Introduction to stochastic differential equations – 2
Stochastic exponentials, Girsanov's theorems

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Spring 2020

April 15, 2020

Abstract

First example of weak solutions; no Lipschitz conditions

A stochastic differential equation in $\mathbb{R}^d$ is considered

$$dX_t = b(t, X_t)dt + dW_t, \ \ t \geq 0, \qquad X_0 = x_0, \tag{1}$$

Or, equivalently in the integral form,

$$X_t = x_0 + \int_0^t b(s, X_s)ds + W_t. \tag{2}$$

Here $(W_t, \mathcal{F}_t)$ is a standard d-dimensional Wiener process, b and σ are vector and matrix Borel functions of corresponding dimensions d and $d \times d$. The initial value x_0 may be non-random, or random but $\mathcal{F}_0$-measurable. Yet, the function b is only Borel measurable and bounded. Is there a solution?

Stochastic exponentials

Let b_t be an adapted bounded stochastic vector-valued d-dimensional process. Denote

$$\rho_t = \rho_t[b] := \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds),$$

where $b^2 := (b, b)$, that is, a scalar product. By Ito's formula,

$$d\rho_t = b_t \rho_t dW_t \quad \text{(here } bdW_t \text{ is also a scalar product).}$$

In other words, ρ_t is a solution of an SDE *with a random diffusion coefficient*

$$dX_t = b_t X_t dW_t, \quad X_0 = 1.$$

In the integral form we have,

$$\rho_t = 1 + \int_0^t b_s \rho_s dW_s.$$

The integral form gives us a hope that possibly ρ_t may be a martingale, as usual for stochastic integral. If this is true, we would have, in particular, $E\rho_t = 1$. In turn, any object which is non-negative and integrates to one can serve as a density. May ρ_t serve as a probability density?

Theorem

If b_t is bounded, then $\rho_t[b]$ is a martingale and $E\rho_t = 1$.

Proof. Let $\tau_N := \inf(t \geq 0 : \rho_t \geq N)$. Then clearly $\int_0^t 1(s \leq \tau_N) b_s \rho_s dW_s$ is a martingale ("mart") because

$$E\int_0^t 1^2(s \leq \tau_N) b_s^2 \rho_s^2 ds \leq t\|b\|_B^2 N^2 < \infty.$$

So, $E\rho_{t\wedge\tau_N} = 1 + E\int_0^{t\wedge\tau_N} b_s \rho_s dW_s = 1$.

Moreover, by virtue of the martingale property for $\rho_{t\wedge\tau_N}$,

$$E(\rho_{t\wedge\tau_N}|\mathcal{F}_s) = \rho_{s\wedge\tau_N}, \quad s < t.$$

Here due to the continuity of ρ_t,

$$\tau_N \to \infty, \quad N \to \infty.$$

Therefore, the right hand side here tends to ρ_s as $N \to \infty$. What happens with the left hand side? We would show the martingale property of ρ_t if we knew that $\rho_{t\wedge\tau_N}$ is uniformly integrable. Indeed, uniform integrability allows to use Lebesgue's analugue of the dominated convergence theorem *for conditional expectations*, under the U.I. condition instead of the domination assumption. *[This is a material for your homework: to repeat all limit theorems for conditional expectatoins.]*

So, it suffices to show that

$$E\rho^2_{t\wedge\tau_N} \le C$$

with some C, independent of N. We estimate,

$$\begin{aligned} E\rho^2_{t\wedge\tau_N}[b] &= E\exp(2\int_0^{t\wedge\tau_N} b_s dW_s - \int_0^{t\wedge\tau_N} b_s^2 ds) \\ &= E\exp(\int_0^{t\wedge\tau_N} 2b_s dW_s - \frac{1}{4}\int_0^{t\wedge\tau_N} (2b)_s^2 ds) \\ = E\exp(\int_0^{t\wedge\tau_N} 2b_s dW_s &- \frac{1}{2}\int_0^{t\wedge\tau_N} (2b)_s^2 ds + \frac{1}{4}\int_0^{t\wedge\tau_N} (2b)_s^2 ds) \\ &\le \exp(\frac{1}{4}t\|(2b)^2\|_B) E\rho_{t\wedge\tau_N}[2b] = \exp(\frac{1}{4}t\|(2b)^2\|_B) < \infty. \end{aligned}$$

Note that the right hand side here does not depend on N.

Thus, for any bounded b the stochastic exponential ρ_t is a (non-negative) martingale with $E\rho_t = 1$. Hence, this exponential may serve as a probability density. Let us define a new measure on $\mathcal{F}$,

$$\tilde{P}(A) = P^{\rho_t}(A) := E\rho_t 1(A).$$

[Homework: check that $\tilde{P}$ is, indeed, a probability measure.]

Can the boundedness of b for the martingale property of ρ be relaxed and how far? The most well-known is Novikov's condition

$$E\exp(\frac{1}{2}\int_0^t b_s^2 ds) < \infty.$$

There were preceding conditions by Gikhman and Skorokhod, and there are extensions due to Krylov. We will learn one small step towards these weaker conditions.

Martingale property of $\rho_t = \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$

Assumption $E\exp(C\int_0^t b_s^2 ds) < \infty$

Let us return to the calculus establishing the uniform integrability of $\rho_{\cdot\wedge\tau_N}$; we will try to improve it a bit. We have,

$$E\rho^2_{t\wedge\tau_N}[b] = E\exp(2\int_0^{t\wedge\tau_N} b_s dW_s - \int_0^{t\wedge\tau_N} b_s^2 ds)$$

$$= E\exp(\int_0^{t\wedge\tau_N} 2b_s dW_s - (4-3)\int_0^{t\wedge\tau_N} b_s^2 ds)$$

$$\overset{CBS}{\leq} \left(E\exp(\int_0^{t\wedge\tau_N} 4b_s dW_s - 8\int_0^{t\wedge\tau_N} b_s^2 ds\right)^{\frac{1}{2}}$$

$$\times \left(E\exp(6\int_0^{t\wedge\tau_N} b_s^2 ds)\right)^{\frac{1}{2}} \leq \left(E\exp(6\int_0^t b_s^2 ds)\right)^{\frac{1}{2}}.$$

A conclusion: the condition $E\exp(6\int_0^t b_s^2 ds) < \infty$ suffices.

Supermart property of $\rho_t = \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$

Assumption $P(\int_0^t b_s^2 ds < \infty) = 1$ (so that a SI $\int_0^t b_s dW_s$ is defined)

Theorem (recall that $\rho_t = 1 + \int_0^t b_s \rho_s dW_s$)

Under the assumption $P(\int_0^t b_s^2 ds < \infty) = 1$ the process ρ is a supermartingale: $\rho_{t_1} \geq E(\rho_{t_2}|\mathcal{F}_{t_1})$, $\forall t_1 < t_2$, & $E\rho_t \leq 1$.

Proof. Return to the beginning of the proof of the last theorem. With a stopping time $\tau_N := \inf(t \geq 0 : \rho_t \geq N)$, the process $\int_0^t 1(s \leq \tau_N) b_s \rho_s dW_s$ is a martingale, so,

$$1 + E(\int_0^{t_2 \wedge \tau_N} b_s \rho_s dW_s | \mathcal{F}_{t_1}) = 1 + \int_0^{t_1 \wedge \tau_N} b_s \rho_s dW_s.$$

In other words, $E(\rho_{t_2 \wedge \tau_N} | \mathcal{F}_{t_1}) = \rho_{t_1 \wedge \tau_N}$. The supermart inequality follows from the Fatou lemma for conditional expectations $E(\liminf_{N\to\infty} \rho_{t_2 \wedge \tau_N} | \mathcal{F}_{t_1}) \leq \liminf_{N\to\infty} \rho_{t_1 \wedge \tau_N}$, since $\rho_{s \wedge \tau_N} \to \rho_s$ due to continuity of ρ.

Corollary

Corollary

For any bounded adapted process b_t,

$$E \exp\left(\int_0^t b_s dW_s\right) < \infty.$$

New measure $\tilde{P} \mapsto$ new WP $\tilde{W}$

Assume ρ is a mart on $[0, t]$; define $\tilde{W}_s := W_s - \int_0^s b_u du$, $s \le t$.

The next question is natural: we changed our measure; it is likely that W_s is no more a WP under this new measure; but is there a new WP instead? Igor Vladimirovich Girsanov proposed a new WP to be:

$$\tilde{W}_s := W_s - \int_0^s b_u du, \quad 0 \le s \le t.$$

Theorem (Girsanov)

Let b_t be bounded. Then $\tilde{W}_s$ is a Wiener process on $[0, t]$ under the measure $\tilde{P} : d\tilde{P}/dP = \rho_t$.

As an immediate consequence, for any bounded Borel drift $b(\cdot)$ we can construct a *weak* solution of an SDE

$$dX_t = b(X_t)dt + dW_t, \quad X_0 = x. \tag{3}$$

Weak solution from Girsanov's theorem

By changing measure!

Denote $X_s = W_s + x$ and

$$\tilde{W}_s = W_s - \int_0^s b(x + W_u)du, \quad s \leq t.$$

This is a new WP under the new probability measure

$$\frac{d\tilde{P}}{dP} = \rho_t := \exp\left(\int_0^t b(x + W_s)dW_s - \frac{1}{2}\int_0^t b^2(x + W_s)ds\right).$$

Then we have,

$$W_s = \tilde{W}_s + \int_0^s b(x + W_u)du, \quad s \leq t.$$

and therefore, X is a solution to the SDE with a new WP on $[0, t]$,

$$X_s = x + \tilde{W}_s + \int_0^s b(X_u)du, \quad s \leq t.$$

Proof of Girsanov's theorem about a new WP

$\tilde{W}_s := W_s - \int_0^s b_u du$, and $d\tilde{P} = \rho_t dP$ with $\rho_t = \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$

For the proof we need one lemma and the definition of a WP via its characteristic function, namely, for any $0 = t_0 < t_1 < t_2 \ldots < t_N$ and real values $\lambda_j, 1 \leq j \leq N$,

$$\tilde{E} \exp(\sum_{j=0}^{N-1} i\lambda_j(\tilde{W}_{t_{j+1}} - \tilde{W}_{t_j})) = \exp(-\frac{1}{2}\sum_j \lambda_j^2(t_{j+1} - t_j)).$$

Lemma

Let $\beta_t = \beta_t^1 + i\,\beta_t^2$ be a bounded adapted random process, where $i = \sqrt{-1}$. Then the (complex-valued) process

$$\rho_t[\beta] := \exp(\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 ds)$$

is a (complex-valued) martingale.

Proof of Theorem

$\rho_t[b] = \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$; $\tilde{W}_s := W_s - \int_0^s b_u du$

Let $\lambda_s := i\lambda_j$ on $[t_j, t_{j+1})$, and $B_s = b_s + \lambda_s$. Then

$$\tilde{E}\exp(\sum_{j=0}^{N-1} i\lambda_j(\tilde{W}_{t_{j+1}} - \tilde{W}_{t_j})) = E\rho_t[b]\exp(\sum_{j=0}^{N-1} i\lambda_j(\tilde{W}_{t_{j+1}} - \tilde{W}_{t_j}))$$
$$= E\exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$$
$$\times \exp(\sum_{j=0}^{N} i\lambda_j(W_{t_{j+1}} - W_{t_j} - \int_{t_j}^{t_{j+1}} b_u du))$$
$$= E\exp(\int_0^t B_s dW_s - \frac{1}{2}\int_0^t B_s^2 ds + \frac{1}{2}\int_0^t \lambda_s^2 ds).$$

Indeed,

$$B_s^2 = (b_s + \lambda_s)^2 = b_s^2 + \lambda_s^2 + 2b_s\lambda_s.$$

End of Proof of Girsanov's theorem

$\lambda_s := i\lambda_j$ on $[t_{j+1} - t_j)$; $B_s = b_s + \lambda_s$

But due to the last Lemma

$$E \exp(\int_0^t B_s dW_s - \frac{1}{2}\int_0^t B_s^2 ds) = 1$$

while $\frac{1}{2}\int_0^t \lambda_s^2 ds$ is non-random and equals

$$\frac{1}{2}\int_0^t \lambda_s^2 ds = -\frac{1}{2}\sum_j \lambda_j^2 (t_{j+1} - t_j).$$

Therefore,

$$\tilde{E} \exp(\sum_{j=0}^{N-1} i\lambda_j(\tilde{W}_{t_{j+1}} - \tilde{W}_{t_j})) = \exp(-\frac{1}{2}\sum_j \lambda_j^2 (t_{j+1} - t_j)),$$

as required. Girsanov's theorem about a new WP under the (Girsanov's) change of measure is proved.

Proof of Lemma

(Recall:)

Lemma

Let $\beta_t = \beta_t^1 + i\,\beta_t^2$ be a bounded adapted random process, where $i = \sqrt{-1}$. Then the (complex-valued) process

$$\rho_t[\beta] := \exp\left(\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 ds\right)$$

is a (complex-valued) martingale.

Proof. It suffices to check for any $A \in \mathcal{F}_{t_1}$, $t_2 > t_1$, and a complex value z,

$$E1(A)exp\left(\int_0^{t_2} (\beta_s^1 + z\beta_s^2) dW_s - \frac{1}{2}\int_0^{t_2} (\beta_s^1 + z\beta_s^2)^2 ds\right)$$

$$= E1(A)exp\left(\int_0^{t_1} (\beta_s^1 + z\beta_s^2) dW_s - \frac{1}{2}\int_0^{t_1} (\beta_s^1 + z\beta_s^2)^2 ds\right).$$

We already know the equality for all real-valued z,

$$E1(A)exp(\int_0^{t_2}(\beta_s^1 + z\beta_s^2)dW_s - \frac{1}{2}\int_0^t(\beta_s^1 + z\beta_s^2)^2 ds)$$

$$= E1(A)exp(\int_0^{t_1}(\beta_s^1 + z\beta_s^2)dW_s - \frac{1}{2}\int_0^t(\beta_s^1 + z\beta_s^2)^2 ds).$$

Hence, the claim will be proved if we show that both sides are analytic functions of z. For the latter, it suffices to show that both sides are continuous in z and that their integrals along any closed bounded contours are equal to zero (Morera's theorem). Because of the analyticity of the expressions **under** the expectations and by Fubini's theorem (i.e., we can change the order of expectation and integration over the contour), we only need to show that for any $R > 0$ and $|z| \leq R$, these expressions are bounded by an integrable r.v. independently of z.

Clearly, to show such domination we only need to care about the stochasic integrals (since Lebesgue's ones are bounded for $|z| \leq R$). By virtue of the clever inequality for any $\alpha, \beta \in R$ with $|\alpha| \leq |\beta|$,

$$\exp(\alpha) \leq \exp(\alpha) + \exp(-\alpha) \leq \exp(\beta) + \exp(-\beta),$$

we have,

$$\begin{aligned} |\exp(\int_0^{t_2} (\beta_s^1 + z\beta_s^2) dW_s)| &= \exp(\int_0^{t_2} (\beta_s^1 + Re(z)\beta_s^2) dW_s) \\ &\leq \exp(\int_0^{t_2} (\beta_s^1 + R\beta_s^2) dW_s) + \exp(\int_0^{t_2} (\beta_s^1 - R\beta_s^2) dW_s). \end{aligned}$$

The latter expression is integrable independently of z (of course, for $|z| \leq R$).

Exponential inequality via stochastic exponential

$\rho_t[b] = \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$

Theorem (SI exp bounds)

Let the adapted process b. be bounded. Then there exist C_1, C_2 such that for any $a > 0$ and for any $T > 0$,

$$P(\sup_{0\leq t\leq T} |\int_0^t b_s dW_s| \geq a) \leq C_1 \exp(-a^2/(C_2 T)).$$

The setting is d-dimensional; b here is a vector. It is also true for b matrices with some slight changes in constants.

Corollary

Under the same assumptions, for any $T > 0$ $\exists \alpha > 0$ such that

$$E\exp(\alpha \sup_{0\leq t\leq T} |\int_0^t b_s dW_s|^2) < \infty.$$

Proof

For any λ, $\rho_t[\lambda b]$ is a continuous martingale. So, with any $\lambda > 0$ by Bienaymé–Chebyshev–Markov's inequality we have,

$$P(\sup_{0\le t\le T} |\int_0^t b_s dW_s| \ge a)$$

$$\le P(\sup_{0\le t\le T} \int_0^t \lambda b_s dW_s \ge \lambda a) + P(\sup_{0\le t\le T} \int_0^t (-\lambda) b_s dW_s \ge \lambda a)$$

$$\le e^{-\lambda a} E e^{\sup_{0\le t\le T} \int_0^t \lambda b_s dW_s} + e^{-\lambda a} e^{\sup_{0\le t\le T} \int_0^t (-\lambda) b_s dW_s}.$$

Consider each term separately and in the same manner.

Proof, ctd.

In the middle of the calculus we use Doob's inequality:

$$e^{-\lambda a} E e^{\sup_{0\le t\le T} \int_0^t \lambda b_s dW_s} = e^{-\lambda a} E \sup_{0\le t\le T} e^{\int_0^t \lambda b_s dW_s}$$

$$= e^{-\lambda a} E \sup_{0\le t\le T} \rho_t[\lambda b] e^{+\frac{1}{2}\int_0^t (\lambda b_s)^2 ds} \le e^{-\lambda a + Ct\lambda^2} E \sup_{0\le t\le T} \rho_t[\lambda b]$$

$$\le e^{-\lambda a + Ct\lambda^2} \sqrt{4 E \rho_T^2[\lambda b]}$$

$$= 2 e^{-\lambda a + Ct\lambda^2} \left(E \rho_T[2\lambda b] \exp\left(\int_0^t (\lambda b_s)^2 ds\right)\right)^{1/2}$$

$$\overset{\text{new } C}{\le} 2 e^{-\lambda a + Ct\lambda^2} \left(E \rho_T[2\lambda b]\right)^{1/2} = 2\exp(-\lambda a + Ct\lambda^2).$$

Taking $\inf_{\lambda > 0}$, obtain with $\lambda = a/(2Ct)$ the bound

$$e^{-\lambda a} E e^{\sup_{0\le t\le T} \int_0^t \lambda b_s dW_s} \le 2\exp(-a^2/(4Ct)).$$

Proof, ctd.

Check yourself that the other term

$$e^{-\lambda a} e^{\sup_{0\le t\le T} \int_0^t (-\lambda) b_s dW_s}$$

admits the same bound,

$$e^{-\lambda a} E e^{\sup_{0\le t\le T} \int_0^t (-\lambda b_s) dW_s} \le 2\exp(-a^2/(4Ct)).$$

Overall, we obtain, as required,

$$P(\sup_{0\le t\le T} |\int_0^t b_s dW_s| \ge a) \le e^{-\lambda a} E e^{\sup_{0\le t\le T} \int_0^t \lambda b_s dW_s}$$

$$+e^{-\lambda a} e^{\sup_{0\le t\le T} \int_0^t (-\lambda) b_s dW_s} \le 4\exp(-a^2/(4Ct)).$$

Proof of Corollary

The idea is to use the bound with $a^2 = z \geq 0$

$$P(\sup_{0\leq t\leq T} |\int_0^t b_s dW_s|^2 \geq a^2) = P(\sup_{0\leq t\leq T} |\int_0^t b_s dW_s| \geq a)$$
$$\leq 4\exp(-a^2/(4Ct)).$$

Now integrate (in the middle by parts) with $\alpha < (4Ct)^{-1}$:

$$E\exp(\alpha \sup_{0\leq t\leq T} |\int_0^t b_s dW_s|^2)$$
$$= \int_0^\infty \exp(\alpha z) d(-P(\sup_{0\leq t\leq T} |\int_0^t b_s dW_s|^2 \geq z))$$
$$= 1 + \int_0^\infty P(\sup_{0\leq t\leq T} |\int_0^t b_s dW_s|^2 \geq z) d\exp(\alpha z)$$
$$\leq 1 + \alpha \int_0^\infty 4\exp(-z[(4Ct)^{-1} - \alpha])dz < \infty.$$