HJB and McKean-Vlasov equations with fractional noise in space and in time

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Highlights

We present a general point of view on a large class of pseudo-differential equations including HJB and McKean-Vlasov arising from fractional noises, forward-backward systems, and other equations of mathematical physics

The general approach is inspired by the fractional analysis. It turns out that the general point of view yields explicit (seemingly new) estimates even for classical equations (without any fractional derivatives) in terms of the Mittag-Leffler functions.

Main results: Well-posedness and Sensitivity: smooth dependence on parameters and initial data with explicit bounds for the derivatives.

Introduction: Fixed-point principles

Generalized contraction principle (also referred to as the Weissinger fixed point theorem):

Theorem.

(i) If Φ is a mapping $X \to X$ in a complete metric space X with metric ρ such that $\rho(\Phi^n(x), \Phi^n(y)) \leq \alpha_n \rho(x, y)$ for all x, y with some α_n such that $A = 1 + \sum_{n=1}^{\infty} \alpha_n < \infty$, then Φ has a unique fixed point x^*, $\Phi^n(x)$ converges to x^* for any x and

$$\rho(x, x^*) \leq A\rho(x, \Phi(x)).$$

(ii) If Φ_1, Φ_2 are two mappings $X \to X$ such that $\rho(\Phi_j^n(x), \Phi_j^n(y)) \leq \alpha_n(j)\rho(x, y)$ for $j = 1, 2$ and all x, y with some $\alpha_n(j)$ such that $A(j) = 1 + \sum_{n=1}^{\infty} \alpha_n(j) < \infty$, and if $\rho(\Phi_1(x), \Phi_2(x)) \leq \epsilon$ for all x, then

$$\rho(x_1^*, x_2^*) \leq \epsilon \min_{j=1,2} A(j)$$

Introduction: Fixed-point principles: 2

Theorem. For $Y \in M$, $\alpha \in B_1$, a mapping $\Phi_{Y,\alpha} : C([\tau, T], M) \to C_Y([\tau, T], M)$ is such that for any t the restriction of $\Phi_{Y,\alpha}(\mu_\cdot)$ on $[\tau, t]$ depends only on the restriction of the function μ_s on $[\tau, t]$. Moreover

$$\|[\Phi_{Y,\alpha}(\mu^1_\cdot)](t) - [\Phi_{Y,\alpha}(\mu^2_\cdot)](t)\| \le L(Y) \int_\tau^t \|\mu^1_\cdot - \mu^2_\cdot\|_{C([\tau,s],B)}\, ds,$$

$$\|[\Phi_{Y_1,\alpha_1}(\mu_\cdot)](t) - [\Phi_{Y_2,\alpha_2}(\mu_\cdot)](t)\| \le \varkappa \|Y_1 - Y_2\| + \varkappa_1 \|\alpha_1 - \alpha_2\|.$$

Then the mapping $\Phi_{Y,\alpha}$ has a unique fixed point $\mu_{t,\tau}(Y, \alpha)$ in $C_Y([\tau, T], M)$ and

$$\|\mu_{t,\tau}(Y, \alpha) - Y\| \le e^{(t-\tau)L(Y)} \|[\Phi_{Y,\alpha}(Y)](t) - Y\|,$$

$$\|\mu_{t,\tau}(Y_1, \alpha_1) - \mu_{t,\tau}(Y_2, \alpha_2)\| \le \varkappa(\|Y_1 - Y_2\| + \|\alpha_1 - \alpha_2\|) \exp\{(t-\tau)L($$

Introduction: Fixed-point principles, 3

Theorem. (i) Suppose that the conditions of the above Theorem hold but with

$$\|[\Phi_Y(\mu^1_\cdot)](t) - [\Phi_Y(\mu^2_\cdot)](t)\| \le L(Y) \int_\tau^t (t-s)^{-\omega} \|\mu^1_\cdot - \mu^2_\cdot\|_{C[\tau,s]}\, ds$$

$$\|[\Phi_{Y_1,\alpha_1}(\mu_\cdot)](t) - [\Phi_{Y_2,\alpha_2}(\mu_\cdot)](t)\| \le \varkappa(\|Y_1 - Y_2\| + \|\alpha_1 - \alpha_2\|),$$

with some $\omega \in (0,1)$.
Then the mapping $\Phi_{Y,\alpha}$ has a unique fixed point and

$$\|\mu_{t,\tau}(Y,\alpha) - Y\| \le E_{1-\omega}(L(Y)\Gamma(1-\omega)(t-\tau)^{1-\omega})\|[\Phi_{Y,\alpha}(Y)](t) - Y\|$$

$$\|\mu_{t,\tau}(Y_1,\alpha_1) - \mu_{t,\tau}(Y_2,\alpha_2)\|$$

$$\le (\varkappa\|Y_1 - Y_2\| + \varkappa_1\|\alpha_1 - \alpha_2\|)E_{1-\omega}(L(Y_j)\Gamma(1-\omega)(t-\tau)^{1-\omega}).$$

Sensitivity of integral equations

$$[\Phi_{Y,\tau}(\mu_\cdot)](t) = G_{t,\tau}Y + \int_\tau^t \Omega_{t,s}(\mu_s)\, ds,$$

act in $C([\tau, t], B)$, where $G_{t,s}$ and $\Omega_{t,s}$, $t > s$ are the families of linear operators and (possibly) nonlinear mappings in B respectively. Starting point: uniformly bounded Ω. Advanced: Ω have a singularity at $t = s$.

Assuming the existence of the unique fixed points $\mu_{t,\tau}(Y) \in C([\tau, T], B)$, what are the derivatives

$$\xi_t = \xi_{t,\tau}(Y)[\xi] = D\mu_{t,\tau}(Y)[\xi],$$

in some direction ξ, where by D is the derivative in B.

Main hint: if well defined, they satisfy the equations

$$\xi_t = G_{t,\tau}\xi + \int_\tau^t D\Omega_{t,s}(\mu_{s,\tau})[\xi_s]\, ds,$$

obtained by differentiation from the fixed-point equation.

Sensitivity of integral equations

Theorem.

Let $G_{t,s}$ and $\Omega_{t,s}$ depend continuously on t and are measurable on s. Let mapping Φ have unique fixed points μ s.t. there exist $M_T(R) > 0$ such that

$$\|\mu_\cdot\|_{C([\tau,T],B)} \leq M_T(R)$$

any $Y \in B_R$. Let $G_{t,t}$ be the identity operator and

$$\|G_{t,\tau}\|_{B\to B} \leq G,$$

Moreover, let $\Omega_{t,s} \in C^1_{uc}(B_R, B)$ uniformly in t, s for any R, Then the mapping $\mu_\tau = Y \to \mu_{t,\tau}(Y)$ belongs to $C^1_{uc}(B_R, B)$ for all t, $\xi_t = D\mu_{t,\tau}(Y)[\xi]$ represents the unique solution of equation above and

$$\|\xi_t\| \leq Ge^{(t-\tau)L(M_T(R))}\|\xi\|, \quad 0 \leq \tau \leq t \leq T.$$

Sensitivity of ODEs

Theorem. Let $F \in C^1_{luc}(B,B)$ for a Banach space B. Then the solution $\mu_0 = Y \to \mu_t(Y)$ constructed in the 1st Theorem above belongs to $C^1_{luc}(B,B)$ for all t and $\xi_t = D\mu_t(Y)[\xi]$ is the unique solution of the linear equation

$$\xi_t = \xi + \int_0^t DF(\mu_s)[\xi_s]\, ds \quad \Longleftrightarrow \quad \dot{\xi}_t = DF(\mu_t)[\xi_t] \quad \text{and} \quad \xi_0 = \xi,$$

with the initial condition $\xi_0 = \xi$. Moreover,

$$\|\xi_t\| \le e^{|t|L}\|\xi\| \le \exp\{|t|\|F\|_{C^1(B)}\}\|\xi\|.$$

If additionally $F \in C^1_{uc}(B,B)$ or $F \in C^1_{bLip}(B,B)$, then the mapping $\mu_0 = Y \to \mu_t(Y)$ belongs to $C^1_{uc}(B,B)$ or $C^1_{bLip}(B,B)$ respectively.

Sensitivity, advanced

Theorem. Suppose the assumptions of the above Theorem on the sensitivity of integral mappings hold with a modification concerning Ω. Namely, $\Omega_{t,s} \in C^1_{uc}(B_R, B)$ for $t > s$ and any R, and there exists $\omega \in (0,1)$ such that for any ϵ, R, there exist $L(R)$ and $\delta = \delta(R, \epsilon)$ such that

$$\|D\Omega_{t,s}(\mu)\|_{B \to B} \leq L(R)(t-s)^{-\omega},$$

$$\|D\Omega_{t,s}(\mu_1) - D\Omega_{t,s}(\mu_2)\|_{B \to B} \leq \epsilon(t-s)^{-\omega},$$

for all t, s, and any μ_1, μ_2 such that $\mu_1 \in B_R$, $\|\mu_1 - \mu_2\| \leq \delta$. Then the mapping $\mu_\tau = Y \mapsto \mu_{t,\tau}(Y)$ belongs to $C^1_{uc}(B_R, B)$, $\xi_t = D\mu_{t,\tau}(Y)[\xi]$ is well defined and

$$\|\xi_t\| \leq E_{1-\omega}(L(M_T(R))\Gamma(1-\omega)(t-\tau)^{1-\omega})G\|\xi\|.$$

HJB-McKean-Vlasov-Ginzburg-Landau

$$\frac{\partial f}{\partial t}(x) = Af(x) + H\left(x, \frac{\partial f}{\partial x}(x), \phi(f,x)\right), \qquad (1)$$

where A is the generator of a strongly continuous semigroup e^{tA} in $C_\infty(\mathbf{R}^d)$ with the *smoothing property*

$$\|e^{tA} f\|_{C^1(\mathbf{R}^d)} \le \varkappa t^{-\omega} \|f\|_{C(\mathbf{R}^d)},$$

uniformly for t from a compact interval $[0,T]$, $\omega \in (0,1)$, and H is a Lipschitz continuous function of its three variables referred to as the *Hamiltonian* of equation (1).
In main examples either ϕ does not enter at all (basic HJB), or $\phi(f,x) = f(x)$ (Ginzburg-Landau), or $\phi(f,x) = \phi(f) = \int f(y)\mu(dy)$ or other integral dependence.
Generally

$$|\phi(f_1,x) - \phi(f_2,x)| \le C\|f_1 - f_2\|.$$

Examples

Generalized HJB:

$$\frac{\partial f}{\partial t}(x) = Af(x) + H\left(x, \frac{\partial f}{\partial x}(x), \phi(f, x)\right),$$

Ginzburg-Landau

$$\frac{\partial f}{\partial t}(x) = \Delta f(x) - \psi'(f(x))$$

with some given function ψ,
McKean-Vlasov

$$\frac{\partial f}{\partial t}(x) = Af(x) + H\left(x, \frac{\partial f}{\partial x}(x), \int f(y)\mu(dy)\right),$$

with H being linear in $\partial f / \partial x$.

Extensions

(i) $A = A_t$ depends on time;
(ii) coefficients of the linear operator A depend on f;
(iii) Higher-order PDEs

$$\dot{f}_t(x) = -\sigma(x)|\Delta|^{\alpha/2} f_t + H\left(x, \{\frac{\partial^m f_t}{\partial x_{i_1} \cdots \partial x_{i_m}}\}(x), f_t(x)\right),$$

where σ is a constant with a positive real part, $H(x, \{p_{i_1,\cdots,i_m}\}, q)$ is a function of $x \in \mathbf{R}^d$ and $p_{i_1,\cdots,i_m}, q \in \mathbf{R}$, where $p_{i_1,\cdots,i_m}$ are parametrized by sequences of m numbers from $\{1, \cdots, d\}$ with $m \in [1, \cdots, k]$, where $k < \alpha$. Thus H is a function of f and all its derivatives of order up to k.
An important example for physics represents the so-called *Cahn-Hilliard equation*:

$$\dot{f} = -\sigma\Delta^2 f + \Delta(\gamma_2 f^3 + \gamma_1 f^2 - f),$$

Theory of HJB

The *mild form* of this equation:

$$f_t = e^{tA}Y + \int_0^t e^{(t-s)A} H\left(., \frac{\partial f_s}{\partial x}(.), f_s(.)\right) ds.$$

Theorem. Let A be an operator in $C_\infty(\mathbf{R}^d)$ generating a strongly continuous semigroup e^{tA} in $C_\infty(\mathbf{R}^d)$ such that e^{tA} is also a strongly continuous semigroup in $C^1_\infty(\mathbf{R}^d)$, so that

$$\|e^{tA}\|_{C_\infty(\mathbf{R}^d)\to C_\infty(\mathbf{R}^d)} \le T_C, \quad \|e^{tA}\|_{C^1_\infty(\mathbf{R}^d)\to C^1_\infty(\mathbf{R}^d)} \le T_D,$$

with constants T_C, T_D and $t \in [0, T]$. Let e^{tA} takes $C(\mathbf{R}^d)$ to $C^1_\infty(\mathbf{R}^d)$ and the smoothing hold with $\varkappa > 0, \omega \in (0, 1)$, and let $H(x, p, q)$ be a continuous function on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}$ such that $h = \sup_x |H(x, 0, 0)| < \infty$ and

$$|H(x, p_1, q_1) - H(x, p_2, q_2)| \le L_H|p_1 - p_2| + L_H|q_1 - q_2|$$

with a constant L_H.

Theory of HJB

Then for any $Y \in C^1_\infty(\mathbf{R}^d)$ there exists a unique solution $f_\cdot \in C([0,T], C^1_\infty(\mathbf{R}^d))$ of the mild equation. Moreover, for all $t \le T$,

$$\|f_t(Y) - Y\|_{C^1(\mathbf{R}^d)} \le E_{1-\omega}(\varkappa L_H \Gamma(1-\omega) t^{1-\omega})$$

$$\times \left(\frac{t^{1-\omega}\varkappa}{1-\omega}(h + L_H \|Y\|_{C^1(\mathbf{R}^d)}) + \|(e^{tA}-1)Y\|_{C^1(\mathbf{R}^d)} \right),$$

and, for the different solutions $f_t(Y_1)$ and $f_t(Y_2)$,

$$\|f_t(Y_1) - f_t(Y_2)\|_{C^1(\mathbf{R}^d)} \le T_D \|Y_1 - Y_2\|_{C^1(\mathbf{R}^d)} E_{1-\omega}(\varkappa L_H \Gamma(1-\omega) t^{1-\omega}).$$

Further: time dependent A_t, classical solutions (i.e. from $C^2(R^d)$), continuous dependence on parameters

Sensitivity for HJB

Theorem. Under the assumptions above let us assume that the derivatives $\partial H(x,p,q)/\partial p$ and $\partial H(x,p,q)/\partial q$ exist and are continuous functions uniformly for $x \in \mathbf{R}^d$ and p, q from any bounded set. Then the mapping $Y \mapsto f_t \in C^1_\infty(\mathbf{R}^d)$ yielding the solution belongs to $C^1_{luc}(C^1_\infty(\mathbf{R}^d), C^1_\infty(\mathbf{R}^d))$ and the derivative $\xi_t = Df_t(Y)[\xi]$ is the unique solution of the equation

$$\xi_t = e^{tA}\xi + \int_0^t e^{(t-s)A}$$

$$\left(\frac{\partial H}{\partial p}(., \frac{\partial f_s}{\partial x}(.), f_s(.))\frac{\partial \xi}{\partial x} + \frac{\partial H}{\partial q}(., \frac{\partial f_s}{\partial x}(.), f_s(.))\xi\right) ds,$$

and is bounded:

$$\|\xi_t\| \le \varkappa(T, \|Y\|)\|\xi\|(1+t).$$

Preliminaries for fractional PDEs: Zolotarev-Pollard formula

$$E_\beta(s) = \frac{1}{\beta}\int_0^\infty e^{sx} x^{-1-1/\beta} G_\beta(1, x^{-1/\beta})\,dx,$$

where

$$G_{\pm\beta}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{ipx - t|p|^\beta \exp\{\pm\frac{i}{2}\pi\beta\,\mathrm{sgn}\,p\}\}\,dp$$

is the (fundamental) solution of the problem

$$\frac{\partial G}{\partial t}(t,x) = -\frac{d^\beta}{d(\pm x)^\beta} G(t,x), \quad t \geq 0, \quad G_{t=0} = \delta(x),$$

with $\beta \in (0,1)$.

Linear fractional equations in Banach spaces

Theorem. Let $\beta \in (0,1)$ and let A be a generator of a strongly continuous semigroup T_t in a Banach space B with the domain $D(A)$. Let $Y \in D(A)$, b_t be a continuous curve in B such that $b_t \in D(A)$ for any t and the norms $\|Ab_t\|$ are bounded on compact intervals of t. Then the fractional linear Cauchy problem

$$D^{\beta}_{a+*}\mu_t = A\mu_t + b_t, \quad \mu_a = Y, \quad t \geq a,$$

has the unique solution:

$$\mu_t = E_\beta(A(t-a)^\beta)Y + \beta \int_a^t (t-s)^{\beta-1} E'_\beta(A(t-s)^\beta) b_s \, ds.$$

This function is also the unique solution to the fractional integral equation

$$\mu_t = Y + \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1}(A\mu_s + b_s)\, ds = Y + I^\beta(A\mu_. + b_.)(t).$$

If T_t has type of growth m_0, so that $\|T_t\| \leq Me^{mt}$ with some

Fractional HJB-McKean-Vlasov-Ginzburg-Landau

$$D^{\beta}_{a+\star} f_t = A f_t + H(x, \frac{\partial f_t}{\partial x}, f_t),$$

where A is the generator of a strongly continuous semigroup e^{tA} in $C_\infty(\mathbf{R}^d)$ with smoothing

$$\|e^{tA} f\|_{C^1(\mathbf{R}^d)} \le \varkappa t^{-\omega} \|f\|_{C(\mathbf{R}^d)},$$

for $t \in (0, S]$ with some finite or infinite S, $D^{\beta}_{a+\star}$ is the CD derivative of order $\beta \in (0,1)$ and H is Lipschitz on its three variables.

More generally

$$\frac{\partial f}{\partial t}(x) = A f(x) + H\left(x, \frac{\partial f}{\partial x}(x), \phi(f, x)\right),$$

$$|\phi(f_1, x) - \phi(f_2, x)| \le C \|f_1 - f_2\|.$$

Fractional HJB-McKean-Vlasov-Ginzburg-Landau

Mild version (fixed point):

$$f_t = E_\beta(A(t-a)^\beta)Y + \beta \int_a^t (t-s)^{\beta-1} E'_\beta(A(t-s)^\beta) H(., \frac{\partial f_s}{\partial x}, f_s)\, ds,$$

Analogously higher order PDEs:

$$D^\beta_{a+\star} f_t = -\sigma(x)|\Delta|^{\alpha/2} f_t + H(x, \{\frac{\partial^m f_t}{\partial x_{i_1} \cdots \partial x_{i_m}}\}, f_t).$$

Fractional HJB-McKean-Vlasov

Theorem.

Let A be an operator in $C_\infty(\mathbf{R}^d)$ generating a strongly continuous semigroup e^{tA} in $C_\infty(\mathbf{R}^d)$ such that e^{tA} is also a strongly continuous semigroup in $C^1_\infty(\mathbf{R}^d)$, with smoothing, and

$$\|e^{tA}\|_{C_\infty(\mathbf{R}^d)\to C_\infty(\mathbf{R}^d)} \le M_C e^{m_C t}, \quad \|e^{tA}\|_{C^1_\infty(\mathbf{R}^d)\to C^1_\infty(\mathbf{R}^d)} \le M_D e^{m_D t},$$

with constants M_C, M_D, m_C, m_D and all t. Let $H(x,p,q)$ be a continuous function on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}$ with the same properties as for the classical case above. Then for any $Y \in C^1_\infty(\mathbf{R}^d)$ there exists a unique solution $f_\cdot \in C([0,T], C^1_\infty(\mathbf{R}^d))$ of the mild equation. Moreover, the solutions $f_t(Y_1)$ and $f_t(Y_2)$ with different initial data Y_1, Y_2 enjoy the estimate

$$\|f_t(Y_1) - f_t(Y_2)\|_{C^1(\mathbf{R}^d)} \le C\|Y_1 - Y_2\|_{C^1(\mathbf{R}^d)},$$

with constant C depending continuously on $t, \varkappa, \omega, \beta, L_H, M_C, M_D, m_C, m_D$.

Generalized fractional derivatives

Sensitivity!
Main formula for E_ν.

THANK YOU

Some texts:

V. N. Kolokoltsov. Differential equations on measures and functional spaces. Birkhäuser Advanced Texts, Birkhäuser, 2019.

V. N. Kolokoltsov and O. A. Malafeyev. Many Agent Games in Socio-economic Systems: Corruption, Inspection, Coalition Building, Network Growth, Security. Springer Series in Operations Research and Financial Engineering, Springer Nature, 2019. (First part of the book on arxiv)

Vassili Kolokoltsov. The probabilistic point of view on the generalized fractional PDEs. FCAA 22:3 (2019), 543-600 (open access mode).

V. N. Kolokoltsov and M. Veretennikova. The fractional Hamilton-Jacobi-Bellman equation. Journal of Applied Nonlinear Dynamics 6:1 (2017), 4556.