Systems of forward and backward nonlinear Kolmogorov equations

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Aim : The aim of this talk is to discuss the possibility to reduce the Cauchy problem for systems of forward nonlinear parabolic equations (we call them forward Kolmogorov equations)

$$\frac{\partial u_m}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{d} G_{ij}(y,u)\nabla^2_{ji}u_m + \sum_{l=1}^{d_1}\sum_{i=1}^{d} B^i_{ml}(y,u)\nabla_i u_l +$$

$$+\sum_{l=1}^{d} c_{ml}(y,u)u_l, \qquad (1)$$

$u_m(0,y) = u_{0m}(y), m = 1,\dots,d_1,$

$$\frac{\partial u_m}{\partial t} = \frac{1}{2}\sum_{i,j}\nabla^2_{ij}\left[\sum_{l=1}^{d}[G^u]^{ml}_{ij}(y,u)u_l\right] - \nabla\cdot[a_m(y,u)u_m] + \sum_{l=1}^{d_1} c^u_{ml}(y)u_l, \qquad (2)$$

with $G_{ij}(y) = \sum_{k=1}^{d} A_{ik}(y,u)A_{kj}(y,u)$, and $a\cdot b = \sum_k a_k b_k$ to correspondent stochastic problems.

Examples: Parabolic conservation and balance laws

$$\frac{\partial u}{\partial t} + div[f(u)] = div[G(u)\nabla u] + c(u)u,\ u(0) = u_0,$$

$u \in R^{d_1}, f \in R^d \otimes R^{d_1}, G \in R^d \otimes R^d \otimes R^{d_1} \otimes R^{d_1}$ or $G \in R^d \otimes R^d$,
1) Shigesada, Kawasaki and Teramoto model of spatial segregation of interacting species (1979). (SKT-model) [1]

$$\frac{\partial u^m}{\partial t} = \Delta(u_m[\alpha_m + \alpha_{m1}u_1 + \alpha_{m2}u_2]) + c^u_m u^m,$$

$$u_m(0,x) = u_{0m}(x),\ m = 1,2,$$

where $c^u_m = c_m - c_{m1}u_1 - c_{m2}u_2$ and $\alpha_m, \alpha_{mk}, c_m, c_{mk}, m, k = 1,2$ are positive constants.

2) Keller-Segel model in chemotaxis [2],

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + \nabla \cdot (u_1 \nabla u_2) + c_1^u u_1,$$

$$\frac{\partial u_2}{\partial t} = \Delta u_2 + c_2^u u_2,$$

$$u_m(0,x) = u_{0m}(x),\ m = 1,2,$$

3) MHD-Burges system [?]

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = \frac{\sigma^2}{2}\Delta v + (\nabla \times B) \times B, \quad v(0,y) = v_0(y), \quad (3)$$

$$\frac{\partial B}{\partial t} = \frac{\mu^2}{2}\Delta B + \nabla \times (v \times B), \quad B(0,y) = B_0(y). \quad (4)$$

$B \in R^3$ is a magnetic field, $v \in R^3$ is the fluid velocity. (3) is the Burgers equation with pressure defined by the magnetic field.

Connections between nonlinear PDEs and SDEs are studied since seminal papers by **H.McKean** [4], [5] and **M.Freidlin** [6],[7].
$d_1 = 1.$ 1) McKean (1966) A stochastic problem

$$d\xi(s) = \int_{R^d} a(\xi(s), y)\mu(s, dy)ds + \int_{R^d} A(\xi(s), y)\mu(s, dy)dw(s), \tag{5}$$

$$\mu(t) = Law(\xi(t)), \quad \xi(0) = \xi_0, \mu_0(dy) = P\{\xi(0) \in dy\}, \tag{6}$$

can be associated with the Cauchy problem

$$\frac{\partial \mu}{\partial t} = \frac{1}{2} Tr\nabla^2 \left[\int_{R^d} G(x, y)\mu(t, dy)\mu\right] - \nabla \cdot \left[\int_{R^d} a(x, y)\mu(t, dy)\mu\right], \tag{7}$$

$$\mu(0, dy) = \mu_0(dy),$$

where $\mu_0(y)dy = P\{\xi(0) \in dy\}$ $Tr\nabla^2 G = \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} G_{ji}$,
$A(x, \mu) = \int_{R^d} \tilde{A}(x, y)\mu(dy)$, $G(x, \mu) = A(x, \mu)A^*(x, \mu)$ hence
$A : R^d \times \mathcal{M}(R^d) \to R^d \otimes R^d$.

A different approach is due to Freidlin (1967). Namely, let coefficients $a(y,v), A(y,v)$ be defined on $R^d \times R$. Then setting $v(T-t,x) = u(t,x)$ one can reduce the forward Cauchy problem

$$\mathbf{u_t = \frac{1}{2} TrG(x, u(t,x))\nabla^2 u + a(x, u(t,x)) \cdot \nabla\rangle u, \quad u(0,x) = u_0(x),} \tag{8}$$

to the backward Cauchy problem

$$\frac{\partial v}{\partial t} + \frac{1}{2} TrG(x, v(T-t,x))\nabla^2 v + a(x, v(T-t,x)) \cdot \nabla v = 0, \tag{9}$$

$v(T,x) = u_0(x)$ and then to a stochastic problem

$$d\xi(\theta) = a(\xi(\theta), v(T-\theta, \xi(\theta)))d\theta + A(\xi(\theta), v(T-\theta, \xi(\theta)))dw(\theta), \xi(t) = x \tag{10}$$

$$v(T-t,x) = E[u_0(\xi_{t,x}(T))]. \tag{11}$$

Freidlin's approach was extended to a wide class of quasilinear backward Kolmogorov equations and systems in papers by Yu.Dalecky and Bel. (1980 -1990) [8] – [10]. In particular, it was proved that the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{1}{2} TrG(x, u(t,x))\nabla^2 u + a(x, u(t,x)) \cdot \nabla u + c(x,u)u = 0, \tag{12}$$

$u(T,x) = u_0(x)$ can be reduced to a stochastic problem

$$d\xi(\theta) = a(\xi(\theta), u(\theta, \xi(\theta)))d\theta + A(\xi(\theta), u(\theta, \xi(\theta)))dw(\theta), \xi(t) = x, \tag{13}$$

$$u(t,x) = E\left[u_0(\xi_{t,x}(T))exp\left\{\int_t^T c(\xi(\theta), u(\theta, \xi(\theta))d\theta\right\}\right] \tag{14}$$

at least if we are interested in classical solutions of (12) and assume all coefficients and u_0 to be smooth enough.

Moreover it was proved that this approach works for systems

$$\frac{\partial u_m}{\partial t} + \frac{1}{2}TrG(y,v)\nabla^2 u_m + \sum_{l=1}^{d_1}\sum_{i=1}^{d} B^i_{ml}(u(t,y))\nabla_i u_l +$$

$$+\sum_{l=1}^{d_1} c_{ml}(u(t,y))u_l = 0, \quad u_m(T,y) = u_{0m}(y). \tag{15}$$

$m = 1, 2, \ldots, d_1, \quad G : R^d \times R^{d_1} \to R^d \otimes R^d$. Namely, one can reduce (12) to an associated stochastic system

$$d\xi(s) = A(u(s,\xi(s)))dw(s), \quad \xi(t) = x, \tag{16}$$

$$d\eta(s) = c(u(s,\xi(s)))\eta(s)ds + C(u(s,\xi(s)))\eta(s)dw(s), \eta(t) = h, \tag{17}$$

$$h \cdot u(t,x) = E[\eta_{t,h}(T) \cdot u_0(\xi_{t,x}(T))], \tag{18}$$

setting $B = CA$.

Theorem 1. [9] *Assume that* $a(x,u), A(x,u)$ *are* C^3*-smooth bounded functions with sublinear growth in* x *and polilinear growth in* u*. Let* $c(x,u), C(x,u)$ *be* C^3 *smooth function bounded in* x *and polylinear in* u*. Then there exists an interval* $[T_1, T] \subset [0, T]$ *such that:*
1) $\exists!$ *a unique solution to the system (16)-(18) defined on* $[T_1, T]$*;*
2) the function v *defined by (15) is a unique classical solution of the Cauchy problem (15) defined on* $[T_1, T]$*.*

Note that we can apply the above approach to a quasilinear scalar equation like a scalar nonlinear heat equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial y}[u^2\frac{\partial u}{\partial y}] + u^\beta = 0, \quad u(0,y) = u_0(y).$$

To this end we rewrite it as follows

$$\frac{\partial u}{\partial t} + \frac{1}{2}u^2\frac{\partial^2 u}{\partial y^2} + u^\beta + g\frac{\partial u}{\partial y} = 0, \quad u(0,y) = u_0(y) \tag{19}$$

and introducing a new unknown function $g(t,y) = \frac{\partial}{\partial y}u(t,y)$ and derive an equation for g

$$\frac{\partial g}{\partial t} + \frac{1}{2}u^2\frac{\partial^2 g}{\partial y^2} + 3ug\frac{\partial g}{\partial y} + g^3 + \beta u^{\beta-1}\frac{\partial u}{\partial x} = 0, \quad g(0,y) = \frac{\partial u_0(y)}{\partial y}. \tag{20}$$

Hence we obtain a system of the form (15) and can apply the above approach.

A similar trick with setting $v(T-t,x) = u(t,x)$ allows to go from (1) to (15) but it does not work for systems of the form (2).

To construct a probabilistic counterpart to the Cauchy problem (2) we have to find generators of corresponding Markov processes. We deduce it assuming that u satisfies

$$\int_{R^d} h(t,y)u_m(t,y)dy - \int_{R^d} h(0,y)u_{0m}(y)dy =$$

$$= \int_0^t \int_{R^d} u_m(s,y)\left[\frac{\partial h}{\partial s} + \frac{1}{2}TrG_m^u(y)\nabla^2 h(y) + a_m^u \cdot \nabla h(y) + \right.$$

$$\left. + c_m(u,\nabla u)h(y)\right] dyds,, \qquad (21)$$

for any $h \in C_0^\infty([0,T] \times R^d; R)$ and consider Markov processes with generators

$$\mathcal{L}_m^u h(y) = \frac{1}{2}TrG_m^u(x)\nabla^2 h(y) + a_m^u(x) \cdot \nabla h(y). \qquad (22)$$

Given a generator $\mathcal{L}_m^u$ we consider SDEs

$$d\xi_m(s) = a_m^u(\xi_m(s))ds + A_m^u(\xi_m(s))dw_m(s), \quad \xi(0) = \xi_0, \qquad (23)$$

and observe that a remaining problem is to derive a closing relation.

To illustrate the approach we start with a simple version of a reaction - diffusion system

$$\frac{\partial u_1}{\partial t} = \frac{1}{2} M_1^2(y) \Delta u_1 + (c_{10} - c_{11} u_1 - c_{12} u_2) u_1, \qquad (24)$$

$$\frac{\partial u_2}{\partial t} = \frac{1}{2} M_2^2(y) \Delta u_2 + (c_{20} - c_{21} u_1 - c_{22} u_2) u_2, \qquad (25)$$

$$u_m(0) = u_{0m}.$$

We say (24), (25) has a (very) weak solution $u = (u_1, u_2)$ if $u_m(t) \in L^2(R^d)$ and for all $h \in C_0^\infty((0, T) \times R^d)$

$$\int_0^{\mathbf{T}} \int_{\mathbf{R^d}} \mathbf{u_m(t, y)} \left[\mathbf{h_t} + \frac{\mathbf{1}}{\mathbf{2}} \mathbf{M_m^2(t, y)^2 \Delta h(t, y)} + \mathbf{c_m^u(t, y) h(t, y)} \right] \mathbf{dydt} = \mathbf{0}, \qquad (26)$$

hold with $c_m^u = c_{10} - c_{m1} u_1 - c_{m2} u_m$

Thus, we may consider $\xi_m(t)$ and $\eta_m(t), m = 1, 2$, which satisfy

$$\xi_m(t) = \xi_{m0} + \int_0^t M_m(s, \xi_m(s))dw_m(s), \qquad (27)$$

$$\eta_m(t) = 1 + \int_0^t c_m^u(\xi_m(s))\eta_m(s)ds, \qquad (28)$$

where $w_m(t) \in R^d$ are independent Wiener processes and ξ_{0m} are independent r.v independent on $w_m(t)$ as well and having distributions μ_{0m} and use

$$\int_{\mathbf{R^d}} \mathbf{h(y)u_m(t,y)dy} = \mathbf{E}\left[\mathbf{h}(\xi_\mathbf{m}(\mathbf{t}))\exp\left\{\int_\mathbf{0}^\mathbf{t} \mathbf{c_m}(\mathbf{u}(\theta, \xi_\mathbf{m}(\theta)))\mathbf{d}\theta\right\}\right] \qquad (29)$$

for closing relations similar to [15] where it was applied in a scalar case.

To justify this possibility we verify that (29) is equivalent to

$$\int_{R^d} h(y)u_m(t,y)dy = \int_{R^d} h(y)\left[\int_{R^d} p_m(0,x,t,y)u_m(0,x)dx + \right.$$

$$\left. + \int_0^t \int_{R^d} p_m(\theta,z_m,t,y)c_m^u(\theta,z)u_m(\theta,z)dzd\theta\right] dy. \qquad (30)$$

where $p_m(0,x,t,y)$ is a density of a transition probability of the process $\xi_m(t)$ satisfying (27).

Assume that c_m and u_{0m} are bounded functions and consider $\mathbf{E}\left[\mathbf{h}(\xi_\mathbf{m}(\mathbf{t}))\exp\left\{\int_0^t \mathbf{c_m}(\mathbf{u}(\theta,\xi_\mathbf{m}(\theta)))\mathbf{d}\theta\right\}\right]$ as a linear bounded functional on the space $C_b(R^d)$. Applying Riesz theorem we get that there exists a measure $\nu_m(t)$ such that

$$\int_{\mathbf{R^d}} \mathbf{h}(\mathbf{y})\nu_\mathbf{m}(\mathbf{t},\mathbf{dy}) = \mathbf{E}\left[\mathbf{h}(\xi_\mathbf{m}(\mathbf{t}))\exp\left\{\int_0^t \mathbf{c_m}(\mathbf{u}(\mathbf{s},\xi_\mathbf{m}(\mathbf{s})))\mathbf{ds}\right\}\right]. \qquad (31)$$

To deduce (30) from (31) we note that the right hand side of (31) can be rewritten in the form

$$E[h(\xi_m(t))] + E\left[\int_0^t c_m(u(s,\xi_m(s)))\eta_m(s)E[h(\xi_m(t)|\xi_m(s)]ds\right] =$$

$$= \int_{R^d}\int_{R^d} h(y)P_m(0,x,t,dy)\mu_{0,m}(dx) +$$

$$+ \int_0^t \int_{R^d}\int_{R^d} h(y)P_m(s,z,t,dy)c_m(u(s,z))\nu_m(s,dz)$$

and thus (31) has the form

$$\int_{R^d} h(y)\nu_m(t,y)dy = \int_{R^d}\int_{R^d} h(y)p_m(0,x,t,y)dy\mu_{0,m}(x)dx +$$

$$+ \int_0^t \int_{R^d}\left(\int_{R^d} p_m(s,z,t,y)h(y)dy\right)c_m(u(s,z))\nu_m(s,z)dzds.$$

Here we have applied the Riesz theorem with test functions $\mathbf{h}$ and $\mathbf{z} \to \int_{\mathbf{R^d}} \mathbf{h(y)P_m(s, z, t, dy)c_m(u(s, z))}$.

By definition the system (24), (25) has a mild solution $u = (u_1, u_2)$ with $u_1(t), u_2(t) \in L^2(R^d)$, $t \in [0, T]$ if $\forall\ h \in C_0^\infty(R^d)$ (30) holds.

Theorem 2. *Assume that c_m are Borel bounded functions. Functions $u_m \in L^1([0, T], W^1(R^d)$ are unique mild solutions of (24), (25) if and only if (29) holds, that is*

$$\int_{\mathbf{R^d}} \mathbf{h(y)u_m(t, y)dy = E\left[h(\xi_m(t))\eta_m(t)\right]}$$

holds for any $h \in C_0^\infty(R^d)$.

We deduce from Th. 1 that (29) can be treated as an equation for the function $u_m(t, y)$ in a weak sense. This means that we can take

$$u_m(t, y) = \int_{R^d} p_m(0, x, t, y)u_{0,m}(x)dx +$$

$$+ \int_0^t \int_{R^d} \int_{R^d} p_m(s, z, t, y)c_m(u(s, z))u_m(s, z)dzds \qquad (32)$$

as a closing relation for (27), (28).

Theorem 3. *Let $A(x), c(u)$ satisfy conditions of Th. 1 and c is a bounded function. Assume $u_0 \in W^1(R^d)$. Then there exists a unique mild solution,*
$u_m \in \mathcal{Q} = L^1([0,T], W^1(R^d)) \cap L^\infty([0,T], W^1(R^d))$ *of the system*

$$\frac{\partial u_m}{\partial t} = \frac{1}{2} M_m^2(y) \Delta u_m + c(u), u_m(0,y) = u_{0m}(y), \quad m = 1,2. \ (33)$$

To prove this theorem we consider the equations (33), rewrite them as $u_m(t) = G_m(t,y,u)$ and prove that the map G is a contraction in $\mathcal{Q}$. To this end we first consider a couple of operators

$$U_m(s,t)\varphi(y) = \int_{R^d} p_m(s,x,t,y)\varphi(y)dy, \quad \varphi \in L^1(R^d) \cap L^\infty(R^d),$$

$$\Gamma_m v_m(t,y) = \int_\tau^t \int_{R^d} p_m(\theta,z,t,y) c(v_m + U_m(\theta,t)\varphi(y)) \times$$
$$\times (v_m + U_m(s,t)\varphi(z)) dx d\theta.$$

We can verify that Γ_m acts in the space $L^1([t,t+\tau], B(0,K)\cap B_\infty(0,K)$ and is a contraction in norm of $W^1(R^d)$ for small enough τ. Here $B(0,K)\subset W^1(R^d)$ and $B_\infty(0,K)\subset L^\infty([t,t+\tau]\times R^d, R^2)$ for a small τ are centered balls with radius K. Then we know that $\exists v_m^0\in\mathcal{Q}$ such that for $(t,y)\in[0,\tau]\times R^d$

$$v_m^0(t,y)=\int_0^t\int_{R^d}p_m(s,x,t,y)[v_m^0+U_m u_{0m}](s,z)c_m(u(s,z))dzds$$

Setting $u_m^0(t)=v_m^0+\int_{R^d}p_m(0,x,t,y)u_{0m}(x)dx$, we get

$$u_m^0(t,y)=\int_{R^d}p_m(0,x,t,y)u_{0m}(x)dx+$$

$$\int_0^t ds\int_{R^d}p_m(s,z,t,y)u_m^0(s,z)c_m(u^0(s,z))dz.$$

Consider a partition
$0 \le \tau \le 2\tau \le \cdots \le k\tau \le (k+1)\tau \le \cdots \le N\tau = T$. We have proved that $\exists!$ a solution of the system for $s, t \in [r\tau, [r+1]\tau]$

$$u_m^k(t,y) = \int_{R^d} p_m(k\tau, z, t, y) u^{k-1}(k\tau, z) dz +$$

$$+ \int_{k\tau}^t \int_{R^d} p_m(s, z, t, y) u^k(s, z) c_m(u^k(s, z)) dz ds,$$

$$v_m^k(t,y) = \int_{k\tau}^t \int_{R^d} p_m(s, z, t, y)[v^k(s, z) + U_m^k u_0(s, z)] c_m(u^k(s, z)) dz ds,$$

for $t \in [k\tau, (k+1)\tau]$. Setting $u^k = U^k u_0 + v^k$ we get

$$u_m^k(t,y) = \int_{R^d} p_m(s, z, t, y) u^{k-1}(k\tau, z) dz +$$

$$+ \int_{k\tau}^t \int_{R^d} p_m(s, z, t, y) u^k(s, z) c(u^k, \nabla u^k) dz.$$

Next we define $u_m : [0, T] \times R^d \to R^d$ setting $u_m = u_m^k$. on each

Consider one more example of a system of type (2), namely SKT system

$$\frac{\partial u_m}{\partial t} = \Delta(u_m[\alpha_m + \alpha_{m1}u_1 + \alpha_{m2}u_2]) + [c_{m0} + c_{m1}u_1 + c_{m2}u_2]u_m, \tag{34}$$

$$u_m(0,x) = u_{0m}(x),\ m = 1,2,$$

We associate to this system a stochastic system of the form

$$d\xi(_m t) = M^u_m(\xi_m(t))dw(t), \quad \xi_m(0) = \xi_{0m}, \tag{35}$$

$$d\eta(t) = c^u_m(\xi_m(t))\eta_m(t)dt, \eta_m(0) = 1, \quad 36$$

where $M^u_m = \sqrt{\alpha_m + \alpha_{m1}u^1 + \alpha_{m2}u^2}$, $c^u_m = c_{m0} + c_{m1}u^1 + c_{m2}u^2$. In order to obtain a closing relation we mollify this system and obtain a system of McKean-Vlasov type equations. Given a finite Borel measure $\mu(dx)$ (or its density $u(x)$) on R^d we denote by $\rho_\epsilon * \mu$ the convolution

$$[\rho_\epsilon * \mu](y) = \int_{R^d} \rho_\epsilon(y-x)\mu(dx) = \int_{R^d} \rho_\epsilon(y-x)u(x)dx$$

Let

$$M^u_{m,\epsilon}(t,y) = \int_{R^d} \rho_\epsilon(y-x_1)[1+\alpha_{m1}u^1(t,x_1)+\alpha_{m2}u^2(t,x_1)]dx_1,$$

$$c^u_{m,\epsilon}(t,y) = \int_{R^d} \rho_\epsilon(y-x_1)[c_{m0}+c_{m1}u^1(t,x_1)+c_{m2}u^2(t,x_1)]dx_1$$

denote mollifications of coefficients $M^u_m(t,x), \quad c^u_m(t,x)$ for a fixed $\epsilon > 0$. Setting $\mu_m(t,dy) = u^m(t,y)dy$ we may consider the above relations as relations for

$M_{m,\epsilon} : [0,T] \times R^d \times [C([0,T];\mathcal{P}_1(R^d) \times C([0,T];\mathcal{P}_1(R^d))] \to R$

and

$c_{m,\epsilon} : [0,T] \times R^d \times [C([0,T];\mathcal{P}_1(R^d)) \times C([0,T];\mathcal{P}_1(R^d))] \to R$

and set

$$M^\mu_{m,\epsilon}(t,y) = \int_{R^d} \rho_\epsilon(y-x_1)[1+\alpha_{m1}\mu^1(t,dx_1)+\alpha_{m2}\mu^2(t,dx_1)],$$

$$c^\mu_{m,\epsilon}(t,y) = \int_{R^d} \rho_\epsilon(y-x_1)[c_{m0}+c_{m1}\mu^1(t,x_1)+c_{m2}\mu^2(t,x_1)].$$

Note that due to properties of ρ_ϵ we know that $M^\mu_{m,\epsilon}(t,y), c^\mu_{m,\epsilon}(t,y)$ are bounded and Lipschitz continuous with respect to y uniformly with respect to μ_m for each $\epsilon > 0$, that is for $y \in R^d$ there exists a constant $K > 0$ such that

$$|M_{m,\epsilon}(t,y,\mu)| \leq K \quad \text{and} \quad |c_\epsilon(t,y,\mu)| \leq K, \forall \mu \in C([0,T];\mathcal{P}_1(R^d)).$$

With these notations the mollification of system (31) can be written in the form

$$\frac{\partial \mu_m}{\partial t} = \frac{1}{2}\Delta[M^2_{m,\epsilon}(t,y,\mu)\mu_m] + c_{m,\epsilon}(t,y,\mu)\mu_m, \quad \mu_m(0,dy) = \mu_{0m}(dy), \tag{37}$$

as a system for $\mu_m \in C([0,T];\mathcal{P}_1(R^d))$.

Theorem 4. *A stochastic counterpart of (34) has the form*

$$d\xi_m^\epsilon(t) = M_{m,\epsilon}(t, \xi_m^\epsilon(t), \mu)dw_m(t), \quad \xi_m^\epsilon(0) = \xi_{0m}, \tag{38}$$

with $\mathcal{L}(\xi_{0m}) = P\{\xi_{0m} \in dy\} = \mu_{0m}(dy)$

$$\mathbf{d}\eta_{\mathbf{m}}^\epsilon = \mathbf{c}_{\mathbf{m},\epsilon}(\mathbf{t}, \xi_{\mathbf{m}}^\epsilon(\mathbf{t}), \mu)\eta_{\mathbf{m}}^\epsilon(\mathbf{t})\mathbf{dt}, \quad \eta_{\mathbf{m}}(\mathbf{0}) = \mathbf{1}. \tag{39}$$

$$\int_{\mathbf{R^d}\times\mathbf{R^d}} \mathbf{h}(\mathbf{y})\mu_{\mathbf{m}}^\epsilon(\mathbf{t}, \mathbf{dy}) = \mathbf{E}[\eta_{\mathbf{m}}^\epsilon(\mathbf{t})\mathbf{h}(\xi_{\mathbf{m}}^\epsilon(\mathbf{t}))]. \tag{40}$$

As a final example, we consider the MHD-Burgers system which can be rewritten in the form

$$\frac{\partial \mathbf{u_m}}{\partial \mathbf{t}} = \frac{1}{2}\mathbf{Tr}\nabla^2[\mathbf{A_m^2(x)u_m}] + \mathbf{c_m^u(t,y)u_m}, \quad \mathbf{u_m(0,y) = u_{0m}(y)}. \tag{41}$$

w.r.t. $u = (v, B) \in R^6$ $u_i = v_i, u_{3+i} = B_i, i = 1, 2, 3,$ $c_m^u(t, y) = c_m(u(t, y), \nabla u(t, y)), A_m^2 = A_m A_m^*,$

$$A_1 = A_2 = A_3 = \sigma, \quad A_4 = A_5 = A_6 = \mu,$$

$$c_1^u = \frac{1}{u_1}[u_4\nabla_3 u_6 - u_6\nabla_1 u_6 - u_5\nabla_1 u_5 + u_5\nabla_2 u_4 - \sum_{i=1}^{3}\nabla_i u_1], \dots$$

$$c_4^u = \frac{1}{u_4}[\nabla_2[u_1 u_5 - u_2 u_4] - \nabla_3[u_1 u_6 - u_3 u_4]], \dots$$

e.t.c.

A function $u : [0, T] \times R^d \to R^{d_1}$, $u = (u_1, \dots, u_{d_1})$, $u_m(t) \in W^{1,1}_{loc}(R^d)$, $t \in [0, T]$, with $d = 3, d_1 = 6$, is a weak solution of (38), if for any $h \in C_0^\infty(R^d)$

$$\int_{R^d} u_m(t,y)h(y)dy = \int_{R^d} u_{0m}(y)h(y)dy + \int_0^t \int_{R^d} u_m(\theta,z)\mathcal{L}_m h(z)dzd\theta +$$

$$+ \int_0^t \int_{R^d} u_m(\theta,z)c_m^u(\theta,z)h(z)dzd\theta. \qquad (42)$$

where $\mathcal{L}_m h = \frac{1}{2} Tr A_m^2(x)\nabla^2 h$. As above a mild solution of (38) is defined by an integral identity valid for any $h \in C_0^\infty(R^d)$

$$\int_{R^d} h(y)u_m(t,y)dy = \int_{R^d} h(y)\left(\int_{R^d} p_m(0,x,t,y)u_{0m}(x)dx\right) +$$

$$+ \int_{[0,T]\times R^d} \left(\int_{R^d} h(y)p_m(\theta,z,t,y)dy\right) c_m^u(\theta,z)u_m(\theta,z)dzd\theta.$$

A stochastic problem associated with (41) has the form

$$\xi_m(t) = \xi_{0m} + \int_0^t A_m(\xi_m(\theta))dw_m(\theta), \quad (43)$$

$m = 1, \dots, 6,$

$$\eta_m(t) = 1 + \int_0^t c_m(u(\theta, \xi_m(\theta)), \nabla u(\theta, \xi_m(\theta)))\eta_m(\theta)d\theta. \quad (44)$$

where ξ_{0m} are independent r.v. which do not depend on $w(t) = (w_1(t), \dots, w_{d_1}(t))$ and $P\{\xi_{0m} \in dy\} = \mu_{0m}(dy) = u_{0m}(y)dy$.

$$\int_{R^d} h(y)\mu_m(t, dy) = \mathbf{E}\left[h(\xi_m(t))\eta_m(t)\right]. \quad (45)$$

Let $c_m(u, v)$ be a bounded function for $u \in R_1^d, v \in R^d \otimes R^{d_1}$ then similar to above we use

$$\int_{R^d} h(y)\mu_m(t, dy) = E\left[h(\xi_m(t)) \exp\left\{\int_0^t c_m^v(\theta, \xi_m(\theta))d\theta\right\}\right]$$

to define closing relations.

Let $v_m \in C([0,T], W^{1,1}(R^d))$ be given and $\xi_m(t)$ solves (39). Then (45) defines a unique mild measure valued solution μ_m of

$$\frac{\partial \mu_m}{\partial t} = \frac{1}{2} Tr \nabla^2 [A_m^2(y)\mu_m] + c_m(v, \nabla v)\mu_m, \qquad (46)$$

$$\mu_m(0, dy) = \mu_{0m}(dy).$$

We can verify that given u, the measure $\mu_m(t)$ defined by (45) satisfies an integral equation

$$\mu_m(t, dy) = \int_{R^d} \mu_{0m}(dx) P_m(0, x, t, dy) +$$

$$\int_0^t \int_{R^d} c_m^u(\theta, z) P_m(\theta, z, t, dy)\mu_m(\theta, dz) d\theta. \qquad (47)$$

where $P_m(0, \xi_0, t, dy) = P\{\xi_{0,\xi_0}(t) \in dy\}$.

To satisfy the condition that $c(u,v)$ is bounded we mollify (38) and consider a system

$$\frac{\partial \mu_{\epsilon,m}}{\partial t} = \frac{1}{2} Tr \nabla^2 [M_m^2(y)\mu_{\epsilon,m}] + c_m^{\rho_\epsilon * \mu}(t,y)\mu_{\epsilon,m} \tag{48}$$

$$\mu_{\epsilon,m}(0,dy) = \mu_{0m}(dy),$$

where $c_m^{\rho_\epsilon * \mu}(t,y) = c_m([\rho_\epsilon * \mu](t,y), [\nabla\rho_\epsilon * \mu](t,y))$, and an associated system of stochastic equations

$$\xi_m(t) = \xi_{m0} + \int_0^t A_m(\xi_m(\theta))dw_m(\theta), \tag{49}$$

$$\eta_{\epsilon,m}(t) = 1 + \int_0^t c_m([\rho_\epsilon * \mu](\theta, \xi_m(\theta)), [\nabla\rho_\epsilon * \mu](\theta, \xi_m(\theta)))\eta_{\epsilon,m}(\theta)d\theta, \tag{50}$$

$$\int_{R^d} h(y)\mu_{\epsilon,m}(t,dy) = \mathbf{E}\left[h(\xi_m(t))\eta_{\epsilon,m}(t)\right] \tag{51}$$

$\forall h \in C_b(R^d), t \in [0, T].$

C 1. Let A_m b$\in$ C^3 and bounded , $u_{0m}(dy)$ be a Borel measure with a density $u_{0m}(y)$, and $c_m : R^{d_1} \times R^d \otimes R^{d_1} \to R$ satisfy

$$|c_m(u,v)| \le K[1 + \|u\|^2 + \|v\|^2],$$

$$|c_m(u,v) - c(u_1,v_1)| \le L_{u,v}[\|u - u_1\| + \|v - v_1\|].$$

Lemma 5. *Let* **C1** *holds, $\xi_m(t)$ solve (41), $\eta_m(t)$ solve (45) and $u_m(t) \in W^{1,1}(R^d) \cap C^1(R^d)$ is a given function. Then $\mu_{\epsilon,m}(t)$ is a mild measure valued solution of (48) iff for all $h \in C_0^\infty(R^d)$ (51) holds.*

To prove the existence and uniqueness of a solution to (48) we consider the space $\mathcal{M}^{d_1}(R^d) = \oplus_{k=1}^{d_1} \mathcal{M}_k(R^d)$ with the norm $\|\mu\|_{TV} = \sum_{m=1}^{d_1} \|\mu_m\|_{TV}$.

Let $\mathcal{M}(R^d)$ be the space of bounded Borel measures, $N([\theta, \theta + \tau], \mathcal{M}(R^d))$ ($\mathcal{N}([\theta, \theta + \tau], \mathcal{M}^{d_1}(R^d))$) be the space of bounded maps defined on $[\theta, \theta + \tau]$ valued in $\mathcal{M}^{d_1}(R^d)$ (in $\mathcal{M}^{d_1}(R^d)$), equipped with total variation norm $\| \cdot \|_{TV}$.

$$\|\mu_m\|_{TV} = sup_{h \in C_b(R^d), \|h\|_\infty \le 1} | \int_{R^d} h(x) \mu_m(dx)|$$

Let $B^{d_1}(0, K)$ ($B(0, r)$) a centered ball with radius K (r) in $\mathcal{M}^{d_1}(R^d)$ (in $\mathcal{M}(R^d)$) and let $\mathcal{N}([\theta, \theta + \tau], B^{d_1}(0, K)) \subset \mathcal{N}([\theta, \theta + \tau], \mathcal{M}^{d_1}(R^d))$ denotes a closed subset of maps from $[\theta, \theta + \tau]$ to $B^{d_1}(0, K)$.

We consider maps

$$U_m(t) : \kappa_m \in \mathcal{M}(R^d) \mapsto U_m(t)\kappa_m \in N([s, s+\tau], \mathcal{M}(R^d))$$

given by

$$[U_m(t)\kappa_m](t, dy) = \int_{R^d} P_m(0, x, t, dy)\kappa_m(dx), \quad t \in [s, s+\tau]$$

and $\Gamma_\epsilon : \mu \mapsto \Gamma_\epsilon\mu$, $\mu \in \mathcal{N}([s, s+\tau], \mathcal{M}^{d_1}(R^d))$, $t \in [s, s+\tau]$
given by

$$[\Gamma_\epsilon\mu]_m(t, dy) = \int_0^t \int_{R^d} P_m(s, z, t, dy)c_m([\rho_\epsilon*\tilde{\mu}](\theta, z), [\nabla\rho_\epsilon*\tilde{\mu}](s, z))\times$$

$$\times[\mu + U\kappa]_m(s, dz)ds,$$

where $m = 1, \ldots, d_1$, $\tilde{\mu} = \mu + U\kappa$ and
$[\rho_\epsilon * \tilde{\mu}](\theta, z) = ([\rho_\epsilon * \tilde{\mu}]_1(\theta, z), \ldots, [\rho_\epsilon * \tilde{\mu}]_{d_1}(\theta, z))$.

Theorem 6. *Let* **C 1** *hold and for given* $r > 0, \epsilon > 0$ *the estimates* $\|\mu_{0m}\|_{TV} \le r$ *and* $\max_{\nu \in B(0,r)} |c_m(\nu_\epsilon, \nabla \nu_\epsilon)| \le K_c^\epsilon$ *hold . Then there exist* $K > 0$ *and* $\tau > 0$ *such that the map* Γ_ϵ *is a contraction in* $\mathcal{N}([s, s+\tau], B^{d_1}(0, K))$.

Let $0 = t_0 \le t_1 \le \cdots \le t_n = T$ be a partition of $[0, T]$, $\tau = \frac{T}{n}$.

Theorem 7. *Assume* **C 3.1**. *For fixed* $\epsilon > 0$ $\mu_m^\epsilon : [0, \tau] \to \mathcal{M}(R^d)$ *satisfy*

$$\mu_{\epsilon,m}(t, dy) = \int_{R^d} P_m(k\tau, z, t, dy)\mu_{\epsilon,m}(k\tau, dz) +$$

$$+ \int_{k\tau}^t \int_{R^d} P_m(s, z, t, dy) c_m^{\mu_\epsilon}(s, z)\mu_{\epsilon,m}(s, dz)ds$$

iff

$$\mu_m^\epsilon(t, dy) = \int_{R^d} P_m(0, x, t, dy)\mu_0(dx) +$$

$$+ \int_0^t \int_{R^d} P_m(s, z, t, dy)) c_m^{\mu_\epsilon}(s, z)\mu_m^\epsilon(s, dz)ds.$$

In addition μ_m^ϵ *is a unique mild solution of (48).*

Theorem 8. *Let* **C 1** *hold and* u_m^ϵ *is a density of a solution* $\mu_m^\epsilon(t)$ *to (48)*

$$u_{\epsilon,m}(t,y)dy = \mu_m^\epsilon(t,dy).$$

Then functions $u_{\epsilon,m}(t,\cdot)$ *converge to a mild solution* $u_m(t,\cdot)$, *of (41) in the norm of* $W^{1,1}(R^d)$.

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