Remarks on asymptotical independence

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Problem.

The talk is devoted to the discussion of the following question:

Under what condition two given sequences (X_n) and (Y_n) of random elements could be considered as asymptotically independent?

Intuitive answer:

The common distribution $\mathcal{P}_{(X_n,Y_n)}$ has to be close to the product of marginal distributions $\mathcal{P}_{X_n} \times \mathcal{P}_{Y_n}$.

Definitions

It is reasonable to define asymptotic independence (AI) as the merging of the distributions $P_{(X_n,Y_n)}$ and $P_{X_n} \times P_{Y_n}$.

AI-2

For each $A \in \mathcal{E}_1, B \in \mathcal{E}_2$,

$$\mathcal{P}_{(X_n,Y_n)}(A \times B) - \mathcal{P}_{X_n}(A) \cdot \mathcal{P}_{Y_n}(B) \to 0, \quad n \to \infty.$$

AI-3

$$\sup_{A\in\mathcal{E}_1, B\in\mathcal{E}_2} |\mathcal{P}_{(X_n,Y_n)}(A \times B) - \mathcal{P}_{X_n}(A) \cdot \mathcal{P}_{Y_n}(B)| \to 0, \quad n \to \infty.$$

AI-4

$$\| \mathcal{P}_{(X_n,Y_n)} - \mathcal{P}_{X_n} \times \mathcal{P}_{Y_n} \|_{var} \to 0, \quad n \to \infty.$$

Obviously, **AI-4**$\Rightarrow$**AI-3**$\Rightarrow$**AI-2**.

Polish spaces, I.

Now we consider the case when E_1 and E_2 are Polish (that is complete separable metric) spaces with metrics d_1 and d_2 respectively.

Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are Borel σ-algebras of E_1 and E_2. Consider the space $E_1 \times E_2$ endowed with the product topology.

We can suppose that it is generated by one of the metrics
$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$ or
$r((x_1, y_1), (x_2, y_2)) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$.

These metrics are equivalent: $r \leq d \leq 2r$.

Polish spaces, II.

In this case we can use the notions of weak convergence of measures and merging of measures.

We say that two sequences $(\mu_n),\ (\nu_n)$ of probability measures are *merging* if

$$\pi(\mu_n,\ \nu_n)\ \to 0, \quad n \to \infty,$$

where π is the *Lévy-Prokhorov metric*

$$\pi(\mu,\nu) = \inf\{\varepsilon : \mu(A^{\varepsilon}) \leq \nu(A) + \varepsilon \ \text{ for all closed sets } A\}. \tag{1}$$

Polish spaces, III.

Now considering asymptotic independence, it is natural to introduce the following condition:

$$(P_{(X_n,Y_n)}) \text{ merges with } (P_{X_n} \times P_{Y_n}) \text{ when } n \to +\infty.$$

Due to [D&Rotar' (2009), Theorem 1] it is equivalent to the following:

Condition 1 (AI-1).

For each bounded **uniformly continuous** *function* $h : E_1 \times E_2 \to \mathbb{R}$,

$$\int h(x,y) P_{(X_n,Y_n)}(dx,dy) - \int h(x,y)(P_{X_n} \times P_{Y_n})(dx,dy) \to 0,$$

when $n \to +\infty$.

Polish spaces, IV.

We can also suggest a weaker condition:

AI-0:

For each bounded **uniformly continuous** *functions* $f : E_1 \to \mathbb{R}, \ g : E_2 \to \mathbb{R}$,

$$\mathbb{E}f(X_n)g(Y_n) - \mathbb{E}f(X_n)\mathbb{E}g(Y_n) \to 0,$$

when $n \to +\infty$.

It is clear that **AI-1** implies **AI-0**.

AI for σ-algebras

In theory, it would be possible to use a different approach by considering the AI of sequences of σ-algebras.

Let $(\mathfrak{M}_n)$, $(\mathfrak{N}_n)$ be two sequences of sub-σ-algebras of the main probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. Consider two conditions:

I. (Analog of **AI-2**). For each $A \in \mathfrak{M}_n, \; B \in \mathfrak{N}_n$,

$$\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \to 0, \quad n \to +\infty.$$

II. (Analog of **AI-3**).

$$\alpha(\mathfrak{M}_n, \mathfrak{N}_n) := \sup_{A \in \mathfrak{M}_n, B \in \mathfrak{N}_n} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \to 0, \quad n \to +\infty.$$

Here $\alpha(\mathfrak{M}_n, \mathfrak{N}_n)$ is nothing but the coefficient of α-mixing.

Weak dependence. Mixing.

Let $X = (\xi_n)_{n\in\mathbb{Z}}$ be a strictly stationary sequence.
Let Y_n be the shifted sequence: $(Y_n)_k = \xi_{n+k}$.
We consider X and Y_n as elements of the space $(E, \mathcal{E})$ where $E = R^{\mathbb{Z}}$ and $\mathcal{E}$ is its natural σ-algebra.

Mixing:

For each $A, B \in \mathcal{E}$

$$P\{X \in A,\ Y_n \in B\} \to P\{X \in A\}P\{X \in B\}$$

If we take $X_n \equiv X$ then it is equivalent to

$$P\{X_n \in A,\ Y_n \in B\} - P\{X_n \in A\}P\{Y_n \in B\} \to 0,$$

which means that **Mixing** is a particular case of **AI-2**.

Strong mixing.

Let X_n be restriction of X on $\{\dots,-1,0\}$ and Y_n be restriction of X on $\{n,n+1,\dots\}$. Let $\mathcal{M}_a^b=\sigma\{\xi_a,\dots,\xi_b\}$.

Strong mixing:

As $n\to\infty$:

$$\sup_{A\in\mathcal{M}_{-\infty}^0,B\in\mathcal{M}_n^\infty}|P\{X_n\in A,\ Y_n\in B\}-P\{X_n\in A\}P\{Y_n\in B\}|\to 0.$$

Strong mixing is a particular case of **AI-3**.

Complete regularity.

As before X_n is the restriction of X on $\{\ldots, -1, 0\}$ and Y_n is the restriction of X on $\{n, n+1, \ldots\}$.

Kolmogorov's condition of complete regularity:

$$\| \mathcal{P}_{(X_n, Y_n)} - \mathcal{P}_{X_n} \times \mathcal{P}_{Y_n} \|_{var} \to 0, \quad n \to \infty.$$

Complete regularity is a particular case of **AI-4**.

Sufficient conditions.

Proposition 2.

Suppose that (X'_n), (Y'_n) *are such that*
1) X'_n *and* Y'_n *are independent for each* n;
2) $P\{X'_n \neq X_n\} \to 0; \;\; P\{Y'_n \neq Y_n\} \to 0.$

Then **AI-4** *is satisfied.*

Proposition 3.

Suppose that X_n, Y_n *are conditionally independent given* Ω_n *and* $P(\Omega_n) \to 1$.

Then **AI-3** *is satisfied.*

Tight sequences, I

Proposition 4.

Suppose that (X_n), (Y_n) *are tight sequences in* $E_1 = \mathbb{R}^{d_1}$, $E_2 = \mathbb{R}^{d_2}$. *The following conditions are equivalent:*

1) **AI-1**;

2) For each $t \in \mathbb{R}^{d_1}$, $s \in \mathbb{R}^{d_2}$, *we have convergence of characteristic functions*

$$f_{(X_n,Y_n)}(t,s) - f_{X_n}(t) f_{Y_n}(s) \to 0.$$

Connections between AI-0 - AI-4

Proposition 5.

The condition **AI-4** implies **AI-1**.

Proposition 6.

The condition **AI-2** implies **AI-0**.

In general **AI-3** does not imply **AI-1**. In particular, **AI-0** does not imply **AI-1**.

Connections between AI-0 - AI-4. Tight sequences, II

Proposition 7.

Suppose that (P_{X_n}) and (P_{Y_n}) are tight. The following implications take place:

$$\textbf{AI-4} \Rightarrow \textbf{AI-3} \Rightarrow \textbf{AI-2} \Rightarrow \textbf{AI-1} \Rightarrow \textbf{AI-0}.$$

Moreover, in this case **AI-0** $\Rightarrow$ **AI-1**.

Counterexamples

Proposition 8.

Let $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}$. There exist sequences (X_n) and (Y_n) of random variables satisfying **AI-3** but not **AI-1**.

As **AI-0** always follows from **AI-3** we immediately deduce

Corollary 1.

Property **AI-0** does not imply **AI-1**.

Remark

Let $U_1,\ U_2$ and U be the spaces of bounded and *uniformly continuous* real functions defined respectively on $E_1,\ E_2$ and $E_1 \times E_2$.

Let H be the closed subspace of U formed by all linear combinations of the form $\sum_1^n f_i(x)g_i(y)$, where $f_i \in U_1,\ g_i \in U_2,\ n \in \mathbb{N}$.

If E_1 and E_2 are compact, it follows from the Stone-Weierstrass theorem (Dunford and Schwartz, [VI.6.Th.16]) that H coincides with U, and it allows to easily deduce **AI-1** from **AI-0**.

Proposition 8 shows that even in locally compact spaces the equality $H = U$ may fail.

AI under transformations.

Let (X_n), (Y_n) be two **AI** sequences and f, g two functions. Does the transformed sequences $(f(X_n))$, $(g(Y_n))$ remain **AI**?

Proposition 6.

1) If (X_n), (Y_n) satisfy **AI-0** then $(u(X_n))$, $(v(Y_n))$ satisfy **AI-0** for all *uniformly continuous* functions u, v.
2) If (X_n), (Y_n) satisfy **AI-1** then $(u(X_n))$, $(v(Y_n))$ satisfy **AI-1** for all *uniformly continuous* functions u, v.
3) If (X_n), (Y_n) satisfy **AI-2** then $(u(X_n))$, $(v(Y_n))$ satisfy **AI-2** for all *measurable* functions u, v.
4) If (X_n), (Y_n) satisfy **AI-3** then $(u_n(X_n))$, $(v_n(Y_n))$ satisfy **AI-3** for all *measurable* functions u_n, v_n.
5) If (X_n), (Y_n) satisfy **AI-4** then $(u_n(X_n))$, $(v_n(Y_n))$ satisfy **AI-4** for all *measurable* functions u_n, v_n.

Concluding remarks.

• It is clear that conditions **AI-0 - AI-4** can naturally be modified for mutual asymptotic independence of several (more than 2) random sequences. At the same time analogs of all main given results will remain true.

Open questions

• Find sufficient conditions for AI of the following type:
If $(f(X_n))$, $(g(Y_n))$ are AI for all f, g belonging to some classes $\mathcal{F}_1$, $\mathcal{F}_2$ of functions, then (X_n), (Y_n) are AI.

• Does **AI-0** imply **AI-1** if only one of the sequences P_{X_n} and P_{Y_n} is tight?

• It is interesting to consider conditions for AI for random elements of concrete spaces (such as: space of sequences, space $C[0,1]$, space of configurations and so on...).

References

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