On the spectrum of the hierarchical Schrödinger-type operator: the case of sparse potentials LSA Winter Meeting - 2019

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This lecture is based on the project "On the spectrum of the hierarchical Schrödinger-type operator acting on a Cantor-like set" joint with S. A. Molchanov (UNC at Charlote) and A. A. Grigor'yan (Bielefeld University)

The concept of the hierarchical Laplacian is going back to N. Bogolubov and his school. This concept was used by F. J. Dyson in his construction of the phase transition in **1D** ferromagnetic model with long range interaction

- F. J. Dyson, Existence of a phase-transition in a one-dimensional Ising ferromagnet, Comm. Math. Phys., 12: 91-107, 1969.
- S. A. Molchanov, Hierarchical random matrices and operators, Application to Anderson model, Proc. of 6th Lucacs Symposium (1996), 179-194.

The concept of hierarchical lattice and hierarchical Laplacian was developed to the high level of generality in the paper

- A. Bendikov, A. Grigoryan, C. Pittet, W. Woess, Isotropic Markov semigroups on ultrametric spaces, Russian Math. Surv. 69:4, 589-680 (2014).

1 Dyson's dyadic model

Let us consider (as a simplest example) the Dyson's dyadic model which realizes the hierarchical Laplacian L as a self-adjoint operator acting in $L^2(0,\infty)$. Let $\{\Pi_r : r \in \mathbb{Z}\}$ be the family of partitians of the set $X = [0,\infty[$ each of which is made of dyadic intervals $I_{r,i} = [(i-1)2^r, i2^r[$. We call r the rank of the partitian Π_r (resp. the rank of the interval $I_{r,i}$). Any point x belongs to exactly one interval of the rank r, we denote it $I_r(x)$. For any $x \neq y$ we set

$$n(x,y) = \min\{r : y \in I_r(x)\}.$$

The hierarchical distance $\mathrm{d}(x,y)$ is defined as follows:

$$\mathrm{d}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{n(x,y)} & \text{if } x \neq y \end{cases}.$$

Observe that for all x, y, z in X,

$$\mathrm{d}(x,y) \leq \max\{\mathrm{d}(x,z), \mathrm{d}(z,y)\},$$

i.e. $\mathrm{d}(x,y)$ is an *ultrametric* on X.

- The couple (X, d) is a complete, locally compact, non-compact and separable metric space. In this metric space the set of all open balls coincides with the set of all dyadic intervals $I_{r,i}$, i.e. it is countable whereas the space $X = [0, \infty[$ by itself is uncountale.

- Thanks to the ultrametric property each open ball in (X, d) is a closed set, each point of a ball can be regarded as its center, any two balls either do not intersect or one is a subset of another etc.

- It is remarckable that the Borel σ-algebra in the metric space (X, d) coinsides with the classical Borel σ-algebra $\mathcal{B}(0, \infty)$ (generated by the metric $d(x,y) = |x - y|$).

For a set $A \in \mathcal{B}(0, \infty)$ we denote $|A|$ its Lebesgues measure. The hierarchical Laplacian L we introduce as a linear combination of "elementary Laplacians" with coefficients depending on the parameter $\varkappa \in]0, 1[$

$$(Lf)(x) = \sum_{r=-\infty}^{+\infty} (1 - \varkappa)\varkappa^r \left(f(x) - \frac{1}{|I_r(x)|} \int_{I_r(x)} f(y)dy \right).$$

This operator is well defined on the set $\mathcal{D}$ of all functions which are locally constant and have compact supports. Thanks to the ultrametric property the set $\mathcal{D}$ belongs to the Banach spaces $C_\infty([0, \infty[, \mathrm{d})$ and $L^p(0, \infty)$, $1 \leq p < \infty$, and is a dense subset there.

The operator $(L, \mathcal{D})$ admits representation as a hypersingular integral operator

$$(Lf)(x) = \int_0^\infty [f(x) - f(y)] J(x,y) dy$$

where

$$J(x,y)=\frac{2(1-\varkappa)}{\varkappa(2-\varkappa)}\frac{1}{\mathrm{d}(x,y)^{1+2/s}}\quad\text{and } s=\frac{2}{\log_2 1/\varkappa}.$$

On each interval $I_{r,i}$ we can define the Haar function

$$\mathcal{X}_{r,i}(x)=\begin{cases}2^{-r/2} & \text{if } \; x\in[(i-1)2^r,(i-1/2)2^r[\\ -2^{-r/2} & \text{if } \; x\in[(i-1/2)2^r,i2^r[\\ 0 & \text{if } \; x\notin I_{r,i}\end{cases}.$$

Simple calculations show that $\{\mathcal{X}_{r,i}\}$ is a complete orthonormal basis in $L^2(0,\infty)$ and that

$$L\mathcal{X}_{r,i}=\varkappa^r\mathcal{X}_{r,i}.$$

In particular, L is essentially self-adjoint non-negative definite operator, its spectrum

$$Spec(L)=\{\varkappa^r : r\in\mathbb{Z}\}\cup\{0\}$$

is pure point and each eigenvalue $\lambda_r=\varkappa^r$ has infinite multiplicity. In particular, the spectrum of L coincides with its essential part

$$Spec(L)=Spec_{ess}(L).$$

The operator L generates a symmetric Markov semigroup $(e^{-tL})_{t>0}$ that admits a continuous heat kernel $p(t,x,y)$ (the fundamental solution of the "parabolic equation" $\partial_t p=Lp$). The heat kernel $p(t,x,y)$ can be estimated as follows

$$p(t,x,y)\asymp\frac{t}{[t^{s/2}+\mathrm{d}(x,y)]^{1+2/s}},\quad s=\frac{2}{\log_2 1/\varkappa},$$

uniformly in t,x and y. In particular, uniformly in t and x,

$$p(t,x,x)\asymp t^{-s/2}.$$

Remark 1.1 *It is remarkable but easy to prove that the hierarchical Laplacian L introduced above is unitary equivalent to the Taibleson-Vladimirov multiplier $\mathfrak{D}^\alpha$, $\alpha=\log_2 1/\varkappa$, acting in $L^2(\mathbb{Q}_2)$ where $\mathbb{Q}_2$ is the field of 2-adic numbers. In particular, it follows that $p(t,x,x)$ does not depend on x.*

Notice that in contrary to the classical case the function $t\to p(t,x,x)$ does not vary regularly, namely

$$p(t,x,x)=t^{-s/2}\mathcal{A}(\log_2 t),$$

where $\mathcal{A}(\tau)$ is a continuous non-constant α-periodic function.

2 The Schröding-type operator

Let $V(x)$ be a measurable function. The operator $H = L + V$ we understand in the sense of quadratic forms, i.e. H is a densly defined self-adjoint operator such that $dom(H) \subset dom(Q)$ where

$$dom(Q) := \{u \in L^2(0,\infty) : \int (|L^{1/2}u(x)|^2 + |V(x)||u(x)|^2)dx < \infty\}$$

and

$$Q(u,u) := \int (|L^{1/2}u(x)|^2 + V(x)|u(x)|^2)dx.$$

Theorem 2.1 *Assume that V is locally bounded. Then the operator $(H, \mathcal{D})$ is essentially self-adjoint.*

Remark 2.2 *In the classical theory Theorem 2.1 does not hold in such a great generality. Indeed, in the case of Schrödinger operator*

$$H\psi = -\psi'' + V \cdot \psi, \;\; \psi \in C^{\infty}_{com}(0,\infty),$$

with $V(x) = -x^{\gamma}$, $\gamma > 2$, there is continuum *of self-adjoint extensions of H.*

Theorem 2.3 *Assume that $V(x) \to 0$ as $x \to \infty$, then*

$$Spec_{ess}(H) = Spec_{ess}(L).$$

In particular, $Spec(H)$ is pure point and the negative part of the spectrum consists of isolated eigenvalues of finite multiplicity.

Remark 2.4 *Notice that in our setting the set $X = [0,\infty[$ is equipped with the topology defined by the ultrametric $\mathrm{d}(x,y)$, which different from the eucledian metric $d(x,y) = |x-y|$(!) This difference gives a number of results which do not have their counterparts in the classical theory, e.g. in the case*

$$H\psi = -\psi'' + V \cdot \psi, \quad \psi \in C^{\infty}_{com}(0,\infty),$$

with $V(x) \to 0$ as $x \to +\infty$ the set $Spec(H)$ may contain non-trivial discrete, absolutely continuous and singular continuous parts. Let us mention here the typical result.

Theorem 2.5 *(S. Kotani) Assume that $V(x)$ has the folloving form*

$$V(x) = \sum_{n=1}^{\infty} n^{-\gamma} X_n \mathbf{1}_{[n-1,n]}(x),$$

where X_n are i.i.d. random variables uniformly distributed in $[-1,1]$. Then a.s. the spectrum of the operator $H\psi = -\psi'' + V \cdot \psi$ is:

- absolutelly continuous, for $\gamma > 1/2$,
- pure point singular, for $\gamma < 1/2$,
- neither absolutely continuous nor pure point singular, for $\gamma = 1/2$.

3 Splitting Lemma

Let $B_i = B(a_i)$ are open balls which all belong to the same horocycle $\mathfrak{T}$ (have the same diameter). Spectral theory of the operator $H = L + V$ with potential V of the form

$$V = -\sum_{i=1}^{\infty} \sigma_i \mathbf{1}_{B_i}$$

can be reduced to Spectral theory of the operator $[H] = [L] + [V]$ defined on the discrete ultrametric space $[X] = \{0, 1, 2, ...\}$ equipped with certain ultrametric [d] (induced by the ultrametric d). Here $[L]$ is a hierarchical Laplacian acting on the ultrametric space $([X], [\mathrm{d}])$ and $[V]$ is given by

$$[V] = -\sum_{i=1}^{\infty} \sigma_i \mathbf{1}_{a_i}.$$

Since V (resp. $[V]$) is a locally bounded function, the operator H (resp. $[H]$) defined on the set of test functions is essentially self-afjoint in $L^2(0,\infty)$ (resp. in l^2).

Let us explain this reduction $H \longmapsto [H]$ in details. Consider the set of dyadic partitians $\{\Pi_r\}$ of the set $[X]$:

$$\begin{aligned} \Pi_0 &= \{0, 1, 2, ..., n, ...\} \text{ - single points} \\ \Pi_1 &= \{(0, 1), (2, 3), (4, 5), ...\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Pi_m &= \{(0, ..., 2^m - 1), (2^m, ..., 2 \cdot 2^m - 1), ...\} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Put $I_{m,i} = \{(i-1)2^m, ..., i2^m - 1\}$ and denote $I_m(x)$ the unique $I_{m,i}$ which contains x. In the Hilbert space l^2 let us define the hierarchical Laplacian $[L]$ as follows

$$([L]f)(x) = \sum_{r=1}^{\infty} (1-\varkappa)\varkappa^r \left(f(x) - \frac{1}{2^r} \sum_{y \in I_r(x)} f(y) \right).$$

The operator $[L]$ is a *bounded symmetric operator* having eigenvalues $\lambda_r = \varkappa^r$, $r = 1, 2, \ldots.$ The corresponding eigenfunctions are discrete versions of the Haar functions as explained above.

The following below *Splitting Lemma* explaines the relation of the operator $H = L + V$ with V as above to the hierarchical Schrödinger-type operator $[H] = [L] + [V]$. We may assume, without loss generality, that each ball B_k is of the form $[k-1, k[$, i.e.

$$V = -\sum_{k=1}^{\infty} \sigma_k \mathbf{1}_{[k-1,k[}.$$

Lemma 3.1 *(Splitting Lemma) Let us define two subspaces of* $L^2(0,\infty)$: $L^2_- = span\{\mathcal{X}_{r,i}(x) : r \leq 0, i \geq 1\}$ *and* $L^2_+ = span\{\mathbf{1}_{I_{r,i}} : r \geq 0, i \geq 1\}$. *Then*

1. $L^2(0,\infty) = L^2_- \oplus L^2_+$.

1. The spaces L^2_- *and* L^2_+ *reduce the operator* $H = L + V$.

2. The Haar functions $\mathcal{X}_{r,i}(x) \in L^2_-$ *are the only eigenfunctions of* H *restricted to* L^2_- *with eigenvalues* $\lambda_{r,i} = \varkappa^r + \sigma_i$.

3. The operator H *restricted to* L^2_+ *can be identified with the operator* $[H] = [L] + [V]$ *with potential* $[V] = -\sum_{i=1}^{\infty} \sigma_i \delta_i$ *and with hierarchical Laplacian* $[L]$.

Proof. The proof of this lemma can be derived by direct inspection. ■

Corollary 3.2 *Assume that* V *as above tend to* ∞ *at* ∞. *Then the spectrum of* $H = L + V$ *is pure point.*

Proof. Let H_- (resp. H_+) be the restriction of the operator H to the space L^2_- (resp. L^2_+), then $Spec(H) = Spec(H_-) \cup Spec(H_+)$. By Splitting lemma, $Spec(H_-)$ is pure point. Let us show that $Spec(H_+)$ is discrete. The operator H_+ is unitary equivalent to the operator $[H] = [L] + [V]$. Observe that the operator $[L]$ is bounded, $||[L]|| = \varkappa < 1$, and, since $\sigma_i \to \infty$, the operator $[V]$ has a compact resolvent $([V] - \lambda)^{-1}$. For λ big enough $||[L]([V] - \lambda)^{-1}|| < 1$ whence the operator $[H]$ has a compact resolvent

$$([H] - \lambda)^{-1} = ([V] - \lambda)^{-1}(1 + [L]([V] - \lambda)^{-1})^{-1}.$$

It follows that $Spec(H_+) = Spec([H])$ is discrete as desired. ■

4 Sparse potentials

We assume that the ultrametric mesure space (X, d, m) is *countably infinite and homogeneous.* Analysis of the finite rank potentials $V = -\sum_{i=1}^{n} \sigma_i \delta_{a_i}$

indicates that in the case of increasing distances between locations $\{a_i\}$ of the *bumps* $V_i = -\sigma_i \delta_{a_i}$ their contributions to the spectrum is close to the union of the contributions of the individual bumps V_i (each bump contributes one eigenvalue in each gap $(\lambda_{m+1}, \lambda_m)$ of the spectrum of the operator L).

The development of this idea leads to consideration of the class of *sparce potentials*

$$V = -\sum_{i=1}^{\infty} \sigma_i \delta_{a_i}$$

where distances between locations $\{a_i : i = 1, 2, ...\}$ form an increasing to ∞ sequence. In the classical spectral theory this idea goes back to D. B. Pearson, S. Molchanov, and A. Kiselev, J. Last, S. and B. Simon.

Throughtout this section we will assume that $\alpha < \sigma_i < \beta$ for all i and for some $\alpha, \beta > 0$. We use the following notation

- $\mathcal{R}(\lambda, x, y)$ is the resolvent kernel of the operator L, i.e. solution of the equation $Lu - \lambda u = \delta_y$. Notice that $\mathcal{R}(\lambda, x, x)$ does not depend on x.
- Σ_* is the set of limit points of the sequence $\{\sigma_i\}$.
- $1/\Sigma_* := \{1/\sigma_* : \sigma_* \in \Sigma_*\}$.
- $\mathcal{R}^{-1}(1/\Sigma_*) := \{\lambda : \mathcal{R}(\lambda, a, a) \in 1/\Sigma_*\}$.

Theorem 4.1 *Assume that the following condition holds*

$$\lim_{n\to\infty} \sup_{i\geq n} \sum_{j:\ j\neq i \ and \ j\geq n} \frac{1}{\mathrm{d}(a_i, a_j)} = 0, \tag{4.1}$$

then

$$Spec_{ess}(H) = Spec(L) \cup \mathcal{R}^{-1}(1/\Sigma_*). \tag{4.2}$$

5 Localization theorem

In this section we consider the Schrödinger-type operator

$$H^\omega = L + V^\omega,\ \omega \in (\Omega, \mathcal{F}, P).$$

Here L, the deterministic part of H^ω, is the hierarchical Laplacian and

$$V^\omega = -\sum_{a\in I} \sigma(a, \omega) 1_{B(a)}$$

is a random potential defined by a family of open balls $\{B(a) : a \in I\}$ and a family $\{\sigma(a,\omega) : a \in I\}$ of i.i.d. random variables. . We assume that all $B(a)$ belong to the same horocycle $\mathfrak{T}$. Notice that the set of all open balls is countably infinite whence the set I of locations is at most countable.

Thanks to the Splitting Lemma the study of the set $Spec_{ess}(H^\omega)$ reduces to the case where the ultrametric measure space (X, d, m) is *countably infinite and homogeneous* and

$$V^\omega = -\sum_{a\in I} \sigma(a,\omega)\delta_a.$$

When $I = X$ the operator

$$H^\omega = L - \sum_{a\in X} \sigma(a,\omega)\delta_a$$

has a pure point spectrum for $P-$a.s. ω provided the distribution function $\mathcal{F}_\sigma(\tau) = P(\sigma(a,\omega) \leq \tau)$ satisfies certain regularity conditions. This statement (*localization theorem*) appeared first in the paper of Molchanov ($\sigma(a,\omega)$ are Cauchy random variables) and later in a more general form in two papers of Kritchevski. The proof essentially uses *self-similarity of* H.

Sparsness destroyes the self-similarity property, the localization theorem 5.1 below complements Theorem 4.1. The proof of this theorem is based on the abstract form of Simon-Wolff theorem for pure point spectrum, technique of fractional moments, decoupling lemma of Molchanov and Borel-Cantelly type arguments.

Theorem 5.1 *Set* $I = \{a_i\}$, $\sigma_i(\omega) := \sigma(a_i,\omega)$, *and assume that the distribution function* $\mathcal{F}_\sigma(x) = P(\omega : \sigma_i(\omega) \leq x)$ *is absolutely continuous and has a bounded density* $f_\sigma(\tau)$ *supported by a finite interval* $[\alpha,\beta]$. *Then the operator* H^ω *has a pure point spectrum for* $P-$*a.s.* ω *provided the following condition holds: For some (whence for all)* $y \in X$ *the sequence* $\mathrm{d}(a_i, y)$ *is eventually increasing and*

$$\lim_{M\to\infty} \sup_{i\geq M} \sum_{j:\ j\geq M,\ j\neq i} \frac{1}{\mathrm{d}(a_i,a_j)^r} = 0 \tag{5.3}$$

for some small enough r *(say,* $0 < r < 1/3$*). Moreover, since the set of limit points of the sequence* $\{\sigma_i(\omega)\}$ *coinsides for* $P-$*a.s.* ω *with the whole interval* $[\alpha,\beta]$, *we obtain: for* $P-a.s.$ ω

$$Spec_{ess}(H^\omega) = Spec(L) \cup \mathcal{R}^{-1}([1/\beta, 1/\alpha]).$$