Gradient estimate for SDEs driven by cylindrical Lévy processes

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Based on a joint work with Zhen-Qing Chen[2,3] and Xicheng Zhang[1]

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Outline

- **Introduction**
- **Main results**
- **Proof**

Part 1 : Introduction

Motivation

- Let $d \geqslant 2$. Consider the following stochastic differential equation :

$$\begin{cases} \mathrm{d}X_t = A(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t, \\ X_0 = x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

where $B_t = (B_t^1, ..., B_t^d)$ is a d-dimensional standard Brownian motion, $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function, and $A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a $d \times d$ matrix-valued measurable function and satisfies

(H) $A \in C(\mathbb{R}^d)$ and for some $c_0 \geqslant 0$, it holds that

$$|\det A(x)| \geqslant c_0, \quad x \in \mathbb{R}^d.$$

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$$|\det A(x)| \geqslant c_0, \quad x \in \mathbb{R}^d.$$

- Under the above assumption and b is bounded, it is well known that for each $x \in \mathbb{R}^d$, SDE (1.1) admits a unique weak solution $X_t(x)$(see [1]) . Furthermore, if A and b have more regularities it admits a density $p_t(x,y)$ enjoying the following estimates(see [2]): for any $T > 0$, there are constants $c_i > 0$ such that for all $0 < t < T$ and $x, y \in \mathbb{R}^d$

$$c_1 t^{-d/2} e^{-c_2|x-y|^2/t} \leqslant p_t(x,y) \leqslant c_3 t^{-d/2} e^{-c_4|x-y|^2/t}.$$

[1] Bass, R.F., Diffusions and Elliptic Operators. Springer-Verlag, New York, 1997
[2] Z.-Q. Chen, E. Hu, L. Xie, and X. Zhang, Heat kernels for non-symmetric diffusion operators with jumps.*J. Differential Equations, 263 (2017), 6576-6634.*

- Notice that B_t^i in $B_t = (B_t^1, ..., B_t^d)$ are i.i.d. 1-dimensional standard Brownian motions.
- Naturally, we consider the standard cylindrical α-stable process $L_t = (L_t^1, ..., L_t^d)$ and the following SDE

$$\begin{cases} \mathrm{d}X_t = A(X_{t-})\mathrm{d}L_t + b(X_t)\mathrm{d}t, \\ X_0 = x \in \mathbb{R}^d, \end{cases} \tag{1.2}$$

  where L_t^i are i.i.d. 1-dimensional standard α-stable processes.
- In fact, L_t admits a density $p_t(x)$ enjoying the following estimates :
  For any $T > 0$, there are constants $c_1, c_2 > 0$ such that for all $0 < s < t < T$ and $x \in \mathbb{R}^d$

$$c_1 \Pi_{i=1}^d \frac{t}{(\sqrt{t} + |x_i|)^{\alpha+1}} \leqslant p_t(x) \leqslant c_2 \Pi_{i=1}^d \frac{t}{(\sqrt{t} + |x_i|)^{\alpha+1}}.$$

- However, there is no result for the density estimate for X_t. Actually, the existence of the solution X_t and the density of X_t are not easy questions.

- More generality, we consider the following SDE driven by the cylindrical α-stable process L_t,

$$\begin{cases} \mathrm{d}X_t^x = \int_{\mathbb{R}^d} \sigma(X_{t-}, z) N(\mathrm{d}t, \mathrm{d}z) + b(X_t)\mathrm{d}t, \\ X_0^x = x \in \mathbb{R}^d, \end{cases} \tag{1.3}$$

where $\sigma = (\sigma_i)_{i=1}^d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function, and $N(\mathrm{d}t, \mathrm{d}z)$ is the Poisson random measure of L_t^α defined as follow

$$N((s,t], E) := \sum_{s<u\leqslant t} \mathbf{1}_{(L_u - L_{u-})\in E}.$$

- Define $\nu(E) := \mathbb{E}N([0,1], E)$. For simplify, we assume that for all $x \in \mathbb{R}^d$ and $0 < r < R < +\infty$

$$\int_{r\leqslant|z|\leqslant R} \sigma(x,z)\nu(\mathrm{d}z) = 0.$$

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Questions:

- In what condition of σ and b, there is a weak(or strong) solution of SDE (1.3)?
- If there is a weak solution, does the solution have a density?
- If there is a density, can we get some precise estimates for it?

- When L_t is a d-dimensional standard α-stable process, the infinitesimal generator of X_t^x has the following form

$$\begin{aligned}\mathscr{L}f(x) &= \text{p.v.}\int_{\mathbb{R}^d}\frac{f(x+\sigma(x,z))-f(x)}{|z|^{d+\alpha}}\mathrm{d}z\\ &= \text{p.v.}\int_{\mathbb{R}^d}\frac{f(x+z)-f(x)}{|z|^{d+\alpha}}\kappa(x,z)\mathrm{d}z,\end{aligned} \tag{1.4}$$

where

$$\kappa(x,z)=\frac{|z|^{d+\alpha}}{|\sigma^{-1}(x,z)|^{d+\alpha}}|\det\nabla_z\sigma^{-1}(x,z)|.$$

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$$\kappa(x,z) = \frac{|z|^{d+\alpha}}{|\sigma^{-1}(x,z)|^{d+\alpha}}|\det \nabla_z \sigma^{-1}(x,z)|.$$

- When L_t is a d-dimensional cylindrical α-stable process, which is our case, the infinitesimal generator of X_t^x has the following form

$$\mathscr{L}f(x) = \sum_{i=1}^{d} \text{p.v.}\int_{\mathbb{R}} \frac{f(x+\sigma(x,ze_i)) - f(x)}{|z|^{1+\alpha}}\mathrm{d}z,$$

where $e_i = (0, .., 1(\text{i-}th), .., 0)$.

- Notice that, it is impossible to find such a κ in (1.4) this time.

- Let $\mathscr{F}$ be the Fourier transform. The infinitesimal generator of d-dimensional cylindrical α-stable process is $\sum_{i=1}^d (\partial_i \partial_i)^{\frac{\alpha}{2}}$ with

$$\mathscr{F}\Big(\sum_{i=1}^d (\partial_i \partial_i)^{\frac{\alpha}{2}} f\Big)(\xi) = c \sum_{i=1}^d |\xi_i|^\alpha \mathscr{F}(f)(\xi) := \psi_1(\xi) \mathscr{F}(f)(\xi),$$

where $\psi_1 \in C^\infty(\mathbb{R}^d \setminus (\cup_{i=1}^d \mathbb{R}_i))$, where

$$\mathbb{R}_i := \{x \in \mathbb{R}^d; x_i = 0\}.$$

- The infinitesimal generator of d-dimensional standard α-stable process is $\Delta^{\frac{\alpha}{2}}$ with

$$\mathscr{F}(\Delta^{\frac{\alpha}{2}} f)(\xi) = c|\xi|^\alpha \mathscr{F}(f)(\xi) := \psi_2(\xi) \mathscr{F}(f)(\xi),$$

where $\psi_2 \in C^\infty(\mathbb{R}^d \setminus 0)$.

- Therefore, compared with standard α-stable process, the cylindrical one is more difficult to be dealed with.

Assumptions

(A^{σ}) $\sigma(x,z) = A(x)z$ for some matrix value map $A = (a_{i,j}) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, there is a positive number c_0 such that for any $x, y, \xi \in \mathbb{R}^d$ and all $i, j = 1, ..., d$

$$c_0^{-1}|\xi| \leqslant |A(x)\xi| \leqslant c_0|\xi|, \tag{1.5}$$

$$|a_{i,j}(x) - a_{i,j}(y)| \leqslant c_0|x - y|. \tag{1.6}$$

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(A^b_β) For $\beta \in [0,1]$,

$$\|b\|_{\mathbf{C}^\beta} := \sup_{x\in\mathbb{R}^d} |b(x)| + \sup_{|x-y|\neq 0} \frac{|b(x)-b(y)|}{|x-y|^\beta} < \infty. \tag{1.7}$$

- If there is a solution X_t^x of SDE (1.3), we define

$$P_t^{\sigma,b}\phi(x) = \mathbb{E}(\phi(X_t^x)), \qquad P_t^\sigma := P_t^{\sigma,0}.$$

Well-known results

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Assume $\sigma(x,z)=\sigma(x)z$ is continuous in variable x and $b\equiv 0$. For any bounded domain $D\subset\mathbb{R}^d$, define $\tau_D:=\inf\{t>0,\ X_t^x\notin D\}$. If any bounded function h satisfies

$$h(x)=\mathbb{E}[h(X^x_{\tau_D})] \quad \text{for every } x\in D,$$

then h is Hölder continuous in D.

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2012 (Debussche-Fournier)
Assume that $\sigma(\cdot,z) = \sigma(\cdot)z \in \mathbf{C}^{\theta_1}$ and $(\mathbf{A}_\beta^b)$ with some conditions of θ_1 and θ_2, the solution admits a density, and density is in some Besov space(we will introduce below).

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2017 (Chen-Zhang-Zhao)
Under the conditions $(\mathbf{A}^\sigma)$ and $(\mathbf{A}_\beta^b)$ with $\beta\in(1-\frac{\alpha}{2},1)$, there is a unique strong solution of (1.3).

2018 (Kulczycki-Ryznar-Sztonyk)
Assume $b \equiv 0$ and $\alpha \in (0,1)$. Under the condition $(\mathbf{A}^\sigma)$, for any $\gamma \in (0,\alpha)$, $T > 0$, there is a constant C such that for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$ and $f \in L^\infty(\mathbb{R}^d)$

$$|P_t^\sigma f(x) - P_t^\sigma f(y)| \leqslant C|x-y|^\gamma t^{-\frac{\gamma}{\alpha}} \|f\|_{L^\infty}. \tag{1.8}$$

For any $\gamma \in (0, \frac{\alpha}{d})$, $T > 0$, there is a constant C such that for all $t \in (0,T]$, $x \in \mathbb{R}^d$ and $f \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$

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- Notice that they can **not** deal the case $\alpha \in [1,2)$.
- Hölder index γ can **not** be 1.

Part 2: Our main results

Littlewood-Paley decomposition and Besov space

- Let ϕ_0 be a radial C^∞-function on $\mathbb{R}^d$ with
$$\phi_0(\xi) = 1 \ \text{ for } \ \xi \in B_1 \ \text{ and } \ \phi_0(\xi) = 0 \ \text{ for } \ \xi \notin B_2.$$

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- For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^d$ and $j \in \mathbb{N}$, define
$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi).$$

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$$\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-(j-1)}\xi).$$
- It is easy to see that for $j \in \mathbb{N}$, $\phi_j(\xi) = \phi_1(2^{-(j-1)}\xi) \geqslant 0$ and
$$\text{supp}\phi_j \subset B_{2^{j+1}} \setminus B_{2^{j-1}}, \quad \sum_{j=0}^{k} \phi_j(\xi) = \phi_0(2^{-k}\xi) \to 1, \;\; k \to \infty.$$

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- Notice that $\{\phi_j\}_{j\in\mathbb{N}_0}$ is a partition of unity of
$$\mathbb{R}^d = B_2 \cup \Big(\cup_{j\in\mathbb{N}} \left(B_{2^{j+1}} \setminus B_{2^{j-1}}\right)\Big).$$

- For given $j \in \mathbb{N}_0$, the block operator Δ_j is defined on $\mathscr{S}'$ by

$$\Delta_j f(x) := \mathscr{F}^{-1}(\phi_j \mathscr{F}(f))(x) = \mathscr{F}^{-1}(\phi_j) * f(x)$$
$$= 2^{\cdot m(j-1)} \int_{\mathbb{R}^d} \mathscr{F}^{-1}(\phi_1)(2^{(j-1)}(x-y)) f(y) \mathrm{d}y.$$

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$$\Delta_j = \Delta_j \widetilde{\Delta}_j, \quad \text{where } \widetilde{\Delta}_j := \Delta_{j-1} + \Delta_j + \Delta_{j+1} \text{ with } \Delta_{-1} \equiv 0, \tag{2.1}$$

and Δ_j is symmetric in the sense that

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- The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=0}^{k-1} \Delta_j f = 2^{dk} \int_{\mathbb{R}^d} \check{\phi}_0(2^k(x-y)) f(y) \mathrm{d}y \to f. \tag{2.2}$$

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- We rewrite (2.2) as

$$f = \sum_{j=0}^{\infty} \Delta_j f,$$

which is called the Littlewood-Paley decomposition.

Definition 1 (Besov space)

For any $s \in \mathbb{R}$, the Besov space $\mathbf{B}^s_{\infty,\infty}$ is defined by

$$\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^d) := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{\mathbf{B}^s_{p,\infty}} := \sup_{j \geqslant 0} \left(2^{sj} \|\Delta_j f\|_{L^\infty} \right) < \infty \right\}.$$

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Proposition 2

For any $s \geqslant 0$ *with* $s \notin \mathbb{N}$,

$$\mathbf{C}^s(\mathbb{R}^d) = \mathbf{B}^s_{\infty,\infty}(\mathbb{R}^d),$$

where $\mathbf{C}^{s_2}(\mathbb{R}^d)$ *is the Hölder space.*
Moreover, for any $n \in \mathbb{N}$,

$$\mathbf{C}^n(\mathbb{R}^d) \subset \mathbf{B}^n_{\infty,\infty}(\mathbb{R}^d).$$

Our assumption for σ

($\mathbf{H}^\sigma$) There is a constant $c_0 > 1$ such that for all $x, y, z \in \mathbb{R}^d$ and all $\lambda > 0$

$$\inf_{\omega\in\mathbb{S}^{d-1}} \inf_{\lambda>0} \lambda \sum_{i=1}^{d} |\omega \cdot \sigma(x, \frac{e_i}{\lambda})| \geqslant c_0^{-1}, \tag{2.3}$$

$$|\sigma(x,z) - \sigma(y,z)| \leqslant c_0|x-y||z|.$$

$$c_0^{-1}|z| \leqslant |\sigma(x,z)| \leqslant c_0|z|.$$

Remark 3

- *Notice that condition $\boldsymbol{A}^\sigma$ implies condition $\boldsymbol{H}^\sigma$ here.*
- $\sigma(x,z) = (2 + sinz_1)z$ *satisfies condition $\boldsymbol{H}^\sigma$ but not satisfies condition $\boldsymbol{A}^\sigma$.*

Main Results

Theorem 4

Let $\alpha \in (0,2)$ *and* $\beta \in [0,1]$ *with* $\alpha + \beta > 1$. *Assume* $(\mathbf{H}^\sigma)$ *and one of the following conditions holds:*

(i) $b = 0, \ \beta = 1$;

(ii) $\alpha \in (\frac{1}{2}, 2)$ *and* $b \in \mathbf{C}^\beta$.

Let $X_t(x)$ *be the unique solution of SDE* (1.3) *and define*

$$P_t^{\sigma,b}\varphi(x) := \mathbb{E}\varphi(X_t(x)).$$

Let $\gamma \in [0, \alpha + \alpha \wedge \beta)$ *and* $\eta \in (-((\alpha + \beta - 1) \wedge 1), \gamma]$. *For any* $T > 0$, *there is a constant* $C > 0$ *such that for all* $0 < t \leqslant T$,

$$\|P_t^{\sigma,b}\varphi\|_{\mathbf{B}^\gamma_{\infty,\infty}} \leqslant C(t)^{\frac{\eta-\gamma}{\alpha}} \|\varphi\|_{\mathbf{B}^\eta_{\infty,\infty}}. \tag{2.4}$$

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$$\|P_t^{\sigma,b}\varphi\|_{\mathbf{B}^\gamma_{\infty,\infty}} \leqslant C(t)^{\frac{\eta-\gamma}{\alpha}}\|\varphi\|_{\mathbf{B}^\eta_{\infty,\infty}}. \tag{2.4}$$

- Notice that (2.4) reduced the restriction of the γ in (1.8) from $(0,\alpha)$ to $(0,\alpha+\alpha\wedge\beta)$ by taking $\eta=0$. In particular, we have gradient estimate. Moreover, we can deal with the case $\alpha\geqslant 1$.
- By a way of interpolation, we also get (1.9).

Main Results

Corollary 5

(A) *Let* $\varphi \in \cup_{\eta<(\alpha+\beta-1)\wedge 1}\mathbf{B}^{-\eta}_{\infty,\infty}$. *For any* $0 < t$, $P^{\sigma,b}_t\varphi \in \cap_{\gamma<\alpha+\alpha\wedge\beta}\mathbf{B}^{\gamma}_{\infty,\infty}$ *solves the following backward Kolmogorov equation: for all* $x \in \mathbb{R}^d$,

$$P^{\sigma,b}_{t-t_0}\varphi(x) = P^{\sigma,b}_{t-t_1}\varphi(x) + \int_{t_0}^{t_1} \mathscr{L}^{\sigma,b}P^{\sigma,b}_{t-s}\varphi(x)\mathrm{d}s, \quad 0 \leqslant t_0 < t_1 < t, \tag{2.5}$$

where $\mathscr{L}^{\sigma,b}u(x) := \text{p.v.}\int(u(x+\sigma(x,z)) - u(x))\nu(\mathrm{d}z) + b\cdot\nabla u(x)$.

(B) *For* $\alpha \in (\frac{1}{2}, 2)$, *the following gradient estimate holds: for* $0 < t \leqslant T$,

$$\|\nabla P^{\sigma,b}_t\varphi\|_\infty \leqslant C_T t^{-\frac{1}{\alpha}}\|\varphi\|_\infty. \tag{2.6}$$

(C) *For each* $t > 0$, *the random variable* $X_t(x)$ *admits a density* $p^X_t(x,\cdot)$ *with*

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- When σ is Lipschitz, (2.7) here is better the result in Debussche-Fournier's.

Part 3: Proof

PDE related to SDE

- Naturally we consider the following PDE,

$$\begin{cases}\partial_t u(t,x) = \mathscr{L}_\sigma^\alpha u(t,x) + b(x)\cdot\nabla u(t,x), \\ u(0,x) = \varphi(x),\end{cases} \tag{3.1}$$

where $\varphi \in \mathbf{C}_b^\infty(\mathbb{R}^d)$ and

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Definition 6

We call a function $u(t,x) \in C([0,+\infty); \mathbf{C}^{\alpha+\varepsilon}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d))$ for some $\varepsilon > 0$ be a classical solution of PDE (3.1) in $[0,T]$ if for all $t \in [0,\infty)$ and $x \in \mathbb{R}^d$

$$u(t,x) = \int_0^t \mathscr{L}_\sigma^\alpha u(s,x) + b(x)\cdot\nabla u(s,x)\mathrm{d}s + \phi(x).$$

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  - Fortunately, we have a priori estimate: under the condition $(\mathbf{H}^\sigma)$ and $(\mathbf{A}^b_\beta)$ with $\beta \in ((1-\alpha)\vee 0, \alpha)$, for any $T > 0$ and $\varepsilon \in (0, \beta \wedge \alpha)$, there is a constant C such that for all $t \in [0, T]$, $\varphi \in \mathbf{C}^\infty_b$ and classical solutions u

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- Let u be a classical solution. By Itô formula, $s \to u(t-s, X_s^x)$ is a martingale for $s \in [0, t]$. Then

$$P_t^{\sigma,b}\varphi(x) = \mathbb{E}(\varphi(X_t^x)) = \mathbb{E}(u(t-s, X_s^x)) = \mathbb{E}(u(t,x)) = u(t,x).$$

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- The equality above tell us that if we want to establish any estimate of $P^{\sigma,b}_t\varphi(x)$, it is enough to establish the estimate of classical solution u.
- Moreover, it tell us that the uniqueness of weak solution of SDE (1.3) is equivalent to the uniqueness of classical solution of PDE (3.1).

Crucial lemma

- Let $\theta : \mathbb{R}_+ \to \mathbb{R}^d$ be a measurable function and $p_{s,t}$ be the transition probability of process

$$Z_{s,t} := \int_s^t \int_{\mathbb{R}^d} \sigma(\theta(r), z)\tilde{N}(dz, dr).$$

Lemma 7 (Crucial Lemma)

For any $\beta \in [0, \alpha)$, $\gamma \in [0, +\infty)$ and $T > 0$, there is a constants C such that for $m \in \mathbb{N}_0$ all $j > 0$, $f \in L^1_{loc}(\mathbb{R}_+)$ and $t \in (0, T]$ $s \in [0, t)$,

$$\int_0^t \int_{\mathbb{R}^d} |x|^\beta |\nabla^m \Delta_j p_{s,t}(x)||f(s)| \mathrm{d}x\mathrm{d}s \leqslant C 2^{(m-\gamma-\beta)j} \int_0^t (t-s)^{-\frac{\gamma}{\alpha}} |f(s)| \mathrm{d}s.$$

The key point of proof

- For simplify, we assume $\sigma(x,z) = A(x)z$ for some matrix value map $A : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $a(t) := A(\theta(t))$. Recall that $p_{s,t}$ is the transition probability of

$$Z_{s,t} = \int_s^t a(r) \mathrm{d}L_t^\alpha \quad \text{with} \quad \lambda^{\frac{1}{\alpha}} L_{\lambda t}^\alpha \stackrel{(d)}{=} L_t^\alpha .$$

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Therefore using the change of variable and the scaling property, we have

$$\begin{aligned}\int_s^t a(r)\mathrm{d}L_r^\alpha &= \int_0^{t-s} a(r+s)\mathrm{d}\Big(L_{r+s}^\alpha - L_s^\alpha\Big)\\ &\overset{(d)}{=} (t-s)^{-\frac{1}{\alpha}}\int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha.\end{aligned}$$

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We denote by $\bar{p}_{0,1}$ the density of $\int_0^1 a(r(t-s)+s)\mathrm{d}L_r^\alpha$, then

$$p_{s,t}(x) = (t-s)^{-\frac{d}{\alpha}} \bar{p}_{0,1}((t-s)^{-\frac{1}{\alpha}} x).$$

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- $\inf_{\omega \in \mathbb{S}^{d-1}} \inf_{\lambda > 0} \lambda \sum_{i=1}^d |\omega \cdot \sigma(x, \frac{e_i}{\lambda})| \geqslant c_0^{-1}$ guarantee that for any $n \in \mathbb{N}_0$ and $\beta \in [0,\alpha)$, there is a constant C such that

$$\int_{\mathbb{R}^d} |x|^\beta |\nabla^n \bar{p}_{0,1}(x)| \mathrm{d}x \leqslant C.$$

Our approach

- Firstly, we introduce the characteristic line θ_t^y, which is a solution of following ODE

$$\theta_t^y = y - \int_0^t b(\theta_s^y)\mathrm{d}s,$$

and get a new equation(for simplification we drop the initial y in some places):

$$\begin{cases} \partial_t \tilde{u}(t,x) = \mathscr{L}_{\tilde{\sigma}}^{\alpha} \tilde{u}(t,x) + \tilde{b}(x) \cdot \nabla \tilde{u}(t,x), \\ \tilde{u}(0,x) = \varphi(x+y), \end{cases} \tag{3.3}$$

where $\tilde{u}(t,x) = u(t, x+\theta_t)$,$\tilde{\sigma}(x,z) = \sigma(x+\theta_t, z)$ and $\tilde{b}(x) = b(x+\theta_t) - b(\theta_t)$. Notice that $|\tilde{b}(x)| \lesssim |x|^{\beta}$ which releases the regularity of spatial x.

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- Then let $\tilde{\sigma}_0(z) = \tilde{\sigma}(0,z) = \sigma(\theta_t, z)$ and we have the following presentation

$$\tilde{u}(t,x) = \int_0^t P_{s,t}\Big(\mathscr{L}_{\tilde{\sigma}}^{\alpha} - \mathscr{L}_{\tilde{\sigma}_0}^{\alpha}\Big)\tilde{u}(s,x)\mathrm{d}s + \int_0^t P_{s,t}(\tilde{b} \cdot \nabla \tilde{u})(s,x)\mathrm{d}s$$
$$+ P_{0,t}\varphi(x+y),$$

where $\mathscr{L}_{\tilde{\sigma}_0}^{\alpha}$ is the infinitesimal generator of some process $Z_{s,t}$ introduced in the crucial lemma and $P_{s,t}f(x) := \mathbb{E}f(x+Z_{s,t})$.

- Next step is a highlight point. We take the block operator Δ_j on both sides and only look at the point zero:

$$\Delta_j \tilde{u}(t,0) = \int_0^t \Delta_j P_{s,t}\Big(\mathscr{L}^{\alpha}_{\tilde{\sigma}} - \mathscr{L}^{\alpha}_{\tilde{\sigma}_0}\Big)\tilde{u}(s,0)\mathrm{d}s + \int_0^t \Delta_j P_{s,t}(\tilde{b}\cdot\nabla\tilde{u})(s,0)\mathrm{d}s + \Delta_j P_{0,t}\varphi(y).$$

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- This time, we turn the convolution $P_{s,t}f$ into an inner product $\langle p_{s,t}, f\rangle$. Therefore, we can use our crucial lemma and get the regularity of the space.
- Notice that $\Delta_j u(t,\theta_t) = \Delta_j \tilde{u}(t,0)$. We take the supremum of the initial point of the θ_t and get the estimate of $\|\Delta_j u(t)\|_\infty$. Then by taking supremum of j, we obtain that for any $\gamma \in [0,\alpha)$, $\delta > (\alpha-1)\vee(1-\beta)$ and$\eta \leqslant \gamma$, there is a $\vartheta > -1$ such that :

$$\|u(t)\|_{B^{\gamma}_{\infty,\infty}} \lesssim \int_0^t (t-s)^{\vartheta}\|u(s)\|_{B^{\delta}_{\infty,\infty}}\mathrm{d}s + t^{-\frac{1}{\alpha}(\gamma-\eta)}\|\varphi\|_{B^{\eta}_{\infty,\infty}}.$$

Volterra-type Gronwall inequality

Lemma 8 (Volterra-type Gronwall inequality)

Assume $A > 0$. *For any* $\theta, \vartheta > -1$ *and* $T > 0$, *there exists a constant* $C = C(A, \theta, \vartheta, T) \geqslant 0$ *such that if locally integrable functions* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy*

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- By this type Gronwall inequality, we obtain the main result for $\gamma \in [0, \alpha)$ and $\eta \in (-(\alpha+\beta-1) \wedge 1, \gamma]$.
- To lift the limitation of γ from $[0, \alpha)$ to $[0, \alpha + \alpha \wedge \beta)$, we need a lift lemma and the semigroup property of Feller process.
- The proof can be found in [1].

[1] X. Zhang, Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. *J.Funct. Anal., 258 (2010), 1361-1425.*

Lift lemma

Lemma 9

Assume one of the following conditions holds,

- $\alpha \in (0,2)$, $b \equiv 0$ *and let* $\beta = 1$.
- $\alpha \in (\frac{1}{2}, 2)$ *and condition* $(\mathbf{A}^b_\beta)$ *holds with* $\beta \in ((1-\alpha) \vee 0, \alpha \wedge 1)$.

Under condition $(\mathbf{H}^\sigma)$, *for any*

$$\gamma \in (\alpha, \alpha + \alpha \wedge \beta), \quad \delta \in [0, \alpha),$$

there is a constant C_T *such that for all* $\phi \in C_0^\infty(\mathbb{R}^d)$ *and all* $t \in (0, T]$,

$$\|P_t^{\sigma,b}\phi\|_{B^\gamma_{\infty,\infty}} \leqslant C_T t^{-\frac{\delta}{\alpha}} \|\phi\|_{B^{\gamma-\delta}_{\infty,\infty}}. \tag{3.4}$$

- Notice that $P_t^{\sigma,b}\phi = P_{\frac{t}{2}}^{\sigma,b} P_{\frac{t}{2}}^{\sigma,b}\phi$ and $(\alpha, \alpha + \alpha \wedge \beta) - \alpha \subset (0, \alpha)$, by this C-K property, we obtain the main result.

Some techniques

Lemma 10 (Chen-Zhang-Zhao 2017)

For $\beta \in (0,1)$ *and* $\theta \in (-\beta, 0]$, *there is a constant* C *such that*

$$\|[\Delta_j, f]g\|_\infty \leqslant C 2^{-j(\beta+\theta)} \|f\|_{C^\beta} \|g\|_{B^\theta_\infty},$$

where $[\Delta_j, f]g := \Delta_j fg - f\Delta_j g$.

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- Step 1: prove it when $\theta = 0$. By definition of block operator, it is easy.
- Step 2: make the Bony decomposition:

$$fg = \sum_{i,j\in\mathbb{N}_0} \Delta_i f \Delta_j g = \sum_{i>j+1} \Delta_i f \Delta_j g + \sum_{j>i+1} \Delta_i f \Delta_j g + \sum_{|i-j|\leqslant 1} \Delta_i f \Delta_j g$$
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- Step 3:

$$[\Delta_j, f]g = \Big(\Delta_j f < g - f < \Delta_j g\Big) + \Big(\Delta_j f \circ g - f \circ \Delta_j g + \Delta_j f > g - f > \Delta_j g\Big),$$

the estimate of the first one from step 1 and the second part has a higher regularity.

- To estimate $\int_0^t \Delta_j P_{s,t}\Big(\mathscr{L}^\alpha_{\tilde\sigma} - \mathscr{L}^\alpha_{\tilde\sigma_0}\Big)\tilde u(s,0)\mathrm{d}s$, we define

$$\mathscr{D}^y_z f(x) = f(x+\sigma(x+\theta^y_t, z)) - f(x+\sigma(\theta^y_t, z)) - \mathbb{1}_{\alpha\geqslant 1}\tilde\sigma(x,z)\cdot\nabla f(x),$$

- and

$$\mu_\theta(h) := \int_{\mathbb{R}^d}(1\wedge|x|)^\theta|h(x)|dx \quad , \quad \langle f, g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx.$$

- To estimate $\int_0^t \Delta_j P_{s,t}\left(\mathscr{L}^{\alpha}_{\tilde\sigma} - \mathscr{L}^{\alpha}_{\tilde\sigma_0}\right)\tilde u(s,0)\mathrm{d}s$, we define

$$\mathscr{D}^y_z f(x) = f(x+\sigma(x+\theta^y_t, z)) - f(x+\sigma(\theta^y_t, z)) - \mathbb{1}_{\alpha\geqslant 1}\tilde\sigma(x,z)\cdot\nabla f(x),$$

- and

$$\mu_\theta(h) := \int_{\mathbb{R}^d}(1\wedge|x|)^\theta|h(x)|dx \quad , \quad \langle f,g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Lemma 11

For any $\theta\in[0,1]$, there exists a constant $C=C(d,\theta)>0$ such that for all $|z|\leqslant \frac{1}{2c_0}$, $f\in C^\theta$ and $g\in C^2$

$$|\langle\mathscr{D}^y_z f, g\rangle| \leqslant C|z|^\theta\|f\|_\infty\left[\mu_0(|g|)+\mu_\theta(|\nabla g|)^\theta\mu_\theta(|g|)^{1-\theta}\right]$$

when $\alpha<1$ and

$$|\langle\mathscr{D}^y_z f, g\rangle| \leqslant C|z|^{1+\theta}\|f\|_{\mathbf{C}^\theta}\left[\mu_0(|g|)+\mu_1(|\nabla g|)+\mu_{1+\theta}(|\nabla^2 g|)^\theta\mu_{1+\theta}(|\nabla g|)^{1-\theta}\right]$$

when $\alpha\geqslant 1$.

The key point of the proof

- For simplicity, we assume $\alpha < 1$ and $\phi_z(x) = \sigma(x + \theta_t^y, z)$. Rewrite

$$\mathscr{D}_z f(x) := \mathscr{D}_z^y f(x) = f(x + \phi_z(x)) - f(x + \phi_z(0)).$$

- We can let $\bar{f}(x) = f(x + \phi_z(0))$. Their C^θ norms are the same. Therefore we assume that $\phi_z(0) = 0$ and there is a constant such that $|\phi_z(x)| \leqslant C(|x| \wedge 1)|z|$.

The key point of the proof

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- Let $\Gamma_z(x) = x + \phi_z(x)$. By change of variable, we have

$$\langle \mathscr{D}_z f, g \rangle = \langle f, \mathscr{D}_z^* g \rangle,$$

where

$$\mathscr{D}_z^* g(x) = \det(\nabla_x \Gamma_z^{-1}(x)) g(\Gamma_z^{-1}(x)) - g(x).$$

- Noticing that

$$|\det(\nabla_x \Gamma_z^{-1}(x)) - 1| \leqslant |z|, \quad \text{and} \quad |\Gamma_z^{-1}(x) - x| \leqslant C^2(|x| \wedge 1)|z|,$$

we complete the proof.

Thanks!