Introduction to stochastic integration with jumps

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Spring 2020

May 29, 2020

[1] Evry University, Paris-Saclay University, Russia online mini-course

The aim of these short lectures is to understand the meaning of :

Short introduction to stochastic integration with jumps

Poisson random measures
Definition and construction
Martingales related to the PRM
Examples of PRM. Jump measure
Jump measure of Poisson process
Jump measure of a compound Poisson process

Levy process
Jump measure of a general Lévy process
Levy measure
Lévy-Itô decomposition

Stochastic integration
Stochastic integral with respect to semi-martingale

$$
\begin{aligned}
X_t = & X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \\
& \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \gamma(X_{s_-}, x)\mathbf{1}_{|x|\leq 1}\tilde{\mu}(ds, dx) + \\
& + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \gamma(X_{s_-}, x)\mathbf{1}_{|x|>1}\mu(ds, dx)
\end{aligned} \tag{1}
$$

or

$$
Z_t = \int_{[0,t]\times\mathbb{R}_+} e^{-\alpha(t-s)}\mathbf{1}_{z\leq f(s-)}d\pi(s, z). \tag{2}
$$

And to learn how to express $f(X_t)$ in an analogous form.

Some books

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- [*CT*] Rama Cont and Peter Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC 2004
- [*IW*] Nobuyuki Ikeda and Shinzo Watanabe. , *Stochastic Differential Equations and Diffusion Processes*; Kodansha: Tokyo, 1981;
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Abstract

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3 Stochastic integration
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  - Stochastic integral of predictable process with respect to a martingale
- Stochastic integral with respect to the PRM
- Quadratic variation
- Ito formula

Poisson random measures (PRM)

Setting:
$(\Omega, \mathcal{F})$ - probability space; $(\bar{E}, \mathcal{B}_{\bar{E}})$ - measurable space.

$$M - \text{ set of all } \mathbb{N} \cup \{+\infty\}\text{-valued measures on } (\bar{E}, \mathcal{B}_{\bar{E}}).$$

$$\mathcal{B}_M = \sigma\left\{M \ni \pi \mapsto \pi(B) \in \mathbb{N} \cup \{+\infty\},\ B \in \mathcal{B}_{\bar{E}}\right\}$$

Definition

A r.v. $\pi : \Omega \to M$ is a PRM on $(\bar{E}, \mathcal{B}_{\bar{E}})$ if

- *for all $B \in \mathcal{B}_{\bar{E}}$, $\pi(B) \sim \mathcal{P}(m(B))$ with $m(B) = \mathbb{E}(\pi(B))$;*
- *if $B_1 \in \mathcal{B}_{\bar{E}}, \ldots, B_n \in \mathcal{B}_{\bar{E}}$ are disjoint, then $\pi(B_1), \ldots, \pi(B_n)$ are independent.*

* A Poisson r.v. with parameter $+\infty$ is equal to $+\infty$ a.s.

Definition

$\mathcal{B}_{\bar{E}} \ni B \mapsto m(B) := \mathbb{E}\pi(B)$ is a measure on $(\bar{E}, \mathcal{B}_{\bar{E}})$, called the **intensity, or average measure** *of π.*

If $\bar{E}$ is a c.s.m.s.. and π is a on $\bar{E}$ with σ-finite and locally bounded intensity measure, and $\pi(\omega, \{x\}) \leq 1$ for all $(\omega, x) \in \Omega \times \bar{E}$, we can write π as (Daley, Vere-Jones , vol II, prop 9.1.III)

$$\pi(\omega, \cdot) = \sum_i \delta_{X_i(\omega)}(\cdot)$$

where $(X_i) = (X_i(\omega))_i$ is a random countable set of points of $\bar{E}$. We then can see π as follows, for any $B \in \mathcal{B}_{\bar{E}}$,

$$\pi(\omega, B) = Card\,(B \cap (X_i(\omega))_i)\,.$$

$$\mathcal{B}_{\bar{E}} \ni B \mapsto \pi(\omega, B) = Card\left(B \cap (X_i(\omega))_i\right).$$

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Construction of PRM

Theorem

For any sigma-finite measure m on $(\bar{E}, \mathcal{B}_{\bar{E}})$ there exists a PRM π on $\bar{E}$ with intensity measure m.

Proof Case where $m(\bar{E}) = M < \infty$.

- Take $\xi_1, \xi_2, \ldots$ i.i.d. with value in E and law m/M.
- Take $N \sim \mathcal{P}(M)$ independent of (ξ_i).
- For all $B \in \mathcal{B}_{\bar{E}}$ define $\pi(B) = \sum_{i=1}^{N} \mathbb{I}_B(\xi_i)$

- *Poisson distribution:*

$$\mathbb{P}(\pi(B)=k)=\sum_{n=k}^{\infty}\mathbb{P}(\sum_{i=1}^{n}\mathbf{1}_B(\xi_i)=k)\mathbb{P}(N=n)=$$

$$\sum_{n=k}^{\infty}C_n^k m(B)^k(1-m(B))^{n-k}\frac{e^{-1}}{n!}=$$

$$\frac{m(B)^k}{k!}\sum_{\ell=0}^{\infty}\frac{(1-m(B))^{\ell}}{\ell!}e^{-1}=\frac{m(B)^k e^{-m(B)}}{k!}.$$

- independence of $\pi(B_1),\dots,\pi(B_n)$ for disjoints $B_1,\dots,B_n$, ans the general (σ-finite)case are left on exercice.

end of the Proof

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For stochastic calculus we need the time variable

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ under usual assumptions. We consider a PRM π on $\bar{E} = [0; \infty[\times E$, ($E$ is $\mathbb{R}^d$, or $\mathbb{R}^d \setminus \{0\}$, or $\mathbb{N}$...) Set, for all $B \in \mathcal{B}_E$, $\pi_t(B) = \pi([0, t] \times B)$.

Definition

We say that π is a **stationnary** *PRM on $\bar{E} = [0; \infty[\times E$, w.r.t. $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ if $\pi(\{0\} \times E) = 0$;*

- *For all $B \in \mathcal{B}_E$, $\pi_t(B)$ is $\mathcal{F}_t$ measurable;*
- *For all $t \geq 0$, $h > 0$, $B \in \mathcal{B}_E$, $\pi_{t+h}(B) - \pi_t(B) \perp\!\!\!\perp \mathcal{F}_t$.*
- *For all ≥ 0, $h > 0$, $B \in \mathcal{B}_E$, $\pi_{t+h}(B) - \pi_t(B) \sim \mathcal{P}(h\vartheta(B))$ where ϑ is a σ-finite measure on $(E, \mathcal{B}_E)$*

$dt \times \vartheta$ is a measure on $\bar{E} = [0; \infty[\times E$, called "compensator" of π. The measure $\tilde{\pi} := \pi - dt \times \vartheta$ is called compensated measure.

Proposition

Let π be a **stationnary** *PRM on $\bar{E} = [0;\infty[\times E$, w.r.t. $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with compensator $dt \times \vartheta$. For all $B \in \mathcal{B}_E$, such that $\vartheta(B) < \infty$ for all $t > 0$,*

1. $(\tilde{\pi}_t(B))_t = (\pi_t(B) - t\vartheta(B))_{t\geq 0}$ *is a martingale;*
2. $(\tilde{\pi}_t(B)^2 - t\vartheta(B))_{t\geq 0} = ((\pi_t(B) - t\vartheta(B))^2 - t\vartheta(B))_{t\geq 0}$ *is a martingale;*
3. $(\tilde{\pi}_t(B)\tilde{\pi}_t(A) - t\vartheta(A \cap B))_{t\geq 0}$ *is a martingale.*

$\triangle$

1. $\mathbb{E}\pi_t(B) = t\vartheta(B) < \infty$; hence $(\tilde{\pi}_t(B))_t$ is integrable.

$$\begin{aligned}\mathbb{E}(\pi_{t+s}(B)|\mathcal{F}_t) &= \mathbb{E}(\pi_{t+s}(B) - \pi_t(B)|\mathcal{F}_t) + \pi_t(B)\\ &= s\vartheta(B) + \pi_t(B);\end{aligned}$$

Hence

$$\mathbb{E}(\pi_{t+s}(B) - (t+s)\vartheta(B)|\mathcal{F}_t) = \pi_t(B) - t\vartheta(B)$$

2. Using $\mathbb{E}(\tilde{\pi}_t(B) - \tilde{\pi}_s(B))^2 = Var(\mathcal{P}((t-s)\vartheta(B)),$

$\mathbb{E}(\tilde{\pi}_t(B)^2|\mathcal{F}_s) = \mathbb{E}(\tilde{\pi}_t(B) - \tilde{\pi}_s(B))^2 + (\tilde{\pi}_s(B))^2 -$
$2\tilde{\pi}_s(B)\mathbb{E}[\tilde{\pi}_t(B) - \tilde{\pi}_s(B)|\mathcal{F}_s] = (t-s)\vartheta(B) + (\tilde{\pi}_s(B))^2 + 0$

Similarly to the previous proposition we can show :

Proposition

If π *is a PRM w.r.t.* $(\mathcal{F}_t)$ *and* $f : [0, \infty[\times E \to \mathbb{R}$, *measurable, such that* $\forall t > 0$, $\int_{[0,t]\times E} |f(s,x)| ds\vartheta(dx) < \infty$. *Then*

- $$\mathbb{E}\int_{[0,t]\times E} |f(s,x)|\pi(ds,dx) = \mathbb{E}\int_{[0,t]\times E} |f(s,x)| ds\vartheta(dx)$$
- *the process* $\left(X_t = \int_{[0,t]\times E} f(s,x)\tilde{\pi}(ds,dx)\right)_{t\geq 0}$ *is a martingale. In particular ,* $\forall t \geq 0$,

$$\mathbb{E}\int_{[0,t]\times E} f(s,x)\pi(ds,dx) = \int_{[0,t]\times E} f(s,x) ds\vartheta(dx).$$

Proposition

- *Assume that* $\int_{[0,t]\times E} f^2(s,x)ds\vartheta(dx) < \infty$*. Then the process* $(Y_t)_{t\geq 0}$ *is a martingale.*

$$Y_t = \left(\int_{[0,t]\times E} f(s,x)\tilde{\pi}(ds,dx)\right)^2 - \int_{[0,t]\times E} f^2(s,x)ds\vartheta(dx)$$

in particular, $\forall t \geq 0$,

$$\mathbb{E}\left(\int_{[0,t]\times E} f(s,x)\tilde{\pi}(ds,dx)\right)^2 = \int_{[0,t]\times E} f^2(s,x)ds\vartheta(dx)$$

If π- PRM on $[0;\infty[\times E$, $\quad \pi_t(B) := \pi([0,t]\times B)$

Then there exists a countable set $(T_n(\omega), Y_n(\omega))_n \in [0;\infty[\times E$ s.t.

$$\pi = \sum_n \delta_{(T_n, Y_n)}.$$

For all $t > 0$, $B \in \mathcal{B}_E$,

$$\pi_t(B) = \sum_n \mathbf{1}_{[0,t]}(T_n)\mathbf{1}_B(Y_n) = \sum_{T_n \le t} \mathbf{1}_B(Y_n).$$

If π is a PRM on $\mathbb{R}^+ \times E$ adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$, then

- T_n are strictly positive $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ stopping times ;
- Y_n are $\mathcal{F}_{T_n}$ measurable.

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Examples of PRM

The probably most important example of PRM is a jump measure of a Levy process. It will be treated in the case of Poisson and CPP process in this section and in general in the next section. If we consider a PRM with value in $\mathbb{R}_+ \times \mathbb{N}$, it is clearly not a jump measure of a Lévy process. Such a PRM appears in the theory of particles. Its intensity measure is a measure on $\mathbb{R}_+ \times \mathbb{N}$, given by $dt \times \kappa(dn)$, where κ is a counting measure on $\mathbb{N}$.
If we consider a PRM π on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $dt \times dx$, it is not a jump measure of a Lévy process niether, because dx is not a Lévy measure. Such a PRM appears in the theory of Hawkes processes, in particular in the neural activity modeling.

The following (Z_t) represents a "thinned" PRM of such a kind.

$$Z_t = \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{x \le f(s-)} d\pi(s,x).$$

There exists a countable set of random points (T_n, X_n) such that

$$\pi = \sum_n \delta_{(T_n, X_n)}$$

hence

$$Z_t = \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{x \le f(s-)} d\pi(s,x) = \sum_n \mathbf{1}_{T_n \le t} \mathbf{1}_{X_n \le f(T_n -)}.$$

Hence Z_t is a counting process which count the number of (T_n) on $[0,t]$, with $X_n < f(T_n-)$. Let's calculate, using "the compensation formula" $\mathbb{E}Z_t$:

$$\mathbb{E}Z_t = \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{x\le f(s-)} d\pi(s,x) = \int_{[0,t]\times\mathbb{R}_+} \mathbf{1}_{x\le f(s-)} dsdx$$
$$= \int_{[0,t]} f(s-)ds.$$

and

$$\mathbb{E}(Z_{t_2} - Z_{t_1}) = \int_{t_1}^{t_2} f(s-)ds.$$

Hence f represents a "density " of points of Z_t.

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Let $(\pi)_j$ be a sequence of i.i.d. PRM on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ with intensity $dt \times dz \times \mu(du)$ where μ is a centered measure. Let

$$X_t^N = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} e^{-\alpha(t-s)} u \mathbf{1}_{z \le f(X_{s-}^N)} d\pi_j(s,z,u).$$

In the framework of neurosciences, X_t^N represents the membrane potential of the neurons at time t. The random jump heights u, chosen according to the measure μ, model random synaptic weights and the jumps of π_j represent the spike times of neuron j. If neuron j spikes at time t, an additional random potential height $u/\sqrt{N}$ is given to all other neurons in the system.

Jump measure

(T_n, Y_n)-jump times and amplitudes,

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Let $X = (X_t)$ be a cadlag process with value in $\mathbb{R}^d$. Let

$$X_{t-} := \lim_{s \uparrow t} X_s \quad \text{and} \quad \Delta X_t = X_t - X_{t-}.$$

A cadlag trajectory can have only countable number of jumps on any compact interval $[t_1, t_2]$. Also, for any $\varepsilon > 0$ the number of jumps on $[t_1, t_2]$ such that $\Delta X_s \geq \varepsilon$ is finite.
The jump measure μ^X of X is a measure on $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$ given by

$$\mu^X(]s, t] \times A) = \text{number of jumps of X on }]s; t]$$
$$\text{whose amplitude belongs to A,} \quad (A \subset \mathbb{R}^d \setminus \{0\})$$

Since (T_n) are jump times of X, and $Y_n = \Delta X_{T_n}$ amplitudes:

$$\mu_t^X(A) = \sum_n \mathbf{1}_{[0,t]}(T_n)\mathbf{1}_B(Y_n) = \sum_{T_n \leq t} \mathbf{1}_A(\Delta X_{T_n}) = \sum_{0 \leq s \leq t} \mathbf{1}_A(\Delta X_s)$$

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In particular, if $\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |x|\mu^X(ds,dx) < \infty$, then then the integral in the l.h.s bellow is well defined and

$$\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} x\mu^X(ds,dx) = \sum_{T_n \le t} \Delta X_{T_n} = \sum_{0<s\le t} \Delta X_s$$

Note that if μ^X is a PRM with the compensator $dt\vartheta(dx)$, then

$$\mathbb{E}\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |x|\mu^X(ds,dx) = t\int_{\mathbb{R}^d\setminus\{0\}} |x|\vartheta(x).$$

hence

$$\int_{\mathbb{R}^d\setminus\{0\}} |x|\vartheta(x) < \infty \Longrightarrow \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |x|\mu^X(ds,dx) < \infty, \forall t > 0.$$

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More generally, if $\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |f(x)|\mu^X(ds,dx) < \infty$, then the integral in the l.h.s bellow is well defined and

$$\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} f(x)\mu^X(ds,dx) = \sum_{T_n\le t} f(\Delta X_{T_n}) = \sum_{0<s\le t} f(\Delta X_s)$$

Note again that if μ^X is a PRM with the compensator $dt\vartheta(dx)$, then

$$\mathbb{E}\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |f(x)|\mu^X(ds,dx) = t\int_{\mathbb{R}^d\setminus\{0\}} |f(x)|\vartheta(x).$$

Hence $\forall t > 0$,

$$\int_{\mathbb{R}^d\setminus\{0\}} |f(x)|\vartheta(x) < \infty \Longrightarrow \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |f(x)|\mu^X(ds,dx) < \infty.$$

Example:

$$\int_{\mathbb{R}^d\setminus\{0\}} x^2\vartheta(x) < \infty \Longrightarrow$$

$$\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |x|^2\mu^X(ds,dx) = \sum_{0\leq s\leq t} |\Delta X_s|^2 < \infty.$$

We will see that for a Lévy process we may have

$$\sum_{0\leq s\leq t} |\Delta X_s| = \infty,$$

but it is always true that

$$\sum_{0\leq s\leq t} |\Delta X_s|^2 < \infty.$$

Jump measure of a Poisson process

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We will show that the jump measure μ^X of a Levy process X is a PRM, and that the jump component of X can be written using an integral w.r.t. μ^X and $\tilde{\mu}^X$.
The first example is the Poisson process.

Definition

Let $(T_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of stopping times, $T_0 = 0$, $\lim_{n\to\infty} T_n = +\infty$. A counting process $N_t = \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$ is a Poisson process if

- *for any $t > 0$ $h > 0$, $N_{t+h} - N_t$ is independent of $\mathcal{F}_t$;*
- *for any $t > 0$ $h > 0$, $N_{t+h} - N_t$ has the same distribution as N_h.*

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It follows from definition that $N_t \sim \mathcal{P}(\lambda t)$ for some $\lambda > 0$, actually $\lambda = \mathbb{E}N_1$.
Important caractherization of a PP:
Non-exploding counting process with independent and stationary increments is a Poisson process

Jump measure of a Poisson process

$N_t = \sum_{n \geq 1} \mathbf{1}_{\{T_n \leq t\}}$

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For any $s < t$, $\quad A \in \mathcal{B}_{\mathbb{R}\setminus\{0\}}$,

$$\mu^N(]s,t] \times A) = \sum_{t_1 < s \leq t_2} \mathbf{1}_A(\Delta N_s) = \sum_{T_n \leq t} \mathbf{1}_A(\Delta N_{T_n})$$

Hence

$$\mu^N(]s,t] \times A) = (N_t - N_s) \times \delta_{\{1\}}(A).$$

Let's show that μ^N is a PRM on $[0,+\infty[\times\mathbb{R}$:

- $\mu^N(]s,t] \times A) \sim \mathcal{P}(\lambda(t-s)\delta_{\{1\}}(A))$;
- Using the independence of increments of N, if $]s,t]\cap]u,v] = \emptyset$, then $\mu^N(]s,t] \times A) \perp\!\!\!\perp \mu^N(]u,v] \times A)$. Clearly, if $A \cap B = \emptyset$, then $\mu^N(]s,t] \times A) \perp\!\!\!\perp \mu^N(]s,t] \times B)$.

$\mu^N(]s,t] \times A) = (N_t - N_s) \times \delta_{\{1\}}(A).$

The intensity measure is $dt\vartheta(dx) = dt \times \lambda\delta_{\{1\}}$ Indeed:

$$]s,t] \times A \mapsto \mathbb{E}(\mu^N(]s,t] \times A)) = \mathbb{E}(N_t - N_s) \times \delta_{\{1\}}(A) =$$
$$= \lambda(t-s) \times \delta_{\{1\}}(A).$$

Note that

$$\int_{[0,t]\times\mathbb{R}} |x|\mu^N(ds,dx) < \infty,$$

because

$$\mathbb{E}\int_{[0,t]\times\mathbb{R}} |x|\mu^N(ds,dx) = \int_{[0,t]\times\mathbb{R}} |x| ds \times \delta_{\{1\}}(dx) = t < \infty.$$

Then

$$\int_{[0,t]\times\mathbb{R}} x\mu^N(ds,dx) = \sum_{0\le s\le t} \Delta N_s = \sum_{n,T_n\le t} \Delta N_{T_n} = N_t$$

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Definition

A compound Poisson process with intensity $\lambda > 0$ and jump size distribution F is defined by

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where $(Y_i)_{i \in \mathbb{N}^*}$ are i.i.d. integrable $\mathbb{R}^d$-valued r.v. with distribution f, and $N = (N_t)_{t \geq 0}$ is a PP with intensity λ, independent of $(Y_i)_i$

Important characterization of a CPP:
(X_t) is a CPP if and only if it has

1. independent and stationary increments,
2. is continuous in probability
3. has piecewise constant simple paths

(Note 1 $+$ 2=Lévy process)

Proposition

The jump measure μ^X of a CPP X is a PRM on $[0, +\infty[\times\mathbb{R}^d$ with intensity measure $dt \times \vartheta(dx) = dt \times \lambda \times f(dx)$

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Proof

Jump measure of a Compound Poisson process $X_t = \sum_{i=1}^{N_t} Y_i$ is a PRM

- **Poisson distribution of $\mu^X(]s,t] \times A)$:**
  $\mu^X(]s,t] \times A) = \sum_{s<u\le t} \mathbf{1}_A(\Delta X_u)$
  Conditionally on N, $\mu^X(]s,t] \times A) \sim \mathcal{B}(N_t - N_s, f(A))$.

$$\text{if} \quad S \sim \mathcal{B}(n,p) \quad \text{then} \quad \mathbb{E}e^{iuS} = (e^{iu}p + q)^n$$

$$\text{if} \quad N \sim \mathcal{P}(\lambda), \quad \text{then} \quad \mathbb{E}e^{iuN} = \exp(\lambda(e^{iu} - 1));$$
$$\text{and} \quad \mathbb{E}z^N = e^{\lambda(z-1)}$$

Hence, using $N_t - N_s \sim \mathcal{P}(\lambda(t-s))$,

$$\mathbb{E}e^{iu\mu^X(]s,t]\times A)} = \mathbb{E}(\mathbb{E}e^{iu\mu^X(]s,t]\times A)}|N)) = \mathbb{E}(e^{iu\mathcal{B}(N_t-N_s,f(A))}) =$$
$$\mathbb{E}(e^{iu}f(A) + 1 - f(A))^{N_t-N_s}) = \exp(\lambda(t-s)f(A)(e^{iu} - 1));$$

Hence $\mu^X(]s,t] \times A) \sim \mathcal{P}(\lambda(t-s)f(A))$

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- **Independence for disjoints sets**
  First let $A \cap B = \emptyset$ two borel sets of $\mathbb{R}^d$. Let's show that $\mu^X(]s,t] \times A) \perp\!\!\!\perp \mu^X(]s,t] \times B)$. Conditionally on N,

$$iu\mu^X(]s,t]\times A)+iv\mu^X(]s,t]\times B) = \begin{cases} iu & f(A) \\ iv & f(B) \\ 0 & 1-f(A)-f(B) \end{cases}$$

$$\mathbb{E}(\mathbb{E}(e^{iu\mu^X(]s,t]\times A)+iv\mu^X(]s,t]\times B)}|N)) =$$
$$\mathbb{E}(e^{iu}f(A)+e^{iv}f(B)+1-f(A)-f(B))^{N_t-N_s}) =$$
$$\exp \lambda(t-s)(f(A)(e^{iu}-1)+f(B)(e^{iv}-1)) =$$
$$\mathbb{E}(e^{iu\mu^X(]s,t]\times A)}) \times \mathbb{E}(e^{iv\mu^X(]s,t]\times B)}).$$

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If $[s,t]\cap]u,v] = \emptyset$, then $\mu^X(]s,t]\times A) \perp\!\!\!\perp \mu^X(]u,v]\times A)$ follows from the independence of increments of X.
Remember $\mu^X(]s,t]\times A) \sim \mathcal{P}(\lambda(t-s)f(A))$,
the intensity, or average measure of CPP is given by
$\mathbb{E}\mu^X(]s,t]\times A) = \mathbb{E}\mathcal{P}(\lambda(t-s)f(A)) = (t-s)\lambda f(A)$,

The intensity is: $dt\vartheta(dx) = dt\lambda f(dx)$

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Here again:

$$\int_{[0,t]\times\mathbb{R}} |x|\mu^X(ds \times dx) < \infty,$$

because

$$\mathbb{E}\int_{[0,t]\times\mathbb{R}} |x|\mu^X(ds \times dx) \int_{[0,t]\times\mathbb{R}} |x| ds \times \lambda \times f(dx) = t\lambda\mathbb{E}Y_1 < \infty$$

Then

$$\int_{[0,t]\times\mathbb{R}} x\mu^X(ds \times dx) = \sum_{0\leq s\leq t} \Delta X_s = X_t$$

X_t is "the sum of its jumps".

Levy Process

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Definition

An adapted to $(\mathcal{F}_t)_{0\leq t\leq\infty}$ process $X = (X_t)_{t\geq 0}$ with $X_0 = 0$ is a Lévy process if

1. *X has independent increments : $X_t - X_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t < \infty$;*
2. *X has stationary increments: $X_t - X_s$ has the same law as X_{t-s};*
3. $\forall\varepsilon >$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

There exists a unique modification of X which is càdlàg.
Brownian motion, PP, CPP are Lévy processes.

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Jump measure of a Lévy process is a PRM, of intensity $dt\nu(dx)$. ν is called Lévy measure .

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Sketch of the proof: Let A such that $\bar{A} \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Let $T_A^1 = \inf\{t > 0;\ \Delta X_t \in A\}; \ldots\ T_A^{n+1} = \inf\{t > T_A^n;\ \Delta X_t \in A\}$. Since X is càdlàg, $0 \notin \bar{A}$ and $\mathbb{F}$ is right-continuous, (T_A^n) is strictly increasing sequence of stopping times, $T_A^1 > 0$ and $\lim_{n\to\infty} T_A^n = +\infty$ *a.s.*. Consider the process

$$N_t^A := \mu^X([0,t] \times A) = \sum_{0<s\leq t} \mathbf{1}_A(\Delta X_s) = \sum_{n=1}^{\infty} \mathbf{1}_{\{T_A^n \leq t\}}$$

Observe that N_t^A is a counting process without an explosion. Also

$$N_t^A - N_s^A \in \sigma\{X_u - X_v;\ s \leq v < u \leq t\},$$

and therefore $N_t^A - N_s^A$ is independent of $\mathcal{F}_s$, that is N^A has independent increments.

$N_t^A - N_s^A$ is the number of jumps of X on $]s;t]$ with the amplitude in A.

By the stationarity of X, $N_t^A - N_s^A \stackrel{\mathcal{L}}{=} N_{t-s}^A$.

For A such that $\bar{A} \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $(N_t^A) = \mu^X([0,t] \times A)$ is a non-exploding counting process with independent and stationary increments. Hence it is a Poisson process.

In particular

$$N_t^A := \mu^X(]0,t] \times A) = \sum_{0<s\leq t} \mathbf{1}_A(\Delta X_s)$$

and

$$N_{t_2}^A - N_{t_1}^A = \mu^X(]t_1,t_2] \times A) = \sum_{t_1<s\leq t_2} \mathbf{1}_A(\Delta X_s)$$

are Poisson r.v.

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$$\mu^X(]t_1, t_2] \times A) \quad \text{and} \quad \mu^X(]t_3, t_4] \times A)$$

are independent if $]t_1, t_2] \cap]t_3, t_4] = \emptyset$.
Also , if $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $A \cap B = \emptyset$, $\mu^X(]t_1, t_2] \times A) \perp\!\!\!\perp \mu^X(]t_1, t_2] \times B)$ because they "does'nt jumps at the same time". The case without condition $\bar{A} \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ is "the limit" of the cases "with".
The intensity measure of μ^X is of the form $dt\nu(dx)$. Indeed, we have shown that for A such that $\bar{A} \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $N_t^A = \mu^X(]0, t] \times A) = \sum_{0<s\leq t} \mathbf{1}_A(\Delta X_s)$ is a Poisson process. Its parameter is $\mathbb{E}N_1^A$, hence the intensity is $\mathbb{E}\mu^X(]s, t] \times A) = \mathbb{E}(N_t^A - N_s^A) = (t-s)\mathbb{E}N_1^A$, and

$$\nu(A) = \mathbb{E}N_1^A,$$

is the expected number, per unit time, of jumps whose size belongs to A.

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Lévy measure

It's a measure describing the distribution of jump's amplitudes.

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Definition

Let X be a Lévy process on $\mathbb{R}^d$ with jump measure μ^X. The expected number, per unit time, of jumps whose size belongs to A is called the Lévy measure of X.

$$\nu(A) = \mathbb{E}\mu^X([0,1] \times A).$$

ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$

Moreover, it will be shown that ν satisfies:

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \qquad \int_{|x|\geq 1} \nu(dx) < \infty$$

When $\nu(\mathbb{R}^d) < +\infty$, X is of "finite activity",
if $\nu(\mathbb{R}^d) = +\infty$, X is of "infinite activity".

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- The Lévy measure ν^N of a Poisson process N with parameter λ is $\nu^N = \lambda\delta_{\{1\}}$.
- The Lévy measure ν^X of a CPP directed by a $PP(\lambda)$ with jump size distribution f is $\lambda f(dx)$.

In both cases the process is the sum of its jumps and can be written as the integral of "amplitude" with respect to the jump measure:

$$X_t = \sum_{0<s\leq t} \Delta X_s = \int_{[0,t]\times\mathbb{R}^d} x\mu^X(ds, dx). \tag{3}$$

Since a PP and a CPP has a.s. a finite number of jumps in any interval $[0, t]$, the last sum is a finite sum.

$X_t \stackrel{?}{=} \int_{[0,t]\times\mathbb{R}^d} x\mu^X(ds,dx) = \sum_{0<s\le t} \Delta X_s \quad (*)$

Is it true that for any Lévy process its jump component can be written as in (3) ? The answer is NO. The condition for the convergence of the integral in (3) is

$$\int_{[0,t]\times\mathbb{R}^d} |x|\mu^X(ds,dx) < \infty \tag{4}$$

Guaranteed by: $\int_{|x|\le 1} |x|\nu(dx) < \infty$. But it may happens that $\int_{|x|\le 1} |x|\nu(dx) = \infty$. Example

$$\nu(dx) = \frac{C}{|x|^{\alpha+1}} \mathbf{1}_{|x|\neq 0} dx, \quad 0 < \alpha < 2.$$

This is a Lévy measure of a α-stable process. But for $1 \le \alpha < 2$, $\int_{|x|\le 1} |x|\nu(dx) = +\infty$ and X_t can not be represented as the sum of its jumps.

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Theorem (Lévy-Itô decomposition)

Let $(X_t)_{t\geq 0}$ be a Lévy process and ν its Lévy measure. Then

- ν is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ and verifies:

$$\int_{|x|\leq 1} |x|^2 \nu(dx) < \infty, \qquad \int_{|x|\geq 1} \nu(dx) < \infty \tag{5}$$

- There exist a $b \in \mathbb{R}^d$ and a d-dimensional Brownian motion (B_t) with covariance matrix A such that

$$X_t = bt + B_t + \int_{0<|x|<1;0\leq s\leq t} x\tilde{\mu}(ds, dx) + \int_{|x|\geq 1;0\leq s\leq t} x\mu(ds, dx) \tag{6}$$

$\int_{|x|<1;0\leq s\leq t} x\tilde{\mu}(ds, dx) := \lim_{\varepsilon\downarrow 0} \int_{\varepsilon<|x|<1;0\leq s\leq t} x\tilde{\mu}(ds, dx)$ a.s. and all the terms in (6) are independent .

sketch of the proof

Lemma

Let (X_t, Y_t) be a Lévy process. If (Y_t) is compound Poisson and (X_t) and (Y_t) never jump together, then they are independent.

This lemma allows to prove that $\int(|x|^2 \wedge 1)\nu(dx) < \infty$. Denote

$$X_t^\varepsilon = \int_{\varepsilon \le |x| < 1; 0 \le s \le t} x\mu(ds, dx) = \sum_{0 \le s \le t} \Delta X_s \mathbf{1}_{\{\varepsilon \le \Delta X_s < 1\}}.$$

(X_t^ε) is a Compound Poisson Process, because a Lévy process with piecewise constant paths. Remember that if $Z_t = \sum_{n=1}^{N_t} Y_i$ is a CPP with (N)-PP(λ) and $Y_i \sim F$, then

$$\mathbb{E}e^{iuZ_t} = \exp\left(\lambda t \int_{\mathbb{R}^d} (e^{iuy} - 1)F(dy)\right).$$

The characteristic function of (X_t^ε) is given by:

$$\mathbb{E}\exp(iuX_t^\varepsilon) = \exp(t\int_{\varepsilon\le|x|<1}(e^{iux}-1)\nu(dx)).$$

Here $F(dx) = \mathbf{1}_{[\varepsilon,1[}(dx)\nu(dx)/\nu([\varepsilon,1[)$.
Let $R_t^\varepsilon = X_t - X_t^\varepsilon$. Then $(X_t^\varepsilon, R_t^\varepsilon)$ is a Lévy process.
Fix u, t such that $|\mathbb{E}\exp(iuX_t)| > 0$. Since by Lemma X_t^ε and R_t^ε are independent,

$$\mathbb{E}\exp(iuX_t) = \mathbb{E}\exp(iuX_t^\varepsilon)\mathbb{E}\exp(iuR_t^\varepsilon)$$

Hence there exists $C > 0$ such that

$$|\mathbb{E}\exp(iuX_t^\varepsilon)| = |\exp(t\int_{1>|x|\geq\varepsilon}(e^{iux}-1)\nu(dx))| \geq C > 0.$$

which implies that $\int_{1>|x|\geq\varepsilon}(1-cos(ux))\nu(dx) \leq C' < \infty$. Making $\varepsilon \to 0$ we obtain the result:

$$\int_{|x|\leq 1}|x|^2\nu(dx) < \infty.$$

Denote

$$\tilde{X}_t^{\varepsilon} := \int_{\varepsilon<|x|<1;0\leq s\leq t} x\tilde{\mu}(ds,dx).$$

We need to show the convergence of $\tilde{X}_t^{\varepsilon}$. Set $Y_n = \tilde{X}_t^{\varepsilon_n} - \tilde{X}_t^{\varepsilon_{n+1}}$ and let $\varepsilon_n \downarrow 0$.

$$Y_n = \int_{0\leq s\leq t} x\mathbf{1}_{\{\varepsilon_{n+1}\leq|x|\leq\varepsilon_n\}}\tilde{\mu}(ds,dx).$$

the r.v. Y_n have zero mean. They are independent because they do not jump together.

$$Var(Y_n) = \mathbb{E}(\int_0^t\int_{\varepsilon_{n+1};\varepsilon_n} x\tilde{\mu}(ds,dx))^2 = \int_0^t\int_{[\varepsilon_{n+1};\varepsilon_n]} x^2 ds\nu(dx)$$

Since $\int_{|x|\leq 1}|x|^2\nu(dx) < \infty$, we have $\sum Var(Y_n) < \infty$. By the Kolmogorov three series theorem $\sum_n Y_n = \sum_n \int_{0\leq s\leq t} x\mathbf{1}_{\{\varepsilon_{n+1}\leq|x|\leq\varepsilon_n\}}\tilde{\mu}(ds,dx) < \infty \quad a.s.$

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Definition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$. *A predictable sigma field* $\mathcal{P}$ *is a sigma-field on* $[0, \infty[\times\Omega$, *generated by all adapted left continuous process* $(X_t(\omega))$.

The integrand should be predictable by practical and theoretical reasons. If ϕ describe the trading strategy, the decisions of trader should be based on the past evolution. Considering the predictable integrands permit to define the stochastic integral using the isometry, offers the martingale-preserving property for the stochastic integral, and permits to avoid the strategies leading to the "arbitrage opportunity. "

Simple predictable process

Definition

A stochastic process $(\phi_t)_{t\in[0,T]}$ is called a simple predictable if is

$$\phi_t = \phi_0 \mathbf{1}_{t=0} + \sum_{i=0}^{n} \phi_i \mathbf{1}_{]T_i;T_{i+1}]}(t),$$

where $T_0 = 0 < T_1 < T_2 < \ldots < T_n < T_{n+1} = T$ are stopping times and ϕ_i are bounded r.v. s.t. for each i, ϕ_i is $\mathcal{F}_{T_i}$ measurable. Denote $\mathbb{S}([0,T])$ the set of simple predictable process.

The simple predictable processes permits to approach any predictable process in uniform w.r.t. (ω, t) topologie.

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Let $(S_t)_{t \geq 0}$ be an adapted cadlag process. Define a stochastic integral of ϕ w.r. to S as

$$\int_0^T \phi_u dS_u := \phi_0 S_0 + \sum_{i=0}^{n} \phi_i (S_{T_{i+1}} - S_{T_i}).$$

For all predictable process ϕ there exists a sequence (ϕ^n) of simple predictable processes such that

$$\sup_{(t,\omega) \in [0,T] \times \Omega} |\phi_t^n(\omega) - \phi_t(\omega)| \underset{n \to \infty}{\longrightarrow} 0. \quad (*)$$

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Following the books of Ph. Protter, R.Cont and P. Tankov,

Definition

We call semi-martingale any adapted cadlag process S such that the linear map defined by the stochastic integral verifies the following continuity property:

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |\phi_t^n(\omega) - \phi_t(\omega)| \underset{n\to\infty}{\longrightarrow} 0.$$

implies

$$\int_0^T \phi^n dS \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^T \phi dS.$$

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The continuity property is also satisfied for the process defined by the stochastic integral:

Proposition

If S is a semimartingale and ϕ^n-simple predictable, ϕ-predictable, then

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |\phi_t^n(\omega) - \phi_t(\omega)| \underset{n\to\infty}{\longrightarrow} 0.$$

implies

$$\sup_{t\in[0,T]} \left| \int_0^T \phi^n dS - \int_0^T \phi dS \right| \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} 0.$$

It can be shown that every finite variation process is a semimartingale, every square integrable martingale and every local martingale is a semimartingale. If S is a Wiener process, or more generally a Lévy process, then it is a semimartingale. The space of all semimartingales with value in $\mathbb{R}^d$ is a vector space. Any semimartingale S_t can be represented as

$$S_t = M_t + A_t$$

wher M_t is a local martingale and A_t is a finite-variation process.

Typical semimartingale

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The most useful forme of semimartingale is:

$$X_t^i = X_0^i + M_t^i + A_t^i + \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} f^i(s,x)\pi(ds,dx) + \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} g^i(s,x)\tilde{\pi}(ds,dx$$

$$i = 1, \ldots, d$$

where

- (M_t^i) are square integrable continuous local martingales,
- A_t^i are continuous adapted processes of locally bounded variation, $A_0^i = 0$;
- π and $\tilde{\pi}$ are respectively PRM and its compensated version w.r.t. $(\mathcal{F}_t)$ and f^i, g^i predictable, $\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |f^i(s,x)|\pi(ds,dx) < \infty$, $\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} |g^i(s,x)|^2 ds\vartheta(dx) < \infty$. $f^i(t,x)g^j(t,x) = 0,\ i,j = 1\dots,d$.

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Proposition

If (S_t) is a martingale, then for any simple predictable process ϕ, the stochastic integral $I_t = \int_0^t \phi_u dS_u$ is also a martingale.

Proof Consider the decomposition:

$$\phi_i (S_{T_{i+1}\wedge t} - S_{T_i\wedge t}) = \phi_i (S_{T_{i+1}\wedge t} - S_{T_i\wedge t}) \mathbf{1}_{t > T_{i+1}} + \phi_i (S_{T_{i+1}\wedge t} - S_{T_i\wedge t}) \mathbf{1}_{T_i < t \le T_{i+1}} + \phi_i (S_{T_{i+1}\wedge t} - S_{T_i\wedge t}) \mathbf{1}_{t \le T_i}.$$

Recall that $\mathbb{F}$ is right continuous, hence $T < t \in \mathcal{F}_t$ iff T is a stopping time.

ϕ_i is $\mathcal{F}_{T_i}$ measurable, and hence $\mathcal{F}_{T_{i+1}}$ measurable. $\phi_i \mathbf{1}_{T_{i+1} < t}$ is $\mathcal{F}_t$ measurable, hence I_t is adapted.

Now let's check the martingale property: Since $T_{i+1} \wedge t$, $T_i \wedge t$ are a stopping times, its is enough to show for two stopping times $T_i < T_{i+1}$ that $\mathbb{E}[\phi_i(S_{T_{i+1}} - S_{T_i})|\mathcal{F}_t] = \phi_i(S_{T_{i+1}\wedge t} - S_{T_i \wedge t})$. Decompose:

$$\phi_i(S_{T_{i+1}} - S_{T_i}) = \phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{t > T_{i+1}} + \phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{T_i < t \le T_{i+1}} + \phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{t \le T_i}.$$

$$\mathbb{E}[\phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{t > T_{i+1}}|\mathcal{F}_t] = \phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{t > T_{i+1}};$$

Using optional stopping in the last line bellow,

$$\begin{aligned}\mathbb{E}[\phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{T_i < t \le T_{i+1}}|\mathcal{F}_t] =& \phi_i \mathbf{1}_{T_i < t \le T_{i+1}} \mathbb{E}[(S_{T_{i+1}} - S_{T_i})|\mathcal{F}_t] \\ =& \phi_i \mathbf{1}_{T_i < t \le T_{i+1}} (S_t - S_{T_i})\end{aligned}$$

for the last term:

$$\mathbb{E}[\phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{t\le T_i}|\mathcal{F}_t] = \mathbf{1}_{t\le T_i}\mathbb{E}[\mathbb{E}[[\phi_i(S_{T_{i+1}} - S_{T_i})|\mathcal{F}_t]|\mathcal{F}_{T_i}]$$
$$= \mathbf{1}_{t\le T_i}\mathbb{E}[[\phi_i\mathbb{E}[(S_{T_{i+1}} - S_{T_i})|\mathcal{F}_{T_i}]|\mathcal{F}_t] = 0$$

Hence

$$\mathbb{E}[\phi_i(S_{T_{i+1}} - S_{T_i})|\mathcal{F}_t] =$$
$$= \phi_i(S_{T_{i+1}} - S_{T_i})\mathbf{1}_{t>T_{i+1}} + \phi_i\mathbf{1}_{T_i<t\le T_{i+1}}(S_t - S_{T_i}) =$$
$$= \phi_i(S_{T_{i+1}\wedge t} - S_{T_i\wedge t}).$$

Stochastic integral for cadlag process

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It turns out that any cadlag process $\phi \in \mathbb{L}[0, T]$ can also be approximated by simple predictable processes:

Proposition

Let S be a semimartingale, ϕ a cadlag process and $\pi^n = (T_0^n = 0 < T_1^n < \ldots < T_{n+1}^n = T)$ a sequence of random partitions of $[0, T]$ such that $|\pi^n| = \sup_k |T_k^n - T_{k-1}^n| \to 0$, a.s. when $n \to \infty$. Then

$$\phi_0 S_0 + \sum_{k=0}^{n} \phi_{T_k}(S_{T_{k+1}\wedge t} - S_{T_k \wedge t}) \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} \int_0^t \phi_{u-} dS_u$$

uniformly on $[O, T]$.

Properties of stochastic itegral for cadlag process with respect to the semimartingale

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If X is a semimartingale and σ an adapted process then

- Semimartingale property: $S_t := \int_0^t \sigma_t dXt$ is also a semimartingale;
- Associativity: if ϕ is an adapted cadlag process, then $\int_0^t \phi_t dS_t = \int_0^t \phi_t \sigma_t dS_t$.
- Martingale preservation property: if (X_t) is a square integrable martingale and ϕ bounded then $M_t = \int_0^t \phi_t dX_t$ is a square integrable martingale.

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Let μ be a PRM on $[0, T] \times \mathbb{R}^d$ w.r.t. $\mathbb{F} = (\mathcal{F}_t)$ with the intensity measure $dt \times \vartheta(dx)$.
By analogy with simple predictable process, consider a simple predictable function $\phi : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$:

$$\phi(\omega, t, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j}(\omega) \mathbf{1}_{]T_i, T_{i+1}]}(t) \mathbf{1}_{A_j}(y);$$

where $0 < T_1 < T_2 < \ldots < T_n < T$ are stopping times and $(\phi_{i,j})$ are bounded $\mathcal{F}_{T_i}$ measurable r.v. and (A_j) are disjoints subsets of $\mathbb{R}^d$ with $\vartheta(A_j) < \infty$.

$\phi(\omega, t, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j}(\omega) \mathbf{1}_{]T_i, T_{i+1}]}(t) \mathbf{1}_{A_j}(y)$

Define the stochastic integral on $[0, T]$ for ϕ by

$$\int_{[0,T]} \int_{\mathbb{R}^d} \phi(t, y) \pi(dt, dy) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j}(\omega)(\pi_{T_{i+1}}(A_j) - \pi_{T_i}(A_j))$$

Where $\pi_t(A) = \pi([0, t] \times A)$. Define

$$\int_0^t \int_{\mathbb{R}^d} \phi(s, y) \pi(ds, dy) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j}[\pi_{T_{i+1} \wedge t}(A_j) - \pi_{T_i \wedge t}(A_j)]$$

This stochastic integral is a cadlag, adapted process. Recall that $\tilde{\pi}_t = \pi - t \times \vartheta$. Define the compensated integral as

$$\int_0^t \int_{\mathbb{R}^d} \phi(s, y) \tilde{\pi}(ds, dy) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{i,j}[\tilde{\pi}_{T_{i+1} \wedge t}(A_j) - \tilde{\pi}_{T_i \wedge t}(A_j)]$$

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Proposition

For any simple predictable process $\phi : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, the stochastic integral w.r.to compensated PRM $\tilde{\pi}$

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s, y)\tilde{\pi}(ds, dy)$$

is a square integrable martingale and verifies the isometry formula:

$$\mathbb{E}[|X_t|^2] = \mathbb{E}[\int_0^t \int_{\mathbb{R}^d} |\phi(s, y)|^2 ds\vartheta(dy)]$$

Proof

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Let $Y^j_t := \tilde{\pi}_t(A_j) = \tilde{\pi}([0,t] \times A_j)$. Since A_j are disjoint, Y^j are independent. We already know that (Y^j_t) is a martingale. Denote $\phi^j = \sum_{i=1}^n \phi_{i,j} \mathbf{1}_{]T_i, T_{i+1}]}$.

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y)\tilde{\pi}(ds,dy) =$$

$$\sum_{i=1}^n \sum_{j=1}^m \phi_{i,j}[\tilde{\pi}_{T_{i+1}\wedge t}(A_j) - \tilde{\pi}_{T_i \wedge t}(A_j)] = \sum_{i=1}^n \sum_{j=1}^m \phi_{i,j}(Y^j_{T_{i+1}\wedge t} - Y^j_{T_i \wedge t}$$

$$= \sum_{j=1}^m \int_0^t \phi^j dY^j.$$

Since ϕ^j is simple predictable, we already know that $\int_0^t \phi^j dY^j$ is a martingale, hence (X_t) is a martingale. By conditioning each term on $\mathcal{F}_{T_i}$ we see that $\mathbb{E}X_t = 0$.

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Since (Y_j) are independent, using in the last ligne the optional sampling theorem and the property: $\mathbb{E}\left[(\tilde{\pi}(]t_i, t_{i+1}] \times A))^2 \,|\mathcal{F}_{t_i}\right] = \mathbb{E}\left[(\tilde{\pi}(]t_i, t_{i+1}] \times A))^2\right] = (t_{i+1} - t_i)\vartheta(A)$,

$$\begin{aligned} \mathbb{E}|X_T|^2 &= Var(\int_0^T \int_{\mathbb{R}^d} \phi(s,y)\tilde{\pi}(ds,dy)) = \\ &\sum_{i=1}^n \sum_{j=1}^m \mathbb{E}\left[|\phi_{i,j}|^2 (Y^j_{T_{i+1}} - Y^j_{T_i})^2\right] \\ &= \sum_{i,j} \mathbb{E}\left[\mathbb{E}\left[|\phi_{i,j}|^2 (Y^j_{T_{i+1}} - Y^j_{T_i})^2 |\mathcal{F}_{T_i}\right]\right] \\ &= \sum_{i,j} \mathbb{E}\left[|\phi_{i,j}|^2 (T_{i+1} - T_i)\vartheta(A_j)\right] \end{aligned}$$

The isometry formula permit to extend the stochastic integral w.r.to the compensated PRM to square integrable predictable function. If $\phi \in \mathcal{P}([0, T]$ such that

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\phi(t, y)|^2 dt \vartheta(dy) < \infty,$$

there exists a sequence (ϕ^n) of elements of $\mathbb{S}([0, T])$ such that

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^d} |\phi^n(t, y) - \phi(t, y)|^2 dt \vartheta(dy) \right] \underset{n \to \infty}{\longrightarrow} 0.$$

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Proposition

For any predictable random function $\phi : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$, *verifyng*

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\phi(t, y)|^2 dt \vartheta(dy) < \infty,$$

the following properties hold: $t \to \int_o^t \int_{\mathbb{R}^d} \phi(s, y) \tilde{\pi}(ds, dy)$ *is a square integrable martingale;*

$$\mathbb{E}\left[\left|\int_o^t \int_{\mathbb{R}^d} \phi(s, y) \tilde{\pi}(ds, dy)\right|^2\right] = \mathbb{E}[\int_0^t \int_{\mathbb{R}^d} |\phi(s, y)|^2 ds \vartheta(dy)]$$

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Quadratic variation

Let (X_t) be a semimartingale, $X_0 = 0$. Let $\pi = \{t_0 = 0 < t_1 < \ldots < t_{n+1} = T\}$. The realized variance of X is defined as

$$V_X(\pi) = \sum_{t_i \in \pi} (X_{t_{i+1}} - X_{t_i})^2.$$

We have

$$(X_{t_{i+1}} - X_{t_i})^2 = X_{t_{i+1}}^2 - X_{t_i}^2 - 2X_{t_i}(X_{t_{i+1}} - X_{t_{i+1}});$$

and then

$$V_X(\pi) == X_T^2 - 2 \sum t_i \in \pi X_{t_i}(X_{t_{i+1}} - X_{t_{i+1}}).$$

If $\sup_i |t_{i+1} - t_i| \to 0$, then

$$V_X(\pi) \xrightarrow{\mathbb{P}} |X_T|^2 - 2 \int_0^T X_{u-} dX_u.$$

Quadratic variation

Definition

The quadratic variation process of a semimartingale X is the adapted cadlag process defined by

$$[X, X]_t = |X_t|^2 - 2\int_0^t X_{u-} dX_u.$$

The convergence

$$\sum_{t_i \in \pi} (X_{t_{i+1}} - X_{t_i})^2 \xrightarrow{\mathbb{P}} |X_T|^2 - 2\int_0^T X_{u-} dX_u$$

is uniform in t.

Properties of quadratic variation

- $([X, X])_t$ is an increasing process
- The jumps of $[X, X]$ are related to the jumps of X by

$$\Delta[X, X]_t = |\Delta X_t|^2.$$

  In particular, $[X, X]$ has continuous paths iff X does.
- If X is continuous and of finite variation then $[X, X] = 0$
- If X is a martingale and $[X, X] = 0$, then $X = X_0$ a.s.

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The quadratic variation of a standard Wiener process $[W, W]_t = t$.

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Proof. Note that using $W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, (t_{i+1} - t_i))$ and independent,
$V_W(\pi^n) - T = \sum_{\pi^n} \left[(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i) \right]$ is a sum of i.i.d. centered r.v.. Hence, with $Z \sim \mathcal{N}(O, 1)$,

$$\mathbb{E}|V_W(\pi^n) - T|^2 = \sum_{\pi^n} \mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i) \right]^2 =$$

$$\sum_{\pi^n} |t_{i+1} - t_i|^2 \mathbb{E}\left[\left(\frac{(W_{t_{i+1}} - W_{t_i})^2}{(t_{i+1} - t_i)} - 1 \right)^2 \right] =$$

$$\sum_{\pi^n} |t_{i+1} - t_i|^2 \mathbb{E}\left[\left(Z^2 - 1 \right)^2 \right] \leq CT \max_i |t_{i+1} - t_i| \to 0.$$

Which implies the convergence in probability $V_W(\pi^n) \to T$.

- **Quadratic variation of a Poisson process or any piecewise continuous process.** It is easy to see that $[N,N]_t = \sum_{0\le s\le T} \Delta N_s = N_t$. More generally, if $X_t = \sum_{i=1}^{N_t} Z_t$, then $[X,X]_t = \sum_{0\le s\le T} \Delta Z_s = Z_t$.
- **Quadratic variation of a Lévy process** If X is a Lévy process with characteristic triplet (σ^2, ν, γ), then

$$[X,X]_t = \sigma^2 t + \sum_{0\le s\le t} |\Delta X_s|^2 = \sigma^2 t + \int_{[0,t]\times\mathbb{R}} y^2 \mu^X(ds,dy),$$

where μ^X is a jump measure of X.
- **Quadratic variation of Brownian stochastic integral:** If $X_t = \int_0^t \sigma_s^2 dW_s$, then $[X,X]_t = \int_0^t \sigma_s^2 ds$.

Since the process $[X,X]$ is non-decreasing and cadlag , and since $\Delta[X,X]_t = (\Delta X_t)^2$, we can decompose $[X,X]$ into its continuous part and pure jump part. For a semimartingale X, denoting $[X,X]^c$ the path-by-path continuous part of $[X,X]$ we can write:

$$[X,X]_t = [X,X]^c_t + \sum_{0 \le s \le t} (\Delta X_s)^2.$$

It is also true that $[X^c, X^c] = [X,X]^c$.

Quadratic covariation

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Given two semimartingales X, Y and a partition $\pi = \{t_0 = 0 < t_1 < \ldots < t_{n+1} = T\}$, the realized covariance of X and Y is defined as

$$Cov_\pi(X, Y) := \sum_{t_i \in \pi} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

We can rewrite each term as

$$X_{t_{i+1}} Y_{t_{i+1}} - X_{t_i} Y_{t_i} - Y_{t_i}(X_{t_{i+1}} - X_{t_i}) - X_{t_i}(Y_{t_{i+1}} - Y_{t_i}),$$

then summing up these terms we see that if $|\pi| \to 0$,

$$Cov_\pi(X, Y) \xrightarrow{\mathbb{P}} X_T Y_T - X_0 Y_0 - \int_0^T X_{u-} dY_u - \int_0^T Y_{u-} dX_u -$$

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Definition

Given two semimartingale X, Y, the quadratic covariation process $[X, Y]$ is the semimartingale defined by

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^T X_{u-} dY_u - \int_0^T Y_{u-} dX_u -$$

The quadratic covariation satisfies the polarisation equality :

$$[X, Y] = \mathbf{1}/4([X + Y, X + Y] - [X - Y, X - Y]).$$

Product differentiation rule

Proposition

If X, Y are semimartingales, then

$$X_t Y_t = X_0 Y_0 + \int_0^T X_{u-} dY_u + \int_0^T Y_{u-} dX_u + [X, Y]_t.$$

- Let $X_t = \int_0^t \sigma_s^1 dW^1 s$ and $Y_t = \int_0^t \sigma^2 s dW_s^2$, where σ^1, σ^2 are two predictable processes, and W^1, W^2 are two correlated Wiener processes with $Cov(W_t^1, W_t^2) = \rho t$ then

$$[X, Y]_t = \int_0^t \sigma_s^1 \sigma_s^2 \rho ds.$$

- If X is a continuous process of finite variation and Y is a square integrable martingale, then $[X, Y] = 0$.
- If π is a PRM and W a Wiener process independent of π, and

$$X_t^i = X_0^i + \int_0^t \phi_s^i dW_s + \int_{[0,t]\times\mathbb{R}^d} \psi^i(s,y)\tilde{\pi}(ds,dy) \quad i = 1, 2,$$

then the quadratic covariation is

$$[X^1, X^2]_t = \int_0^t \phi_s^1 \phi_s^2 ds + \int_{[0,t]\times\mathbb{R}^d} \psi^1(s,y)\psi^2(s,y)\pi(ds,dy).$$

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Proposition

Let (X_t) be a real semimartingale . For any function $f \in C^{1,2}$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$,

$$f(t,X_t) - f(0,X_0) = \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_{s-})dX_s$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_{s-})d[X,X]^c_s$$
$$+ \sum_{0\leq s\leq t} [f(s,X_s) - f(s,X_{s-}) - \Delta X_s \frac{\partial f}{\partial x}(s,X_{s-})].$$

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When the jump part is of finite variation, $\sum_{0\le s\le t}|\Delta X_s| < \infty$, then the previous formula became

$$f(t,X_t) - f(0,X_0) = \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial x}(s,X_{s-})dX_s^c$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s,X_{s-})d[X,X]_s^c$$
$$+ \sum_{0\le s\le t} [f(s,X_s) - f(s,X_{s-}))].$$

When the semimartingale is given in the form

$$X_t^i = X_0^i + M_t^i + A_t^i + \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} f^i(s,x)\pi(ds,dx) + \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} g^i(s,x)\tilde{\pi}(ds,dx$$

$$i = 1,\ldots,d$$

then Ito formula for $F \in C^2(\mathbb{R}^d)$ is

$$F(X_t) - F(X_0) = \sum_{i=1}^{d} \int_0^t F'(X_s) sM_s^i + \sum_{i=1}^{d} \int_0^t F_i'(X_s) dA_s^i +$$

$$+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t F''_{i,j}(X_s)) d[M^i, M^j]_s +$$

$$\int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} \left[F(X_{s-} + f(s,x)) - F(X_{s-})\right] d\pi(ds,dx)$$

$$+ \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}} \left[F(X_{s-} + g(s,x)) - F(X_{s-})\right] d\tilde{\pi}(ds,dx)$$

$$+ \int_{[0,t]\times\mathbb{R}^d\setminus\{0\}}$$

$$\left[F(X_{s-} + g(s,x)) - F(X_{s-}) - \sum_{i=1}^{d} g^i(s,x) F_i'(X_s)\right] ds\vartheta(dx)$$