Single jump filtrations and local martingales

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LSA winter meeting 2019
December 04, 2019
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Outline

- Inroduction, definition of the model and preliminaries
- Characterization and description of local martingales
- Examples

Introduction, definition of a single jump filtration and preliminary results

Notation

$$\mathbb{R}_+ = [0, +\infty), \quad \overline{\mathbb{R}}_+ = [0, +\infty].$$

We always assume that there are given a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and a random variable γ with values in $\overline{\mathbb{R}}_+$ on it. Then

$$G(t) = \mathsf{P}(\gamma \leqslant t), t \in \mathbb{R}_+,$$

stands for the distribution function of γ and $\overline{G}(t) = 1 - G(t)$. Put also

$$t_G = \sup \{t \in \mathbb{R}_+ \colon G(t) < 1\}, \quad \mathfrak{T} = \{t \in \mathbb{R}_+ \colon \mathsf{P}(\gamma \geqslant t) > 0\}.$$

We tacitly assume that $\mathsf{P}(\gamma > 0) > 0$.

Dellacherie (1970)

Assumptions

$\mathrm{P}(0 < \gamma < \infty) = 1$, $t_G = +\infty$.

A filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is defined as the smallest (completed) filtration with respect to which γ is a stopping time (or, equivalently, the process $\mathbb{1}_{\{t\geqslant\gamma\}}$ is adapted), $\mathscr{F} = \mathscr{F}_\infty$, i.e. $\mathscr{F}$ is the completion of $\sigma\{\gamma\}$.

It is shown that the compensator of $\mathbb{1}_{\{t\geqslant\gamma\}}$ has the form

$$A_t = \int_0^{\gamma\wedge t} \frac{dG(s)}{\overline{G}(s-)}.$$

In particular, if G is continuous, then

$$A_t = \log \frac{1}{\overline{G}(\gamma\wedge t)}.$$

Chou and Meyer (1975)

Assumptions

$\mathsf{P}(\gamma > 0) = 1.$

A filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ *is defined as the smallest (completed) filtration with respect to which* γ *is a stopping time (or, equivalently, the process* $\mathbb{1}_{\{t\geqslant\gamma\}}$ *is adapted),* $\mathscr{F} = \mathscr{F}_\infty$*, i.e.* $\mathscr{F}$ *is the completion of* $\sigma\{\gamma\}$*.*

It is shown that every local martingale can be represented as a Lebesgue-Stieltjes integral with respect to the local martingale $\mathbb{1}_{\{t\geqslant\gamma\}} - A_t$.

The main idea: if (M_t) is a uniformly integrable martingale, then M_∞ can be represented as $H(\gamma)$ with some function H. Then the structure of the filtration implies that $M_t = H(\gamma)$ for $t \geq \gamma$, and M_t coincides with a deterministic function for $t < \gamma$. Hence, it is enough to prove the representation only at time γ.

Main definition

Our purpose is to define a single jump filtration in such a way that all randomness appears at a random time γ but there are more random events in $\mathscr{F}$ than in $\sigma\{\gamma\}$.

Definition

$\mathscr{F}_t$, $t \in \mathbb{R}_+$, is the collection of subsets A of Ω such that $A \in \mathscr{F}$ and $A \cap \{t < \gamma\}$ is either $\varnothing$ or coincides with $\{t < \gamma\}$.

Lemma

(i) *$\mathscr{F}_t$ is a σ-field and a random variable ξ is $\mathscr{F}_t$-measurable, $t \in \mathbb{R}_+$, if and only if ξ is constant on $\{t < \gamma\}$. ξ is $\mathscr{F}_\infty$-measurable if and only if ξ is constant on $\{\gamma = \infty\}$.*

(ii) *The family $(\mathscr{F}_t)_{t \in \mathbb{R}_+}$ is increasing and right-continuous, i.e. $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathbb{R}_+}$ is a filtration.*

(iii) *γ is a stopping time and $\mathscr{F}_\gamma = \mathscr{F}_\infty$.*

We call this filtration a single jump filtration generated by γ and $\mathscr{F}$.

Proposition

(i) *If* $X = (X_t)_{t\in\mathbb{R}_+}$ *is a right-continuous adapted process with almost all càdlàg trajectories, then there is a càdlàg deterministic function* $F(t)$, $0 \leqslant t < t_G$, *such that* $X_t = F(t)$ *on* $\{t < \gamma \wedge t_G\}$.

(ii) *A random variable* T *with values in* $\overline{\mathbb{R}}_+$ *is a stopping time if and only if satisfies the following property: if the set* $\{T < \gamma\}$ *is not empty, then there is a number* r *such that* $\{T < \gamma\} = \{T = r < \gamma\} = \{r < \gamma\}$.

(iii) *If* $M = (M_t)_{t\in\mathbb{R}_+}$ *is a local martingale then there is a càdlàg function* $F(t)$, $0 \leqslant t < t_G$, *and a finite random variable* L *such that,* P-*a.s.,* $M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}$. *Moreover,* $(M_t)_{t<t_G}$ *is a martingale.*

Characterization and description of local martingales

Our main goal is to provide a complete description of all local martingales in this model. According to previous proposition, a necessary condition is that it is represented as

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}.$$

Thus, it is enough to study only processes of this form. Then, necessarily, L is integrable over $\{\gamma \leqslant t\}$ for any $t < t_G$. Note also that values of L on the set $\{\gamma = \infty\}$ do not affect the right-hand side of the above equality, and that, given (M_t), we can reconstruct L a.s. on the set $\{\gamma < \infty\}$ and the function $F(t)$, $0 \leqslant t < t_G$, from $M_\gamma = L$ a.s. on $\{\gamma < \infty\}$ and $\mathsf{E}M_t = F(t)\overline{G}(t) + \mathsf{E}[L\mathbb{1}_{\{t\geqslant\gamma\}}]$, $0 \leqslant t < t_G$.

First theorem

Theorem

Let $F(t)$, $0 \leqslant t < t_G$, *be a deterministic càdlàg function,* L *a random variable, and a process* $M = (M_t)_{t\in\mathbb{R}_+}$ *be given by*

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}.$$

The following statements are equivalent:

(i) $M = (M_t)_{t\in\mathbb{R}_+}$ *is a local martingale.*

(ii) $(M_t)_{t\in\mathcal{T}}$ *is a martingale.*

(iii)

$$\mathsf{E}\big[|L|\mathbb{1}_{\{\gamma\leqslant t\}}\big] < \infty, \quad t \in \mathcal{T},$$

and

$$\mathsf{E}M_t = \mathsf{E}M_0, \quad t \in \mathcal{T}.$$

Proof of (ii)⇔(iii)

In the case where $\mathscr{F} = \sigma\{\gamma\}$, equivalence (i)⇔(ii) is proved by Chou and Meyer (1975).

(ii)⇒(iii) is trivial. Let (iii) hold. The process $(M_t)_{t\in\mathcal{T}}$ is adapted and integrable due to the first part of (iii). Moreover,

$$M_t - M_s = 0 \quad \text{on} \quad \{s \geqslant \gamma\},$$

where $0 \leqslant s < t \in \mathcal{T}$. Hence,

$$\mathsf{E}[M_t - M_s|\mathscr{F}_s] = 0 \quad \text{on} \quad \{s \geqslant \gamma\}.$$

But $\mathsf{E}[M_t - M_s|\mathscr{F}_s]$ is $\mathscr{F}_s$-measurable and, thus, equals a constant on $\{s < \gamma\}$. And this constant must be zero since $\mathsf{E}[M_t - M_s] = 0$ by the second part of (iii).

Definition of H

Case A $\mathsf{P}(\gamma = t_G < \infty) = 0$ or, equivalently, $\mathcal{T} = [0, t_G)$.

Case B $\mathsf{P}(\gamma = t_G < \infty) > 0$ or, equivalently, $\mathcal{T} = [0, t_G]$.

If $\mathsf{E}\big[|L|\mathbb{1}_{\{\gamma\leqslant t\}}\big] < \infty$, $t \in \mathcal{T}$, holds, then one can define the conditional expectation $H(t)$ of L given that $\gamma = t$ for $t \in \mathcal{T}$:

$$H(t) = \mathsf{E}[L|\gamma = t].$$

More precisely, $H(t)$ is a Borel function on $\mathcal{T}$ with finite values such that for any $t \in \mathcal{T}$

$$\mathsf{E}L\mathbb{1}_{\{\gamma\leqslant t\}} = \int_{[0,t]} H(s)\, dG(s).$$

Note that the function H is dG-a.s. unique and is dG-integrable over any closed interval in $\mathcal{T}$. It is convenient to introduce a notation for such functions.

Additional notation

Let $L^1_{\mathrm{loc}}(dG)$ be the set of all Borel functions z on $\mathcal{T}$ such that

$$\int_{[0,t]} |z(s)|\, dG(s) < \infty \quad \text{for all } t \in \mathcal{T}.$$

Given a function $Z\colon [0, t_G) \to \mathbb{R}$, let us write $Z \overset{\mathrm{loc}}{\ll} G$ if there is $z \in L^1_{\mathrm{loc}}(dG)$ such that $Z(t) = Z(0) + \int_{(0,t]} z(s)\, dG(s)$ for all $t < t_G$; in this case we put $\frac{dZ}{dG}(t) := z(t)$ for $0 < t < t_G$. Let us emphasize that in Case B this definition implies that z is dG-integrable over $[0, t_G]$ and, hence, the function Z has a finite variation over $[0, t_G)$ and there is a finite limit $\lim_{t \uparrow\uparrow t_G} Z(t) = Z(0) + \int_{(0,t_G)} z(s)\, dG(s)$. Note also that in this definition the value $z(0)$ can be chosen arbitrarily even if $G(0) > 0$; the same refers to the value $z(t_G)$ in Case B. Correspondingly, dZ/dG is defined only for $0 < t < t_G$.

Second theorem, part (a)

Theorem

(a) *Let* $F(t)$, $0 \leqslant t < t_G$, *be a deterministic càdlàg function,* L *a random variable,* $H(t) = \mathsf{E}[L|\gamma = t]$, *and let a local martingale* M *satisfy*

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}.$$

Then $H \in L^1_{\mathrm{loc}}(dG)$, $F \overset{\mathrm{loc}}{\ll} G$, *and*

$$F(t)\overline{G}(t) + \int_{(0,t]} H(s)\,dG(s) = F(0)\overline{G}(0), \quad t < t_G.$$

Additionally, in Case B,

$$\lim_{t\uparrow\uparrow t_G} F(t) = H(t_G).$$

Second theorem, part (b)

Theorem (continued)

(b) *Let* $H \in L^1_{\mathrm{loc}}(dG)$. *Define*

$$F(t) = \overline{G}(t)^{-1}\Big[F(0)\overline{G}(0) - \int_{(0,t]} H(s)\,dG(s)\Big], \quad 0 < t < t_G,$$

where $F(0)$ *is an arbitrary real number in Case A and*

$$F(0) = \overline{G}(0)^{-1}\int_{(0,t_G]} H(s)\,dG(s)$$

in Case B. If L *is any random variable satisfying*

$$\mathsf{E}\big[|L|\mathbb{1}_{\{\gamma\leqslant t\}}\big] < \infty \quad \text{and} \quad H(t) = \mathsf{E}[L|\gamma = t], \quad t \in \mathcal{T},$$

then the process $M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}$ *is a local martingale.*

Second theorem, part (c)

Theorem (continued)

(c) *Let a function* $F\colon [0, t_G) \to \mathbb{R}$ *be given such that* $F \overset{\text{loc}}{\ll} G$. *Define* $H(0)$ *arbitrarily,*

$$H(t) = F(t) - \overline{G}(t-)\frac{dF}{dG}(t), \quad 0 < t < t_G,$$

and $H(t_G) = \lim_{t \uparrow\uparrow t_G} F(t)$ *in Case B. If* L *is any random variable satisfying*

$$\mathsf{E}\big[|L|\mathbb{1}_{\{\gamma \leqslant t\}}\big] < \infty \quad \textit{and} \quad H(t) = \mathsf{E}[L|\gamma = t], \quad t \in \mathcal{T},$$

then the process $M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t \geqslant \gamma\}}$ *is a local martingale.*

Remarks

1. *A random variable* L *as required in parts (b) and (c) always exists, e.g.* $L = H(\gamma)\mathbb{1}_{\{\gamma<\infty\}}$.
2. *The function* $F(t)$ *given in (b) satisfies*

$$F(t) = \overline{G}(t)^{-1}\Big[F(0)\overline{G}(0) - \int_{(0,t]} H(s)\, dG(s)\Big], \quad 0 < t < t_G,$$

i.e. it is a unique solution of this equation with unknown F *and given* G, H *and* $F(0)$. *Similarly, integrating this equation by parts, we can obtain the represntation of* H *from part (c).*

Comments

Part (c) of this theorem is essentially due to Herdegen and Herrmann (2016), though they deal with the case $\mathscr{F} = \sigma\{\gamma\}$ and formulate (4) in an equivalent form:

$$H(t) = F(t-) - \overline{G}(t)\frac{dF}{dG}(t), \quad 0 < t < t_G.$$

They prove that if H satisfies the condition in part (c) for given F and G and the assumptions part (b) are satisfied, then $M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + H(\gamma)\mathbb{1}_{\{t\geqslant\gamma\}}$ is a local martingale. However, they do not have a statement that all local martingales constructed in this way. They also prove that, in Case B, if H is not dG-integrable over $(0, t_G)$, then M defined as above is not a semimartingale. We add that, also in Case B, if H is dG-integrable over $(0, t_G)$, $F(t)$ satisfies the corresponding condition but $F(0)$ is not as prescribed, the process M constructed in the same way, is a supermartingale or a submartingale, but not a local martingale.

Comments (continued)

The fact that $H(0)$ can be chosen arbitrarily in (c) says only that L can be an arbitrary integrable random variable on the set $\{\gamma = 0\}$, which is evident ab initio. On the contrary, the fact that $F(0)$ can be chosen arbitrarily in (b) in Case A is an interesting feature of this model. It says that, given the terminal value M_∞ of M (on $\{\gamma < \infty\}$), one can freely choose the initial value M_0 of M (on $\{\gamma > 0\}$) to keep the property of being a local martingale for M.

Corollary

Every local martingale has trajectories of finite variation on compact intervals a.s.

Corollary

Let a local martingale $M = (M_t)_{t\in\mathbb{R}_+}$ *be given by* (1), *where* $F(t)$, $0 \leqslant t < t_G$, *is a deterministic càdlàg function,* L *is a random variable, and let* $H(t)$, $t \in \mathcal{T}$, *be defined in* (14). *Then* $M = M' + M''$, *where*

$$M'_t = F(t)\mathbb{1}_{\{t<\gamma\}} + H(\gamma)\mathbb{1}_{\{t\geqslant\gamma\}} \quad \text{and} \quad M''_t = \big(L - H(\gamma)\big)\mathbb{1}_{\{t\geqslant\gamma\}}$$

are local martingales.

The last decomposition corresponds to the decomposition of a local martingales according to a random measure, see Jacod (1976).

Compensator of an increasing process with a single jump

Let V be a finite random variable, $\{\gamma = 0\} \subseteq \{V = 0\}$, $X_t = V\mathbb{1}_{\{t \geqslant \gamma\}}$ is an increasing process with a single jump. We assume that

$$\mathsf{E}\big[|V|\mathbb{1}_{\{\gamma \leqslant t\}}\big] < \infty, \quad t \in \mathcal{T}.$$

Then we can introduce a function K by

$$K(t) = \mathsf{E}[V|\gamma = t], \quad t \in \mathcal{T}.$$

It is clear that $K \in L^1_{\mathrm{loc}}(dG)$ and $K(0) = 0$ if $\mathsf{P}(\gamma = 0) > 0$. Now define

$$F(t) = \int\limits_{(0,t]} \overline{G}(s-)^{-1} K(s)\, dG(s), \quad 0 \leqslant t < t_G,$$

in particular, $F(0) = 0$. It follows that, in Case B, the function F has a bounded variation on $[0, t_G)$ and has a finite limit as $t \upuparrows t_G$, so we put $F(t_G) = \lim_{t \upuparrows t_G} F(t)$.

Formula for the compensator

The next corollary takes its origin in Dellacherie (1970), where the case $V = 1$, γ is finite and $t_G = +\infty$ is considered. The general case can be deduced from the formula for the compensator of a multivariate point process, see Jacod (1975).

Corollary

Let V, X, K and F be as above. Then X is locally integrable and the compensator A_t of the process X_t is given by

$$A_t = F(t \wedge \gamma) \qquad \textit{in Case A}$$

and

$$A_t = F(t \wedge \gamma) + K(t_G)\mathbb{1}_{\{\gamma \geqslant t_G\}}\mathbb{1}_{\{t \geqslant t_G\}} \qquad \textit{in Case B.}$$

Classification

Let us say that a local martingale $M = (M_t)_{t \in \mathbb{R}_+}$ has

type 1 if the limit $M_\infty = \lim_{t\to\infty} M_t$ does not exist with positive probability or exists with probability one but not integrable: $\mathsf{E}|M_\infty| = \infty$;

type 2a if M is a closed supermartingale (in particular, $\mathsf{E}|M_\infty| < \infty$) and $\mathsf{E}M_\infty < \mathsf{E}M_0$;

type 2b if M is a closed submartingale (in particular, $\mathsf{E}|M_\infty| < \infty$) and $\mathsf{E}M_\infty > \mathsf{E}M_0$;

type 3 if M is a uniformly integrable martingale (in particular, $\mathsf{E}|M_\infty| < \infty$ and $\mathsf{E}M_\infty = \mathsf{E}M_0$) and $\mathsf{E}\sup_t |M_t| = \infty$;

type 4 if M has an integrable variation: $\mathsf{E}\,\mathrm{Var}(M)_\infty < \infty$.

Third theorem

The next theorem complements the classification of the limit behaviour of local martingales that was considered in Herdegen and Herrmann (2016) in the case where $\mathscr{F} = \sigma\{\gamma\}$.

Theorem

Let $M = (M_t)_{t\in\mathbb{R}_+}$ be a local martingale with a representation

$$M_t = F(t)\mathbb{1}_{\{t<\gamma\}} + L\mathbb{1}_{\{t\geqslant\gamma\}}, \quad t \in \mathbb{R}_+,$$

and let $H(t) = \mathsf{E}[L|\gamma = t]$. Then in Case B M has type 4. In Case A all types are possible. Namely,

(i) *M has type 1 if and only if $\mathsf{E}|L - H(\gamma)|\mathbb{1}_{\{\gamma<\infty\}} = \infty$ or $\int_{[0,t_G)} |H(s)|\, dG(s) = \infty$.*

(ii) *If $\mathsf{P}(\gamma = \infty) > 0$, $\mathsf{E}|L - H(\gamma)|\mathbb{1}_{\{\gamma<\infty\}} < \infty$, and $\int_{\mathbb{R}_+} |H(s)|\, dG(s) < \infty$ then M has type 4.*

Third theorem

Theorem (continued)

(iii) *If* $\mathsf{P}(\gamma = \infty) = 0$, $\mathsf{E}|L - H(\gamma)| < \infty$, *and* $\int_{[0,t_G)} |H(s)|\,dG(s) < \infty$ *then*

- (iii.i) *M has type* 2a (*resp.*, 2b) *if and only if* $\lim_{t \uparrow\uparrow t_G} F(t)\overline{G}(t) > 0$ (*resp.*, $\lim_{t \uparrow\uparrow t_G} F(t)\overline{G}(t) < 0$);
- (iii.ii) *M has type* 3 *if and only if*

$$\lim_{t \uparrow\uparrow t_G} F(t)\overline{G}(t) = 0 \quad \textit{and} \quad \int_{[0,t_G)} \overline{G}(s)\left|\frac{dF}{dG}(s)\right| dG(s) = \infty;$$

- (iii.iii) *M has type* 4 *if and only if*

$$\int_{[0,t_G)} \overline{G}(s)\left|\frac{dF}{dG}(s)\right| dG(s) < \infty \tag{1}$$

Examples

Example

Assume that $H\colon (0,1) \to \mathbb{R}$ is a monotone nondecreasing and right-continuous function. Then it is the upper quantile function of $H(\gamma)$, where γ is uniformly distributed on $(0,1)$. Assume also that H is integrable on $(0,1)$ and $\int_0^1 H(s)\,ds = 0$, that is to say that $H(\gamma)$ has zero mean. Put, according to the second theorem,

$$F(t) = -(1-t)^{-1}\int_0^t H(s)\,ds = (1-t)^{-1}\int_t^1 H(s)\,ds.$$

We see that F is the Hardy–Littlewood maximal function corresponding to H.

Example (continued)

If we define M as above with $L = H(\gamma)$, then, by the third theorem, M is a uniformly integrable martingale with $M_\infty = H(\gamma)$ and $\sup_t M_t = F(\gamma)$. This example is essentially the example of Dubins and Gilat (1978) of a uniformly integrable martingale with a given distribution of its terminal value, having maximal (with respect to the stochastic partial order) maximum (in time).

Example 3.14 in Herdegen and Herrmann (2016)

Example

Let $\Omega = (0, 1]$ be equipped with the Borel σ-field $\mathscr{F}$, and let P be the Lebesgue measure, $\gamma(\omega) = \omega$. Put $H(t) \equiv 0$. Then $F(t) = (1 - t)^{-1}$ satisfies the main equation with $F(0) = 1$. By the third theorem, M defined as above is a supermartingale and local martingale but not a martingale. This seems to be the simplest example of a local martingale with continuous time, which is not a martingale. Note that, for $\omega = 1$, the trajectory $M_t(\omega) = (1 - t)^{-1}\mathbb{1}_{\{t<1\}}$ has not a finite left-hand limit at 1. Moreover, if N is a modification of M, for $t < 1$, the values of $M_t(\omega)$ and $N_t(\omega)$ must coincide on the atom $\{t < \gamma\} = (t, 1]$ of $\mathscr{F}_t$, having the positive measure. Hence, $N_t(\omega) = M_t(\omega)$ for $\omega = 1$ for all $t < 1$. This is an example of a right-continuous supermartingale which has not a modification with *all* cádlág paths. Of course, the usual assumptions are not satisfied in this example.

Thank you for your attention

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