x|\le 1; \epsilon\le t\le 1)$, i.e. $(\Gamma_\epsilon = \Gamma_\epsilon^{(t=1)} \cup \Gamma_\epsilon^{(t<1)})$, $\Gamma_\epsilon = ((t,x) : (|x| = 1 \& \epsilon \le t \le 1) \cup (|x|\le 1 \& t = 1))$. Let $\mu_1(d\gamma) = P(X_\tau^{0,x_1}\in d\gamma)$, $\mu_2(d\gamma) = P(X_\tau^{\epsilon,x_2}\in d\gamma)$. Then $\mu_1(|x| = 1\& t = 1)\wedge \mu_2(|x| = 1\& t = 1) \ge q > 0$, so a version of Markov-Dobrushin's condition holds,

$$\inf_{|x_1|\le 1/4, |x_2|\le 1/2} \int_{\Gamma_\epsilon^{(t=1)}} \left(\frac{\mu_1(d\gamma)}{\mu_2(d\gamma)}\wedge 1\right)\mu_2(d\gamma) \ge \frac{q}{N} > 0.$$

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# Embedded Markov chain for SDE



Denote $Z_n := X_n$. It is a Markov and strong Markov process, and it is recurrent if $X_t$ is. Suppose we are able to estimate

$$\|\mu_n^Z - \mu^Z\|_{TV} \le \phi(x,n) \to 0, \quad n \to \infty. \tag{6}$$

Let us show what it means for the process $(X_t, t \ge 0)$. Firstly, if we already know that there exists a unique invariant measure for $X_t$, then it coincides with the one for $X_n = Z_n$. Hence,

$$\mu^Z = \mu^X =: \mu.$$

Further, we shall show that for any $t \ge n$,

$$\|\mu_t^X - \mu^X\|_{TV} \le \|\mu_n^X - \mu^X\|_{TV}. \tag{7}$$

So, whatever already established bound (6) for discrete time MC $Z_n$ implies a very close bound for the continuous time process $X_t$. It remains to notice that if $X$ is recurrent, then $Z$ is also recurrent with very close recurrence properties.

# Proof of (7): $\|\mu_t^X - \mu^X\|_{TV} \le \|\mu_n^X - \mu^X\|_{TV}$.

Proof is based on the markovian property



## Lemma

$$\|\mu_t^X - \mu^X\|_{TV} \le \|\mu_n^X - \mu^X\|_{TV}, \quad t \ge n.$$

*(Here $\mu^X$ is the invariant measure for the process $X$.)*

Proof. By Markov's property of $X$ (or, more precisely due to the Chapman – Kolmogorov's equation),

$$\frac{1}{2}\|\mu_t^X - \mu^X\|_{TV} = \sup_A (P_x(X_t \in A) - P_\mu(X_t \in A))$$

$$= \sup_A \int 1(z \in A)(P_x(X_n \in dy) - P_\mu(X_t \in dy))P_y(X_{t-n} \in dz)$$

$$\le \frac{1}{2}\|\mu_n^X - \mu^X\|_{TV}, \qquad \text{as required.} \qquad QED$$

# Convergence rate implied by $X_n$ suffices

i.e., consider the process $X_t$ only at integer times

Hence, with a very small discrepancy, to verify some rate of convergence for $X_t$ it suffices to establish this rate of convergence for the process $X_n$, that is, only at integer times. Indeed, if, say,

$$\|\mu^X_{x,n} - \mu^X\|_{TV} \le \phi(x)\psi(n) \to 0, \quad n \to \infty,$$

then it follows from the lemma that

$$\|\mu^X_{x,t} - \mu^X\|_{TV} \le \phi(x)\psi([t]) \to 0, \quad t \to \infty.$$

Usually, in such bounds $\psi([t])$ and $\psi(t)$ differ a little, providing practically the same rate of convergence.

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# Markov – Dobrushin's condition

## and "markovian coupling" leading to bounds for convergence rate



Now, instead of verifying convergence rate for $X_t$, we are going to show how to establish it for $X_n$, that is, only at integer times $t = n$, assuming that we may check Markov – Dobrushin's condition

$$\kappa(R, T) := \inf_{x_0, x_1 \in B_R} \int_{B_R} \left( \frac{\mu_T^{x_0}(dy)}{\mu_T^{x_1}(dy)} \wedge 1 \right) \mu_T^{x_1}(dy) > 0,$$

combined with a "good recurrence": namely, we assume for simplicity any polynomial moment

$$E_x \tau_R^m \le C(1 + |x|^m)$$

for the hitting time $\tau_R := \inf(t \ge 0 : |X_t| \le R)$, where $\forall k \ge 1$ $\exists C > 0, m > 0$. The diffusion $\sigma\sigma^*$ is assumed uniformly nondegenerate, and both $b$ and $\sigma$ bounded. The unique invariant measure $\mu$ exists; we want to estimate the rate.

# Markovian coupling



For simplicity in the sequel we assume that for all $x_0, x_1$ the measures $\mu_T^{x_0}(dy)$, $\mu_T^{x_1}(dy)$ are not singular and not equal; moreover (for simplicity!), that

$$0 < \kappa(R,T) = \inf_{x_0,x_1 \in B_R} \int_{B_R} \left( \frac{\mu_T^{x_0}(dy)}{\mu_T^{x_1}(dy)} \wedge 1 \right) \mu_T^{x_1}(dy)$$

$$\leq \sup_{x_0,x_1 \in B_R} \int_{B_R} \left( \frac{\mu_T^{x_0}(dy)}{\mu_T^{x_1}(dy)} \wedge 1 \right) \mu_T^{x_1}(dy) < 1.$$

Then we may couple (or, try to couple) $X_0$ and $Y_0$ by using the coupling lemma (see next two pages); at each further step, if the processes are not yet coupled (and then one joins the other forever), we try to couple them again if they are both in $B_R$ chosen in advance. If both are on $B_R$, then coupling is always possible with a probability $\geq \kappa(T;R)$.

# Reminder from lecture 8: Coupling lemma

(Lemma on two r.v.)



## Lemma ("Of two random variables")

*Let $X^1$ and $X^2$ be two random variables on their (without loss of generality different, which will be made independent after we take their direct product) probability spaces $(\Omega^1, \mathcal{F}^1, \mathbb{P}^1)$ and $(\Omega^2, \mathcal{F}^2, \mathbb{P}^2)$ and with densities $p^1$ and $p^2$ with respect to some reference measure $\Lambda$, correspondingly. Then, if*

$$1 - p := q = \int \Big(p^1(x) \wedge p^2(x)\Big) \Lambda(dx) > 0,$$

*then there exists one more probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two random variables on it $\tilde X^1, \tilde X^2$ such that*

$$\mathcal{L}(\tilde X^j) = \mathcal{L}(X^j), j = 1,2, \ \& \ \frac{\|\mathcal{L}(X^1) - \mathcal{L}(X^2)\|_{TV}}{2} \stackrel{!}{=} P(\tilde X^1 \neq \tilde X^2) = p.$$

# Reminder from lecture 8: Proof of Coupling Lemma

Assume $q < 1$, otherwise the Lemma is trivial



Let r.v. $\eta_1$, $\eta_2$, $\xi$, have the following densities:

$$p_{\eta^1}(t) = (1-q)^{-1}\left(p^1(t) - p^1(t) \wedge p^2(t)\right),$$

$$p_{\eta^2}(t) = (1-q)^{-1}\left(p^2(t) - p^1(t) \wedge p^2(t)\right),$$

$$p_{\xi}(t) = q^{-1}\left(p^1(t) \wedge p^2(t)\right).$$

Let $\zeta$ be a random variable independent of $\eta^1$, $\eta^2$ and $\xi$ taking values in $\{0, 1\}$ such that

$$P(\zeta = 0) = q, \quad P(\zeta = 1) = 1 - q.$$

Assume that $q \neq 0$ and $q \neq 1$ and let

$$\tilde{X}^1 := \eta^1 1(\zeta = 1) + \xi 1(\zeta = 0),$$

$$\tilde{X}^2 := \eta^2 1(\zeta = 1) + \xi 1(\zeta = 0).$$

Then $\tilde{X}^1 \overset{d}{=} X^1$, $\tilde{X}^2 \overset{d}{=} X^2$, and $P(\tilde{X}^1 = \tilde{X}^2) = q$. QED.

# Coupling, an easy example

## time discrete



Let $(\xi_n)$ be Bernoulli trials, $P(\xi_n = 0) = p$, $P(\xi_n = 1) = q$, $p + q = 1, 0 < q < 1$. Assume our MC $(X_n)$ is, in fact, a sequence of random variables $X_0 = 1$,

$$X_n = X_{n-1} 1(\xi_n = 1), \; n \geq 1.$$

This is a MC with a unique stationary distribution $\mu = \delta_0$ (homework). Consider its stationary version $Y_n = 0, n \geq 0$. We have,

$$\|\mu_n^X - \mu\| = 2 \sup_A (P(X_n \in A) - P(Y_n \in A))$$

Since the state space $S = \{0; 1\}$ here, take $A = \{1\}$. Then

$$P(X_n = 1) - P(Y_n = 1) = P(X_n = 1) = P(\bigcap_{k \leq n} (\xi_k = 1))$$

$$= \prod_{k \leq n} P(\xi_k = 1) = q^n \quad \Longrightarrow \quad \|\mu_n^X - \mu\|_{TV} = 2q^n \to 0, n \to \infty.$$

# Markovian coupling, ctd; assume $T = 1$

Now $X_t$ is a solution of our SDE, consider it at integer times $X_n$; the idea

Assume there exists a stationary measure, and $Y_n$ is a stationary version of our MC. If not coupled earlier, on each step $n \mapsto n+1$ where both $X_n, Y_n \in B_R$, by applying the coupling lemma we couple them on this transition with probability at least $\kappa(1; R)$. So, probability of no coupling after $n$ steps is at most

$$(1 - \kappa(1; R))^k,$$

where $k$ (random) is the number of integer times $t = 0, \dots n$ where both $X_t, Y_t \in B_R$. Of course, we prefer to have a deterministic bound; nevertheless, note that this random number as a function of $n$ is growing approximately linearly, due to some version of a LLN (specified below). So, let us use the LLN and appropriate bounds in its statement.

# LLN for recurrent process $(X_t, Y_t)$

Idea: $P(\text{no coupling after } n \text{ steps}) \lessapprox (1 - \kappa(1; R))^k$; what is known of $k$?



Let

$$T_1 = \tau_R = \inf(t \ge 0 : X_t, Y_t \in B_R), \; T_{n+1} := \inf(t > T_n : X_t, Y_t \in B_R).$$

**Lemma (LLN for $T_n$)**

*Under the assumptions*

$$E_x \tau_R < \infty \quad \& \quad \sup_{|x|=R+1} E_x \tau_R < \infty,$$

*the following convergence holds true ($\mu$ is the invariant measure),*

$$\frac{T_n}{n} \stackrel{P}{\to} \kappa = E_\mu(T_2 - T_1).$$

Clearly, $\kappa \ge 1 > 0$. So, we may expect $k \approx n/\kappa$.

# LLN bounds $P(\text{no coupling after } n \text{ steps}) \lesssim (1-\kappa(1;R))^k$

$k \approx n/\kappa$; but it may not lead to an exponential rate of convergence

In fact, we have now to split unit as

$$1 = 1(k < (\kappa^{-1} - \delta)n) + 1(k \ge (\kappa^{-1} - \delta)n).$$

Then it follows ($Y$ is an independent from $X$ stationary version of the process)

$$\begin{aligned}
|P_x(X_n \in A) - \mu(A)| &= |E_x 1(X_n \in A) - E_\mu(Y_n \in A)| \\
&\le |E_{x,\mu}(1(X_n \in A) - 1(Y_n \in A))1(k < (\kappa^{-1} - \delta)n)| \\
&\quad + |E_{x,\mu}(1(X_n \in A) - 1(Y_n \in A))1(k \ge (\kappa^{-1} - \delta)n)| \\
&\le (1-\kappa(1;R))^{(\kappa^{-1}-\delta)n} \\
&\quad + |E_{x,\mu}(1(X_n \in A) - 1(Y_n \in A))1(k < (\kappa^{-1} - \delta)n)| \\
&\le (1-\kappa(1;R))^{(\kappa^{-1}-\delta)n} + E_{x,\mu}1(k < (\kappa^{-1} - \delta)n).
\end{aligned}$$

It remains to evaluate the last term as $n \to \infty$.

# Proof, ctd. here $k = k(n)$

It remains to evaluate the term $E_{x,\mu} 1(k < (\kappa^{-1} - \delta)n)$ as $n \to \infty$

The LLN for a MC [see, e.g., [AYV, Lecture notes on ergodic MC]] only tells us that it goes to zero,

$$E_{x,\mu} 1(\frac{k}{n} < (\kappa^{-1} - \delta)) \to 0.$$

Denote

$$m = m(n) := (\kappa^{-1} - \delta)n.$$

By Bienaymé – Chebyshev – Markov's inequality,

$$E_{x,\mu} 1(k(n) < (\kappa^{-1} - \delta)n) = E_{x,\mu} 1(k(n) < m)$$

$$= P_{x,\mu}(T_m - \kappa m > n - \kappa m) \le E_{x,\mu} \frac{(T_m - \kappa m)^\ell}{(n - \kappa m)^\ell}$$

Now it remains to evaluate $E_{x,\mu}(T_m - \kappa m)^\ell$, under a condition

$$E_x \tau_R^\ell \le C(x).$$

# Ctd: overall bound $(1-q)^n + C(x)n^{-\ell/2}$

$E_x\tau_R^\ell \le C(x)$; $P_{x,\mu}(T_m > n) \le E_{x,\mu}(T_m-\kappa m)^\ell/(n-\kappa m)^\ell$; $\ell = 2m$ (even)



We have, $$(T_m - \kappa m)^\ell = (\sum_{i=1}^m (T_i - T_{i-1} - \kappa))^\ell.$$

So, $$P_{x,\mu}(T_m > n) = P_{x,\mu}(T_m - \kappa m > n - \kappa m)$$
$$\le E_{x,\mu}\frac{(T_m-\kappa m)^\ell}{(n-\kappa m)^\ell} = \frac{E_{x,\mu}(\sum_{i=1}^m (T_i - T_{i-1} - \kappa))^\ell}{(n-\kappa m)^\ell}$$

The trick is that $(n-\kappa^{-1}m)^\ell \sim n^\ell(1-(\kappa^{-1}-\delta)/\kappa^{-1})^\ell$, while **it may be proved that** (a bit involved but well-known Khasminsky's bounds)

$$E_{x,\mu}(\sum_{i=1}^m (T_i - T_{i-1} - \kappa))^\ell \sim Cm^{\ell/2} \sim C_1 n^{\ell/2}.$$

Therefore,

$$\boxed{P_{x,\mu}(T_m > n) \le C_1 n^{-\ell/2}}; \quad \text{in fact, } C_1 = C_1(x).$$

# Exponential bounds

## Polynomial bounds are known but involved; exponential ones are easier



Assume instead that an exponential bound is known for $\tau_R$:

$$E_x \exp(\alpha\tau_R) \le C(x).$$

Then we estimate (let $n - \kappa m \sim cn$, $c = 1 - \delta\kappa$),

$$P_{x,\mu}(T_m - \kappa m > n - \kappa m) \le \exp(-\lambda cn) E_{x,\mu} \exp(\lambda(T_m - \kappa m)).$$

The trick here is that

$$\frac{1}{n} \ln \exp(-\lambda cn) = -\lambda c \qquad \text{(of order } \lambda\text{)},$$

while *(as it may be proved)*

$$\frac{1}{n} \ln E_{x,\mu} \exp(\lambda(T_m - \kappa m)) \sim \ln E_{\mu,\mu} \exp(\lambda(T_1 - \kappa))$$

$$\sim \ln(1 + \frac{\lambda^2}{2} E_{\mu,\mu}(T_1 - \kappa)^2) \sim c_1\lambda^2 \quad \text{(of order } \lambda^2\text{)}.$$

In the estimates we will have also the multiplier $C(x)$



Hence, choosing $\lambda > 0$ small enough (in any case, it must be $\le \alpha$), we obtain a resulting exponential bound (I drop the precise explanation how $C(x)$ shows up here)

$$P_{x,\mu}(T_m - \kappa m > n - \kappa m) \le C(x) \exp(-\lambda c n + c_1 \lambda^2 n)$$
$$\le C(x) \exp(-(\lambda c - \lambda^2 c_1) n) = C(x) \exp(-\lambda c_3 n).$$

In this case the overall bound for will be exponential:

$$|P_x(X_n \in A) - \mu(A)| \le (1-q)^n + C(x) \exp(-\lambda c_3 n),$$

and, hence, a similar exponential bound holds true for $X_t$:

$$\|\mu_t^x - \mu\| \le 2(1-q)^{[t]} + 2C(x) \exp(-\lambda c_3 [t]).$$

*The next lecture on SDEs with Poisson random measures will be delivered by Dr Dasha Loukianova, Université d'Evry, France. Please, send your email addresses (just an email with a simple greeting) to her address dloukianova@gmail.com*

THE END OF THE "DIFFUSION" PART OF THE COURSE