Regularization by noise for SDEs and related systems: revisiting the Zvonkin-Veretennikov strategy

Oleg Butkovsky

Based on joint works with Siva Athreya, Konstantinos Dareiotis, Máté Gerencsér, Khoa Le, Leonid Mytnik

Weierstrass Institute

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Problem

Consider an ODE with a "bad" drift

$$dX_t = b(X_t)dt, \quad t \geqslant 0,$$

or a PDE with a "bad" drift

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + b(u(t,x)), \quad t \geqslant 0,\ x \in \mathbb{R}.$$

- If the drift is non smooth (e.g., $b(s) = 2\sqrt{|s|}$), then these ODE & PDE might have multiple solutions.
- Will the ODE/PDE regularize if we add random noise? Will it then have a unique solution?
- What about flow?
- Can we construct this solution numerically? Will the Euler method work?

Outline

- Big picture: Regularization by noise for ODEs with different type of noise
- Classical method: PDE approach of Zvonkin and Veretennikov
- New method: Stochastic sewing
- Further directions

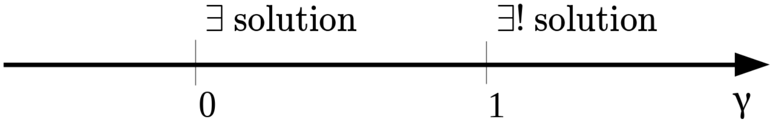
Big picture

ODEs

- We begin with an ODE

$$dX_t = b(X_t)dt, \quad t \geqslant 0.$$

- If b is Lipschitz, then this ODE has a unique solution.
- If $b \in \mathcal{C}^\gamma$, $\gamma < 1$, then this ODE has a solution but it might be non-unique ($\dot{x} = \sqrt{|x|}$, $x(0) = 0$).
- If b is just bounded, then this ODE might have no solutions ($\dot{x} = -\operatorname{sign} x$, $x(0) = 0$, where $\operatorname{sign}(0) := 1$).



SDEs

- Now let us add noise and consider an SDE

$$dX_t = b(X_t)dt + dW_t, \quad t \geqslant 0,\ x \in \mathbb{R}^d.$$

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Theorem (A.K. Zvonkin, 1974; A.Yu. Veretennikov, 1981)

If the drift b is measurable and bounded, then this SDE has a unique strong solution.

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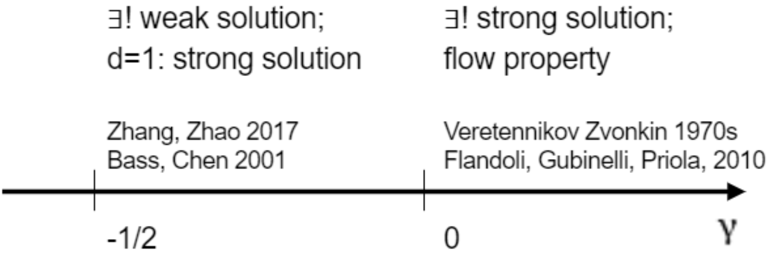
Theorem (A.K. Zvonkin, 1974; A.Yu. Veretennikov, 1981)

If the drift b is measurable and bounded, then this SDE has a unique strong solution.

- "Unique" means that if X^1 and X^2 are two adapted solutions to this SDE, then $X^1(\omega) = X^2(\omega)$ a.s.



SDEs driven by fBM

- For some applications it is more relevant to have an ODE driven by a fractional Brownian motion W^H, $H \in (0,1)$

$$dX_t = b(X_t)dt + dW^H_t, \quad t \geqslant 0,\ x \in \mathbb{R}^d.$$

- Recall that W^H is a Gaussian process with mean 0 and covariance $\mathsf{E} W^H_t W^H_s = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$.
- For $H = 1/2$ fBM is just BM; for $H \neq 1/2$ it is not a Markov process nor a martingale.

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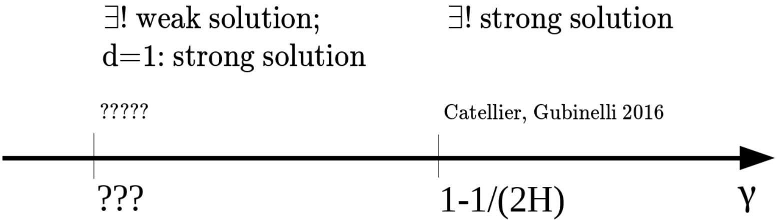

SDEs driven by α-stable noise

- Another research area is SDEs driven by α-stable process L^α, $\alpha \in (0,2)$
$$dX_t = b(X_t)dt + dL_t^\alpha, \quad t \geqslant 0,\ x \in \mathbb{R}^d.$$
- Recall that L^α is a Levy process (= stationary, independent increments) with jump measure $\nu(A) := \mathsf{E}\sum_{s\leqslant 1} \mathbb{1}(\Delta L_s^\alpha \in A) = c_\alpha \int_A |x|^{-1-\alpha}\,dx$.
- It is a pure jump Markov process and $\mathsf{E}(L_t^\alpha)^2 = \infty$ but $\mathsf{E}|L_t^\alpha| < \infty$ for $\alpha > 1$.

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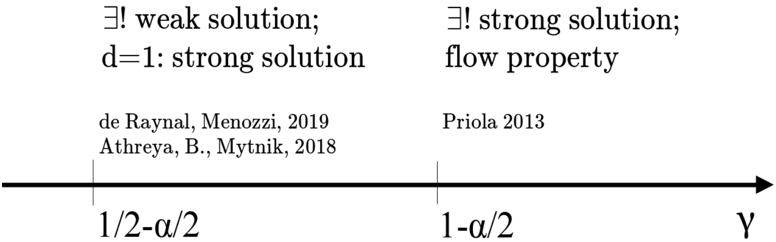

Numerical methods

- The existence and uniqueness of solutions is a pure theoretical result. But how can one construct the solutions on practice?
- Let n be a large integer; put $\kappa_n(t) := \lfloor nt \rfloor / n$. Consider the standard Euler scheme

$$dX_t^n = b(X_{\kappa_n(t)}^n)dt + dW_t^H, \quad t \geqslant 0,\ x \in \mathbb{R}^d.$$

Theorem (B., Dareiotis, Gerencsér, 2019)

Let $\alpha \in [0,1]$ satisfy $\alpha > 1 - 1/(2H)$. Suppose that $b \in \mathcal{C}^\alpha$. Then there exists a random variable $K(\omega)$ such that for all $n \in \mathbb{N}$ one has

$$\sup_{t\in[0,1]} |X_t - X_t^n| \leqslant K(\omega) \frac{1}{n^{1/2+\alpha(H\wedge(1/2))}}.$$

- This result is optimal for fBM. For $H = 1/2$ the rate is $1/2 + \alpha/2$ and do not go to zero as $\alpha \to 0$.

Naive approach

Naive approach

- To fix the ideas consider 1D SDE with "bad" drift $b \in \mathcal{C}^\gamma$, $\gamma < 1$.
$$dX_t = b(X_t)dt + dW_t.$$
- Let us try to prove strong uniqueness of solutions to this equation. Let X and $\widetilde{X}$ be two solutions to this equation. Denote $Z := \widetilde{X} - X$. We have
$$\begin{aligned}|Z_t| &= \Big|\int_0^t [b(\widetilde{X}_s) - b(X_s)]ds\Big|\\ &= \Big|\int_0^t [b(X_s + Z_s) - b(X_s)]ds\Big|\\ &\leqslant \int_0^t |Z_s|^\gamma ds.\end{aligned}$$
- We would like to use the Gronwall inequality but cannot. Thus, this method does not work for $\gamma < 1$:-(

Proof strategy 1:
classical PDE approach
(aka the Zvonkin transformation)

Zvonkin's transformation method

- We had SDE with "bad" drift $b \in \mathcal{C}^\gamma$, $\gamma < 1$. Let $B' = b$.

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$$dX_t = b(X_t)dt + dW_t.$$

- Zvonkin–Veretennikov's idea: let f be arbitrary smooth function. Let $Y_t := f(X_t)$. Then by Ito's formula

$$dY_t = df(X_t) = [\frac{1}{2}f''(X_t) + f'(X_t)b(X_t)]dt + f'(X_t)dW_t.$$

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- Choose f such that $\frac{1}{2} f'' + f'b = 0$. Then the SDE for Y will have no drift.

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- Choose f such that $\frac{1}{2}f'' + f'b = 0$. Then the SDE for Y will have no drift.
- One can take

$$f(x) := \int_0^x e^{-2B(y)}\,dy.$$

- Then $f' > 0$ everywhere, f is invertible and $f \in \mathcal{C}^{\gamma+2}$.
- We have

$$dY_t = f' \circ f^{-1}(Y_t)dW_t.$$

Zvonkin's transformation method

$$dX_t = b(X_t)dt + dW_t. \tag{$*$}$$

$$dY_t = f' \circ f^{-1}(Y_t)dW_t. \tag{$**$}$$

$$Y_t = f(X_t); \quad f(x) := \int_0^x e^{-2B(y)}\, dy.$$

- The function $f' \circ f^{-1}$ is in $\mathcal{C}^{\gamma+1}$. This SDE has a unique strong solution whenever $\gamma + 1 > 1/2$ (in particular if b is bounded).
- Thus if X and $\widetilde{X}$ solve $(*)$, then $f(X_t)$ and $f(\widetilde{X}_t)$ solve $(**)$. By uniqueness, $f(X_t) = f(\widetilde{X}_t)$, and thus $X_t = \widetilde{X}_t$.

Zvonkin's method: summary

- We want to say something "nice" about SDE
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$
- Zvonkin's idea: apply to (t, X_t) a certain "nice" one-to-one mapping $(t, x) \mapsto f(t, x)$.
- We pick f such that the new process $Y_t := f(t, X_t)$ has "good" drift and diffusion
$$dY_t = \widetilde{b}(Y_t)dt + \widetilde{\sigma}(Y_t)dW_t.$$
- If this new SDE has uniqueness/flow property/etc then so does the old SDE.
- The method is very robust but relies on a good Ito formula + careful PDE estimates for f.
- Challenging, but possible if b is a distribution (see Athreya, B., Mytnik, 2018).
- Does not work for SPDEs or SDEs driven by fBM.

Proof strategy 2:
new stochastic sewing approach

Naive approach: revisited

- We had 1D SDE with "bad" drift $b \in \mathcal{C}^\gamma$, $\gamma < 1$.

$$dX_t = b(X_t)dt + dW_t.$$

and two solutions X and $\widetilde{X}$. $Z := \widetilde{X} - X$. Then we want to show something like this

$$\|Z_t\|_{L_p} = \Big\| \int_0^t [b(X_s + Z_s) - b(X_s)] ds \Big\|_{L_p}$$

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- At least we want to show that

$$\Big\| \int_0^t [b(W_s + z) - b(W_s)] ds \Big\|_{L_p} \leqslant C t^\delta |z|.$$

Sewing Lemma of Gubinelli

- Let $f \in \mathcal{C}^{\alpha}$, $g \in \mathcal{C}^{\beta}$. Then it is well–known that $\int fdg$ exists and can be defined as a Riemann integral if $\alpha + \beta > 1$.
- This can be proved using sewing lemma.
- Fix $T > 0$ and let $A_{st} \in \mathbb{R}$, $0 \leqslant s \leqslant t \leqslant T$ are given.
- For $0 \leqslant s \leqslant u \leqslant t \leqslant T$ define $\delta A_{sut} := A_{st} - A_{su} - A_{ut}$.

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Theorem (Gubinelli, 2004)

Suppose that there exists a constant $\Gamma > 0$ *such that for any* $0 \leqslant s \leqslant u \leqslant t \leqslant T$ *one has*

$$|\delta A_{sut}| \leqslant \Gamma |t - s|^{1+\varepsilon}.$$

Then there exists a unique process $\mathcal{A}\colon [0, T] \to \mathbb{R}$ *such that* $\mathcal{A}_0 = 0$ *and*

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Furthermore, $\mathcal{A}_t = \lim \sum A_{t_i, t_{i+1}}$.

Young integral

- Recall that if for $0 \leqslant s \leqslant u \leqslant t \leqslant T$

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- Let $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$. We want to define $\int f dg$.
- Set $A_{st} := f_s(g_t - g_s)$.
- Then $|\delta A_{sut}| = |(f_s - f_u)(g_t - g_u)| \leqslant \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta} (t-s)^{\alpha+\beta}$.
- Thus, by SL, if $\alpha + \beta > 1$, then $\int f dg$ is well–defined.
- Does not work for $\int W_s dW_s$.

Stochastic Sewing Lemma of Le

- Recall, SL needs $|\delta A_{sut}| \leqslant \Gamma |t-s|^{1+\varepsilon}$. This does not hold for stochastic integrals.
- Fix now $T > 0$, $p \geqslant 2$ and filtration $(\mathcal{F}_s)_{s \leqslant T}$. We will write $\mathsf{E}^s[\ldots] := \mathsf{E}[\ldots | \mathcal{F}_s]$.
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Theorem (Le, 2018)

Suppose that there exist constants $\Gamma_1, \Gamma_2 > 0$ *such that for any* $0 \leqslant s \leqslant u \leqslant t \leqslant T$ *one has*

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Then there exists a unique process $\mathcal{A} \colon [0,T] \to \mathbb{R}$ such that $\mathcal{A}_0 = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$ measurable and

$$\|\mathsf{E}^s[\mathcal{A}_t - \mathcal{A}_s - A_{st}]\|_{L_p} \leqslant C\Gamma_1 |t-s|^{1+\varepsilon},$$

$$\|\mathcal{A}_t - \mathcal{A}_s - A_{st}\|_{L_p} \leqslant C\Gamma_2 |t-s|^{1/2+\varepsilon}.$$

Furthermore, $\mathcal{A}_t = \lim \sum A_{t_i, t_{i+1}}$ in L_p.

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  then $\lim \sum A_{t_i,t_{i+1}}$ exists in L_p.
- Let $f \in \mathcal{C}^\alpha$. We want to define $\int f(W)dW$.
- Set $A_{st} := f(W_s)(W_t - W_s)$.
- Then $\delta A_{sut} = (f(W_s) - f(W_u))(W_t - W_u)$.
- We have $\mathsf{E}^s[\delta A_{sut}] = 0$; $\|\delta A_{sut}\|_{L_p} \leqslant C\|f\|_{\mathcal{C}^\gamma}|t-s|^{1/2+\gamma/2}$.
- Thus, by SSL, if $\gamma > 0$, then $\int f(W)dW$ is well–defined.

Back to uniqueness for SDEs

- We want to proof that for $b \in \mathcal{C}^\gamma$

$$\Big\| \int_0^t [b(W_s + z) - b(W_s)] ds \Big\|_{L_p} \leqslant C t^\delta |z|.$$

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- Set $A_{st} := \mathsf{E}^s[\int_s^t (b(W_r + z) - b(W_r)) dr]$. Then

$$\delta A_{sut} = \mathsf{E}^s\Big[\int_u^t (b(W_r + z) - b(W_r)) dr\Big] - \mathsf{E}^u\Big[\int_u^t (b(W_r + z) - b(W_r)) dr\Big].$$

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- We have $\mathsf{E}^s[\delta A_{sut}] = 0$; $\|A_{st}\|_{L_p} \leqslant C \|b\|_{\mathcal{C}^\gamma} |z| |t - s|^{1/2 + \gamma/2}$.
- Thus, by SSL, if $\gamma > 0$, then $\mathcal{A}_t := \int_0^t [b(W_s + z) - b(W_s)] ds$ and
$$\Big\| \int_0^t [b(W_s + z) - b(W_s)] ds \Big\|_{L_p} = \|\mathcal{A}_t - \mathcal{A}_s\|_{L_p} \leqslant C t^{1/2 + \gamma/2} |z|.$$

Back to uniqueness for SDEs

- In general, one has the following "Krylov-type" bound.

Theorem (Athreya, B., Le, Mytnik, 2019)

Let $b \in \mathcal{C}^\gamma$, $\gamma > 0$. Let ψ, φ be a.s. Lipschitz functions. Then

$$\Big\| \int_s^t [b(W_r + \psi_r) - b(W_r + \varphi_r)] ds \Big\|_{L_p} \leqslant C \|b\|_{\mathcal{C}^\gamma} \sup_{r \in [s,t]} \|\psi_r - \varphi_r\|_{L_p} |t - s|^{1/2+\gamma/2} + o(t - s).$$

Back to uniqueness for SDEs

$$\Big\| \int_s^t [b(W_r + \psi_r) - b(W_r + \varphi_r)] ds \Big\|_{L_p} \\ \leqslant C\|b\|_{\mathcal{C}^\gamma} \sup_{r\in[s,t]} \|\psi_r - \varphi_r\|_{L_p} |t-s|^{1/2+\gamma/2} + o(t-s).$$

- Now we can finish the proof of uniqueness. Let X, $\widetilde{X}$ be solutions to $dX_t = b(X_t)dt + dW_t$.
- Put $\psi_t := \int_0^t b(X_s)\, ds$, $\widetilde{\psi}_t := \int_0^t b(\widetilde{X}_s)\, ds$.
- Then $X = W + \psi$, $\widetilde{X} = W + \widetilde{\psi}$.

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- Then $X = W + \psi$, $\widetilde{X} = W + \widetilde{\psi}$.
- We get

$$\|X_t - \widetilde{X}_t\|_{L_p} = \|\psi_t - \widetilde{\psi}_t\|_{L_p} \leqslant C t^{1/2+\gamma/2} \sup_{r \leqslant t} \|\psi_r - \widetilde{\psi}_r\|_{L_p}.$$

- By taking t small enough we get $X = \widetilde{X}$.

Rate of convergence for Euler scheme

$$\Big\|\int_s^t [b(W_r+\psi_r)-b(W_r+\varphi_r)]ds\Big\|_{L_p}$$
$$\leqslant C\|b\|_{\mathcal{C}^\gamma}\sup_{r\in[s,t]}\|\psi_r-\varphi_r\|_{L_p}|t-s|^{1/2+\gamma/2}+o(t-s).$$

- Let X^n solve $dX^n_t = b(X^n_{\kappa_n(t)})dt + dW_t$.
- Put $\psi^n_t := \int_0^t b(X^n_s)\,ds$.
- Then $X = W + \psi$, $X^n = W + \psi^n + O(n^{-1/2-\gamma/2})$.

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$$\Big\| \int_s^t [b(W_r + \psi_r) - b(W_r + \varphi_r)] ds \Big\|_{L_p} \leqslant C \|b\|_{\mathcal{C}^\gamma} \sup_{r \in [s,t]} \|\psi_r - \varphi_r\|_{L_p} |t-s|^{1/2+\gamma/2} + o(t-s).$$

- Let X^n solve $dX_t^n = b(X^n_{\kappa_n(t)})dt + dW_t$.
- Put $\psi_t^n := \int_0^t b(X_s^n)\, ds$.
- Then $X = W + \psi$, $X^n = W + \psi^n + O(n^{-1/2-\gamma/2})$.
- We get

$$\|\psi_t - \psi_t^n\|_{L_p} \leqslant Ct^{1/2+\gamma/2} \sup_{r \leqslant t} \|\psi_r - \widetilde{\psi}_r\|_{L_p} + Cn^{-1/2-\gamma/2}.$$

- This implies (B., Dareiotis, Gerencser, 2019)

$$\|X_t - X_t^n\|_{L_p} \leqslant Cn^{-1/2-\gamma/2}.$$

Remarks

- We see that SSL approach is quite general. It does not rely on a process being a semimartingale, or on careful PDE estimates.
- However it is not very robust (yet!) For example, for $H > 1/2$ to get the rate in the Euler scheme, one has to develop a new version of SSL (B., Dareiotis, Gerencser, 2019).
- Similar approach works for SPDEs (Athreya, B., Le, Mytnik, 2019)
$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + b(u(t,x)) + \dot{W}(t,x), \quad t \geqslant 0,\ x \in \mathbb{R}.$$
- One requires a new version SSL with propagators which is inspired by SL with propagators of Gubinelli, Tindel, 2010.
- With this new SSL in hand one get uniqueness for $b \in \mathcal{C}^\gamma$, $\gamma > -1$. Work in progress: δ-function
- Work in progress with Sasha Shaposhnikov: Bass–Burdzy conjecture on local time of BM along the non-smooth curves.

SDEs with distributional drift

Distributional drift

- Let's go back to the SDE

$$dX_t = b(X_t)dt + dW_t$$

  and to the standard concept of uniqueness (strong uniqueness).
- Assume now that b is not a function, but just a distribution from the Hölder-Besov space $\mathcal{C}^\gamma$, $\gamma < 0$.
- Then this equation is not well–posed in the classical sense: what is $b(X_t)$?

Definition of a solution

$$X_t = x + \int_0^t b(X_s)ds + W_t, \quad t \in [0, T]. \qquad (*)$$

Definition (Bass, Chen, 2000; Athreya, B., Mytnik, 2018)

We say that $X = (X_t)_{t \in [0,T]}$ solves SDE $(*)$ if there exists a continuous $A = (A_t)_{t \in [0,T]}$ such that:

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1. $X_t = x + A_t + W_t$, $t \in [0, T]$;
2. For any approximating sequence $(b_n)_{n\in\mathbb{Z}_+}$ such that $b_n \in \mathcal{C}^\infty$ and $\|b_n - b\|_\gamma \to 0$ we have
$$A_t^n := \int_0^t b_n(X_s)\, ds \to A_t, \text{ as } n \to \infty$$
in probability uniformly over $[0, T]$.

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in probability uniformly over $[0, T]$.
3. For any $\kappa < (1 + \frac{\gamma}{2}) \wedge 1$ there exists $C = C(T, \kappa) > 0$ such that
$$\mathsf{E}|A_t - A_s|^2 \leqslant C|t - s|^{2\kappa}, \quad s, t \in [0, T].$$

Bass–Chen result

$$X_t = x + \int_0^t b(X_s)ds + W_t, \quad t \in [0, T]. \qquad (*)$$

Theorem (Bass, Chen, 2000)

In case $d = 1$ SDE $(*)$ has a unique strong solution whenever $b \in \mathcal{C}^\gamma$, $\gamma > -1/2$.

- Work in progress (B., Mytnik): the constraint $\gamma > -1/2$ is optimal.

Regularization for α-stable noise

- What if instead of W_t we have a pure jump process L_t? Will we have any regularization?

$$dX_t = b(X_t)dt + dL_t$$

- No improvement in regularity if L is the Poisson process: SDE will already have multiple solutions while still "waiting" for the first jump of L.
- L should have "sufficiently many" small jumps.
- The bigger the intensity of the small jumps the rougher drift b can be.
- We take L to be an α–stable process, $\alpha \in (0, 2)$.

Regularization for α-stable noise

$$dX_t = b(X_t)dt + dL_t$$

- Tanaka, Tsuchiya, Watanabe, 1975: there exists a unique strong solution in $d = 1$ if b is measurable bounded and $\alpha > 1$.
- Priola, 2016: PBP uniqueness for $\gamma > 1 - \alpha/2$, $d \geqslant 1$.
- What if $\gamma < 0$?

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- What if $\gamma < 0$?

Theorem (Athreya, B., Mytnik)

In the case $d = 1$ this SDE has a unique solution if $b \in \mathcal{C}^\gamma$, $\gamma > 1/2 - \alpha/2$.

- This extends the Bass–Chen result to the stable case.

Summary

- Using classical PDE approach one can get results on uniqueness of solutions of SDEs with bad drift.
- Even when the drift is a distribution (and thus SDE is not a semi-martingale).
- Stochastic sewing is an alternative approach which does not require a good Ito formula.
- Using SSL we got uniqueness of solutions to the stochastic heat equation with bad drift.
- We also obtained rate of convergence for the Euler scheme for SDEs driven by fBM.
- Work in progress. Rate of converegence for the Euler scheme for SDEs driven by α-stable noise. We are planning to improve Mikulevicius–Xu, 2016. for the stochasic heat equation with distributional drift.
- Big open question: Weak uniqueness for SDEs driven by fBM.

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