Front Propagation for Reaction-Diffusion Equations in Composite Structures

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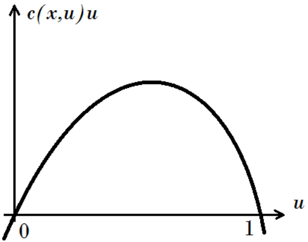
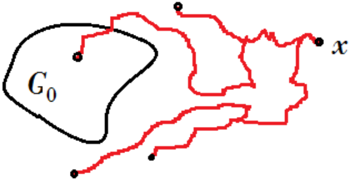
Reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x,u)u, \quad t > 0, \ x \in \mathbb{R}^n,$$

$$u(0,x) = g(x) \in [0,1].$$

The nonlinear term is of Kolmogorov–Petrovskii–Piskunov (KPP) type. The latter means that $c(x,1) = 0, c(x,u) < 0$ for $u > 1$, and $c(x,0) > c(x,u) > 0$ for $u \in (0,1)$ and $x \in \mathbb{R}^n$. Assume that $0 \leq g \leq 1$ is continuous with compact support G_0. (We could also allow g to be continuous everywhere except a smooth hypersurface.)





Y_t^x - diffusion with generator $Mu = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}$.
X_t^x - branching diffusion (branching rate $\alpha(x)$).

$$v(t,x) = \mathrm{P}(X_t^x \bigcap G_0 \ \text{ is empty}).$$

$$v(t+\delta, x) \approx (1-\alpha(x)\delta)\mathrm{E}v(t, Y_\delta^x) + \delta\alpha(x)v^2(t,x).$$

$$\frac{\partial v}{\partial t} = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + \alpha(x)v(v-1), \quad t>0, \ x \in \mathbb{R}^n,$$

$$u(t,x) = 1 - v(t,x) = \mathrm{P}(X_t^x \bigcap G_0 \ \text{ is nonempty}).$$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \alpha(x)u(1-u), \quad t>0, \ x \in \mathbb{R}^n,$$

$$u(0,x) = \chi_{G_0}(x).$$

This is the familiar RDE with $c(x,u)u = \alpha(x)u(1-u)$ and $g = \chi_{G_0}$.

We'll be interested in RDEs with small diffusion term. Need to look at large times to observe the non-trivial evolution of solutions.

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + c(x, u^\varepsilon) u^\varepsilon, \quad t > 0, \; x \in \mathbb{R}^n,$$

$$u^\varepsilon(0, x) = g(x) \geq 0.$$

This is a small perturbation of the system (dynamical system on the space of functions)

$$\frac{\partial u}{\partial t} = c(x, u) u, \quad t > 0, \; x \in \mathbb{R}^n,$$

for which the set of limit points $\mathcal{M}$ consists of indicator functions, i.e.,

$$\lim_{t \to \infty} u(t, \cdot) = \chi_G \in \mathcal{M}.$$

Observe that $\lim_{\varepsilon \downarrow 0} \|u^\varepsilon(t, \cdot) - u(t, \cdot)\| = 0$ on long (but ε-independent) time intervals, while

$$\lim_{t \to \infty} u(t, \cdot) = \chi_G \in \mathcal{M}.$$

Thus, the evolution of $u^\varepsilon(t, \cdot)$ on longer (time-dependent) time intervals can be considered as a process on $\mathcal{M}$. (In general, the long-time evolution of a perturbed dynamical system or a Markov family can be considered on the space of limiting invariant probability measures of the unperturbed system).

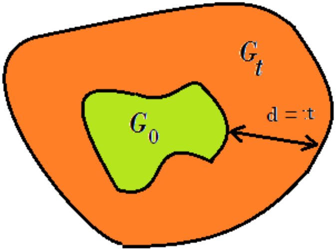
Rescale time in the RDE (by factor $1/\varepsilon$), so that the effective evolution can be observed at finite time scales.

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{1}{\varepsilon} c(x, u^\varepsilon) u^\varepsilon, \quad t > 0, \ x \in \mathbb{R}^n,$$

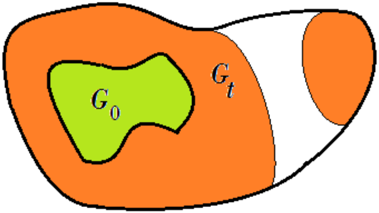
$$u^\varepsilon(0, x) = g(x) \geq 0.$$

There are closed sets G_t such that

$$\lim_{t \to \infty} u^\varepsilon(t, x) = \begin{cases} 1, & x \in \mathrm{Int}(\mathrm{G_t}) \\ 0, & x \notin G_t, \end{cases} \quad .$$

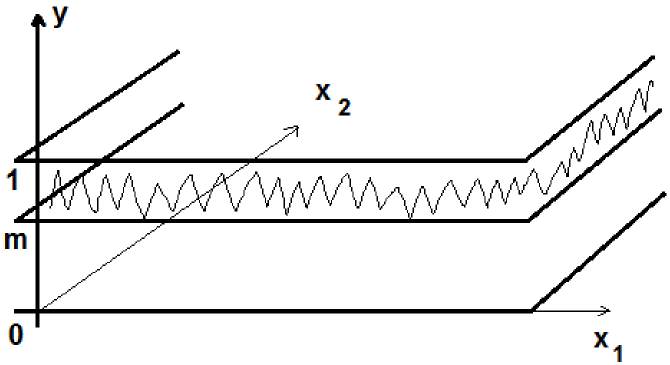


c does not depend on x



c depends on x

Asymptotically thin layer is equivalent to a fixed layer, but with fast diffusion across.



The RDE in a structure with two layers has the form

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} a_{ij}(x,y)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{\varepsilon^{-\beta}}{2}\alpha(x,y)\frac{\partial^2 u^\varepsilon}{\partial y^2} +$$

$$+\frac{1}{\varepsilon}c(x,y,u^\varepsilon)u^\varepsilon,\ t>0,\ x\in\mathbb{R}^n,\ y\in I_1\cup I_2,$$

where $I_1=(0,m)$, and $I_2=(m,1)$. To account for different layers, we assume that

$$a_{ij}(x,y)=\begin{cases} a^1_{ij}(x),\ y\in I_1 \\ a^2_{ij}(x),\ y\in I_2, \end{cases} \qquad \alpha(x,y)=\begin{cases} \alpha^1(x),\ y\in I_1 \\ \alpha^2(x),\ y\in I_2. \end{cases}$$

$$c(x,y,u)=\begin{cases} c^1(x,u),\ y\in I_1, \\ c^2(x,u),\ y\in I_2. \end{cases}$$

We still need initial (and now also boundary) conditions:

$$\frac{\partial u^\varepsilon}{\partial y}|_{y=0,1} = 0, \quad u^\varepsilon(0, x, y) = g(x), \quad \mathrm{supp}(g) = G_0.$$

What happens at the interface? Equation is not satisfied there. Need 'gluing' conditions. Best to introduce a diffusion process and relate u^ε to itself using the Feynman-Kac formula. Let

$$dX_t^\varepsilon = \sqrt{\varepsilon} A(X_t^\varepsilon, Y_t^\varepsilon) dW_t, \quad X_0^\varepsilon = x,$$

$$dY_t^\varepsilon = \frac{1}{\sqrt{\varepsilon^\beta}} \sigma(X_t^\varepsilon, Y_t^\varepsilon) dV_t, \quad Y_0^\varepsilon = y,$$

where the generator of $(X_t^\varepsilon, Y_t^\varepsilon)$ is the diffusion term in the RDE. The process Y_t^ε is reflected at the end points and satisfies a gluing condition at $y = m$.

We define the solution of the RDE as the bounded continuous function on $[0, \infty) \times \mathbb{R}^n \times [0, 1]$ that sastisfies, for each t, x, y,

$$u^\varepsilon(t, x, y) =$$

$$\mathrm{E}_{(x,y)} \left(g(X_t^\varepsilon) \exp(\varepsilon^{-1} \int_0^t c(Y_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon, Y_s^\varepsilon) ds) \right).$$

We study the asymptotics of u^ε as $\varepsilon \downarrow 0$ for various values of β.

New effects: The metric (describing the growth of solution to the linearized problem) is not Riemannian. The evolution of G_t may be non-local even if c does not depend on x (if a_{ij} depend on x).

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{\varepsilon^{-\beta}}{2}\alpha(y)\frac{\partial^2 u^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon}c(y)u^\varepsilon, \;\; t > 0,\; x \in \mathbb{R}^n,\; y \in (0,1)\setminus\{m\},$$

$$\frac{\partial u^\varepsilon}{\partial y}|_{y=0,1} = 0, \quad u^\varepsilon(0,x,y) = g(x).$$

We will show that there is a continuous function $\lambda(t,x)$ such that $\varepsilon \ln u^\varepsilon(t,x,y) \to \sup_{x' \in G_0} \lambda(t, x - x')$. The expressions for $\lambda(t,x)$ are different, depending on whether $\beta = 1$, $\beta > 1$, or $-1 < \beta < 1$. (If $\beta = -1$, there is no need to distinguish between the x and y variables. If $\beta < -1$, then, in order to find the asymptotics of $u^\varepsilon(t,x,y)$ with $y \neq m$, the equation can be viewed in the (t,x) space, with the diffusion in the y variable ignored, and the y variable in the coefficients treated as a parameter.)

Let

$$Lu(y) = \frac{1}{2}\alpha(y)u''(y)$$

be the operator on $C([0,1])$ with the domain $D(L)$ that consists of functions satisfying

$$u \in C^1([0,1]), \quad \alpha u'' \in C([0,1]), \quad u'(0) = u'(1) = 0.$$

Let Y_t^ε be the process s with values on $[0,1]$, whose generator is $\varepsilon^{-\beta}L$. Thus, if the initial value of the process Y_t^ε is y, the process formally satisfies

$$dY_t^\varepsilon = \frac{1}{\sqrt{\varepsilon^\beta}}\sigma(Y_t^\varepsilon)dV_t, \quad Y_0^\varepsilon = y,$$

where $\sigma = \sqrt{\alpha}$ and V_t is a one-dimensional Brownian motion. (Y_t^ε is reflected at the end points of the segment and satisfies a gluing condition at $y = m$.)
Given initial values $X_0^\varepsilon = x$ and $Y_0^\varepsilon = y$, define

$$X_t^\varepsilon = x + \sqrt{\varepsilon}\int_0^t A(Y_s^\varepsilon)dW_s,$$

where A is the positive-definite symmetric square root of the matrix $a = (a_{ij})$ and W_t is an n-dimensional Brownian motion independent of V_t. Note that X_t^ε also depends on β, although this is not reflected in the notation.

For $f \in C([0,1])$, let $H(f)$ be the top eigenvalue of the operator $L_f u = Lu + fu$.

$\mathcal{M}_{[0,1]}$ – the space of probability measures on $([0,1], \mathcal{B}([0,1]))$.

$\mu^\varepsilon_{t,y}$ – the normalized occupation measure on $([0,1], \mathcal{B}([0,1]))$ of the process Y^ε (with $Y^\varepsilon_0 = y$) on the interval $[0,t]$, i.e.,

$$\mu^\varepsilon_{t,y}(B) = \frac{1}{t}\int_0^t \chi_B(Y^\varepsilon_s)ds, \quad B \in \mathcal{B}([0,1]).$$

For $\mu \in \mathcal{M}_{[0,1]}$, define

$$I(\mu) = \sup_{f \in C([0,1])} \left(\int_0^1 f d\mu - H(f)\right).$$

Then tI is the action functional for $\mu^\varepsilon_{t,y}$ (Gartner).
We're mostly interested in $\mu^\varepsilon_{t,y}(I_1)$ and $\mu^\varepsilon_{t,y}(I_2)$.

Let

$$J = \{p = (p_1, p_2) : p_1 + p_2 = 1,\ \ p_1, p_2 \geq 0\}.$$

This space is endowed with the metric $d_J((p_1', p_2'), (p_1'', p_2'')) = |p_1' - p_1''|$. For $p \in J$ and $\mu \in \mathcal{M}_{[0,1]}$ with $\mu(\{m\}) = 0$, define $p_\mu = (\mu(I_1), \mu(I_2))$ and

$$S(p) = \inf_{\mu : p_\mu = (p_1, p_2)} I(\mu).$$

Thus tS is the action functional, uniformly in $(t, y) \in [a, b] \times [0, 1]$, for the family of measures on J induced by the random vectors $(\mu^\varepsilon_{t,y}(I_1), \mu^\varepsilon_{t,y}(I_2))$. Such measures (which also depend on β) will be denoted by $\Lambda^\varepsilon_{t,y}$, i.e.,

$$\Lambda^\varepsilon_{t,y}(A) = \mathrm{P}(p_{\mu^\varepsilon_{t,y}} \in A), \quad A \in \mathcal{B}(J).$$

Main idea: Assuming that p_1 and p_2 are known, derive the expression for the contribution to the expectation in the Feynman-Kac formula, and then maximize the expression under the condition that $p_1 + p_2 = 1$.

Let $a^1 = (a^1_{ij}), a^2 = (a^2_{ij})$. For $v \in \mathbb{R}^n$, define

$$R(p,v) = \frac{1}{2}((p_1 a^1 + p_2 a^2)^{-1} v, v), \quad T(p) = p_1 c^1 + p_2 c^2.$$

Theorem:

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln(u^\varepsilon(t,x,y)) = \sup_{x' \in G_0} \lambda(t, x - x'),$$

where

$$\lambda(t,x) = \sup_p (t(T(p) - S(p) - R(p, \frac{x}{t}))), \qquad \beta = 1,$$

$$\lambda(t,x) = t(T(p_\pi) - R(p_\pi, \frac{x}{t})), \qquad \beta > 1,$$

where π – invariant measure for $Y^{y,\varepsilon}_t$ (doesn't depend on ε or β),

$$\lambda(t,x) = \sup_p (t(T(p) - R(p, \frac{x}{t}))), \qquad -1 < \beta < 1.$$

$M_t^{x,p,\varepsilon}$ – the measure on $\mathcal{C} = C([0,t], \mathbb{R}^n)$ induced by the process X_t^ε conditioned on $\Lambda^\varepsilon_{t,y}(\{p\}) = 1$ (there is no dependence on y or β in $M_t^{x,p,\varepsilon}$).
For $\beta = 1$, by the Feynman-Kac formula,

$$u^\varepsilon(t,x,y) = \mathrm{E}_{(x,y)}\left(g(X_t^\varepsilon)\exp(\varepsilon^{-1}\int_0^t c(Y_s^\varepsilon)ds)\right) =$$

$$\int_J \exp(\varepsilon^{-1} t(c^1 p_1 + c^2 p_2)) \int_{\mathcal{C}} g(\varphi_t) dM_t^{x,p,\varepsilon}(\varphi) d\Lambda^\varepsilon_{t,y}(p).$$

The asymptotics of the interior integral:

$$\lim_{\varepsilon\downarrow 0}\left(\varepsilon \ln \int_{\mathcal{C}} g(\varphi_t) dM_t^{x,p,\varepsilon}(\varphi)\right) = -\inf_{x'\in G_0} tR(p, \frac{x-x'}{t}),$$

Substituting this in the formula above, we get

$$\lim_{\varepsilon\downarrow 0}\varepsilon\ln(u^\varepsilon(t,x,y)) = \lim_{\varepsilon\downarrow 0}\varepsilon\ln\int_J \exp\left(\varepsilon^{-1} t(T(p) - \inf_{x'\in G_0} R(p, \frac{x-x'}{t}))\right) d\Lambda^\varepsilon_{t,y}(p).$$

When $\beta = 1$, since tS is the action functional for the family $\Lambda^\varepsilon_{t,y}$,

$$\lim_{\varepsilon\downarrow 0}\varepsilon\ln(u^\varepsilon(t,x,y)) = \sup_{x'\in G_0}\sup_p(t(T(p) - S(p) - R(p, \frac{x-x'}{t}))).$$

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{\varepsilon^{-\beta}}{2}\alpha(y)\frac{\partial^2 u^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon}c(y,u^\varepsilon)u^\varepsilon, \ t>0, \ x\in\mathbb{R}^n, \ y\in I_1\cup I_2,$$

$$\frac{\partial u^\varepsilon}{\partial y}|_{y=0,1} = 0, \quad u^\varepsilon(0,x,y) = g(x).$$

We can consider the corresponding linear problem, replacing the nonlinearity by $\frac{1}{\varepsilon}c(y,0)u^\varepsilon$. The function λ (exponential growth rate) has been defined for the linear equation.

Define the norm $\|x\|$ via the condition $\lambda(\|x\|,x)=0$. Let $d(x_1,x_2)=\|x_1-x_2\|$. Define

$$G_t = \{x\in\mathbb{R}^n : d(x,G_0)\le t\}.$$

Theorem: If $u^\varepsilon(t,x,y)$ is the solution of the RDE of KPP type (with x-independent coefficients and non-linearity), then, for each $t > 0$,

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x,y) = 0$$

uniformly on every compact $K \subset (\mathbb{R}^n \setminus G_t) \times [0,1]$, and

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x,y) = 1$$

uniformly on every compact $K \subset \mathrm{Int}(G_t) \times [0,1]$.

Main steps of the proof: (a) If the solution of the linearized equation is small, then the solution of the RDE (which is smaller) is also small. So the first statement is simple.

(b) Now consider a compact K such that $K \subset \mathrm{Int}(G_t) \times [0,1]$. Let $\eta > 0$ and $(x_0, y_0) \in K$. Assume that $x_0 \notin G_0$. Let $t_0 = d(x_0, G_0) < t$. Let us show that there is $\delta > 0$ such that

$$u^\varepsilon(t_0, x, y) \geq \exp(-\varepsilon^{-1}\eta)$$

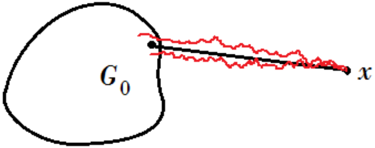
for all sufficiently small ε when $\|x - x_0\| \leq \delta$. **In other words, at a slightly earlier time, the solution of RDE is almost of order one** - it is then easy to grow from a not-too-small value to 1.

(c) In the linear problem, the main contribution to the solution at (t_0, x, y) comes from the near-linear paths, in the Feynman-Kac formula, connecting x to the appropriate point in G_0. Now consider the contribution from a neighborhood of a slightly sub-optimal path.

$$u^\varepsilon(t_0, x, y) =$$

$$\mathrm{E}_{(x,y)}\left(g(X^\varepsilon_{t_0})\exp(\varepsilon^{-1}\int_0^{t_0} c(Y^\varepsilon_s, u^\varepsilon(t_0 - s, X^\varepsilon_s, Y^\varepsilon_s))ds)\right) \geq$$

$$\mathrm{E}_{(x,y)}\left(g(X^\varepsilon_{t_0})\exp(\varepsilon^{-1}\int_0^{t_0} c(Y^\varepsilon_s, u^\varepsilon(t_0 - s, X^\varepsilon_s, Y^\varepsilon_s))ds), \sup_{s\in[0,t_0]} \|X^\varepsilon_s - \hat{\varphi}(s)\| \leq \delta\right).$$



t_0 is chosen so that the solution of the linear problem at (t_0, x, y) is close to 1. So

$$\mathrm{E}_{(x,y)}\left(g(X^\varepsilon_{t_0})\exp(\varepsilon^{-1}\int_0^{t_0} c(Y^\varepsilon_s, 0)ds), \sup_{s\in[0,t_0]} \|X^\varepsilon_s - \varphi(s)\| \leq \delta\right) \approx 1.$$

Now we allow the coefficients a^k, α^k, c^k, $k = 1, 2$, to depend on x.

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{\varepsilon^{-\beta}}{2} \alpha(x, y) \frac{\partial^2 u^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon} c(x, y) u^\varepsilon,$$

$$\frac{\partial u^\varepsilon}{\partial y}|_{y=0,1} = 0, \quad u^\varepsilon(0, x, y) = g(x).$$

In the case of x-independent coefficients, the main contribution to the expectation in the Feynman-Kac formula comes from the event that the trajectoris of X_t^ε (starting at x) closely follow the linearly parametrized segment connecting x with x', where x' is one of the points in G_0. Since the coefficients were spatially homogeneous, the contribution to the expectation from such an event depended only on the difference between x and x'. Now there is an optimal path (not a line) $\varphi : [0, t] \to \mathbb{R}^n$ such that the trajectories of X_t^ε following in its vicinity give the main contribution to the expectation. Also, the optimal behavior of the fast component depends on time (through the x-variable).

Let $K = \{1, 2\}$. Consider the random occupation measure on $(K \times [0, t], \mathcal{B}(K) \times \mathcal{B}([0, t]))$:

$$\nu^{\varepsilon}_{t,x,y}(\{1\} \times \Delta) = \int_{\Delta} \chi_{[0,m)}(Y^{\varepsilon}_s) ds, \quad \nu^{\varepsilon}_{t,x,y}(\{2\} \times \Delta) = \int_{\Delta} \chi_{(m,1]}(Y^{\varepsilon}_s) ds,$$

where $\Delta \in \mathcal{B}([0, t])$ and $(X^{\varepsilon}_0, Y^{\varepsilon}_0) = (x, y)$. The space of measures on $(K \times [0, t], \mathcal{B}(K) \times \mathcal{B}([0, t]))$ whose marginals ν_s, $s \in [0, t]$, are probability measures on K, will be denoted by $\mathcal{M}$. It is endowed with the Levy-Prohorov distance denoted by ρ.

Let $\mathcal{C}$ be the space of continuous functions on $[0, t]$ endowed with the distance d. Thus X^{ε} can be viewed as a random element of $\mathcal{C}$.

For $x, v \in \mathbb{R}^n$, define

$$R(p, x, v) = \frac{1}{2}((p_1 a^1(x) + p_2 a^2(x))^{-1} v, v).$$

For $\varphi \in \mathcal{C}$ and $\nu \in \mathcal{M}$, define

$$\bar{R}(\nu, \varphi) = \int_0^t R(\nu_s, \varphi_s, \dot{\varphi}_s) ds,$$

For $f \in C([0,1])$, let $H^x(f)$ be the top eigenvalue of the operator $L^x_f u = \frac{1}{2}\alpha(x,y)u''(y) + fu$ (with the gluing condition at $y = m$ and reflection at the end points). Let $\pi(x)$ be the invariant measure for the process governed by L^x_0. For $\mu \in \mathcal{M}_{[0,1]}$, define

$$I^x(\mu) = \sup_{f \in C([0,1])} \left(\int_0^1 f d\mu - H^x(f)\right).$$

For $x \in \mathbb{R}^n$ and $p \in J$, define

$$S(p,x) = \inf_{\mu : p_\mu = (p_1,p_2)} I^x(\mu).$$

For $\varphi \in \mathcal{C}$ and $\nu \in \mathcal{M}$, define

$$\bar{S}(\nu,\varphi) = \int_0^t S(\nu_s,\varphi_s)ds.$$

Let $\tilde{\Lambda}^{\varepsilon}_{t,x,y}$, be the measure on $(\mathcal{M} \times \mathcal{C}, \rho \times d)$ induced by $(\nu^{\varepsilon}_{t,x,y}, X^{\varepsilon})$ (with $X^{\varepsilon}_0 = x$ and $Y^{\varepsilon}_0 = y$). Note that $\tilde{\Lambda}^{\varepsilon}_{t,x,y}$ also depends on β because of the dependence of Y^{ε}_t on β. The following theorem follows from the results of Liptser.

Theorem: If $\beta = 1$, the family $\tilde{\Lambda}^{\varepsilon}_{t,x,y}$ obeys the large deviations principle with the action functional

$$L(\nu, \varphi) = \bar{R}(\nu, \varphi) + \bar{S}(\nu, \varphi)$$

If $\beta > 1$, the family $\tilde{\Lambda}^{\varepsilon}_{t,x,y}$ obeys the large deviations principle with the action functional

$$L(\nu, \varphi) = \begin{cases} \bar{R}(\tilde{\nu}^{\varphi}, \varphi) & if \ \nu = \tilde{\nu}^{\varphi}, \\ \infty & otherwise, \end{cases}$$

where $\tilde{\nu}^{\varphi}$ is such that $\tilde{\nu}^{\varphi}_s = \pi(\varphi_s)$ for each s.
If $-1 < \beta < 1$, the family $\tilde{\Lambda}^{\varepsilon}_{t,x,y}$ obeys the large deviations principle with the action functional

$$L(\nu, \varphi) = \inf_{\nu' \in \mathcal{M}} \bar{R}(\nu', \varphi).$$

Define

$$T(p,x) = p_1 c^1(x) + p_2 c^2(x)$$

and

$$\bar{T}(\nu,\varphi) = \int_0^t T(\nu_s,\varphi_s)ds.$$

Let $\mathcal{C}(x,x') = \{\varphi \in \mathcal{C} : \varphi(0) = x, \varphi(t) = x'\}$. For $\beta = 1$, define

$$\lambda(t,x,x') = \sup_{\varphi \in \mathcal{C}(x,x')} \sup_{\nu \in \mathcal{M}} (\bar{T}(\nu,\varphi) - \bar{S}(\nu,\varphi) - \bar{R}(\nu,\varphi)).$$

For $\beta > 1$,

$$\lambda(t,x,x') = \sup_{\varphi \in \mathcal{C}(x,x')} (\bar{T}(\tilde{\nu}^{\varphi},\varphi) - \bar{R}(\tilde{\nu}^{\varphi},\varphi)).$$

For $-1 < \beta < 1$,

$$\lambda(t,x,x') = \sup_{\varphi \in \mathcal{C}(x,x')} \sup_{\nu \in \mathcal{M}} (\bar{T}(\nu,\varphi) - \bar{R}(\nu,\varphi)).$$

Theorem:

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln(u^{\varepsilon}(t,x,y)) = \sup_{x' \in G_0} \lambda(t,x,x')$$

uniformly on every compact $K \subset (0,\infty) \times \mathbb{R}^n \times [0,1]$.

Consider the Cauchy problem for the RDE, but now allow a^k, α^k, c^k, $k = 1, 2$, to depend on x.

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{\varepsilon^{-\beta}}{2} \alpha(x, y) \frac{\partial^2 u^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon} c(x, y, u^\varepsilon) u^\varepsilon,$$

$$\frac{\partial u^\varepsilon}{\partial y}|_{y=0,1} = 0, \quad u^\varepsilon(0, x, y) = g(x),$$

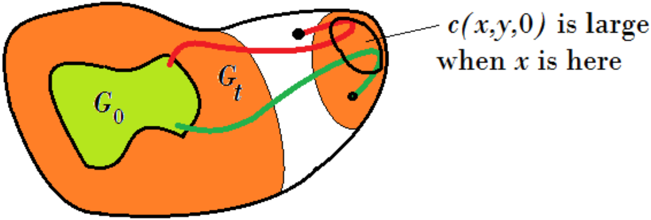
We can consider the corresponding linear problem, replacing the nonlinearity by $\frac{1}{\varepsilon} c(x, y, 0) u^\varepsilon$. The function $\lambda(t, x, x')$ (exponential growth rate) has been defined for the linear equation.

Define

$$G_t = \{x \in \mathbb{R}^n : \lambda(s, x, \varphi(s)) \geq 0 \text{ for}$$

$$\text{all } s \in [0, t], \text{ for some } \varphi \in \mathcal{C} \text{ with } \varphi(0) = x, \ \varphi(t) \in G_0\}.$$

It may happen that $\lambda(t, x, x') > 0$ for $x' \in G_0$ but $x \notin G_t$.



Large contour – set $\{x : \sup_{x' \in G_0} \lambda(t, x, x') = 0\}$.
Orange and green domain – G_t.

Theorem: For each $t > 0$,

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x, y)) = 0$$

uniformly on every compact $K \subset (\mathbb{R}^n \setminus G_t) \times [0, 1]$, and

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t, x, y)) = 1$$

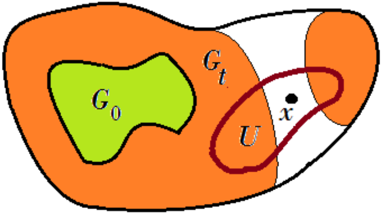
uniformly on every compact $K \subset \mathrm{Int}(G_t) \times [0, 1]$.

Main idea: (a) For x such that $\lambda(t, x, x') < 0$ for all $x' \in G_0$, the solution of the RDE is estimated from above by the solution of the linear equation.

(b) For $x \in G_t$, the argument is similar to the earlier case (use a neighborhood of a slightly sub-optimal path).

(c) What about $x \notin G_t$ such that $\lambda(t, x, x') > 0$ for some $x' \in G_0$?

For $x \notin G_t$ such that $\lambda(t, x, x') > 0$, find a small $\delta > 0$ and a smooth domain U with $x \in U$ and $U \bigcap G_0 = \emptyset$ such that $\lambda(t + \delta, x, x') < 0$ for $x' \in \partial U$.



Estimate the solution of the Cauchy problem by the solution of the initial-boundary value problem.

$$\frac{\partial u^\varepsilon}{\partial t} = \frac{\varepsilon}{2} \sum_{i,j=1}^{n} a_{ij}(x, y) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{\varepsilon^{-\beta}}{2} \alpha(x, y) \frac{\partial^2 u^\varepsilon}{\partial y^2} +$$

$$+ \frac{1}{\varepsilon} c(x, y, 0) u^\varepsilon, \quad t > 0, \; x \in U_\delta, \; y \in (0, 1) \setminus \{m\}.$$

$$\frac{\partial u^\varepsilon}{\partial y}|_{y=0,1} = 0, \quad u^\varepsilon(0, x, y) = 0, \quad x \in U; \quad u^\varepsilon(t, x, y) = 1, \quad x \in \partial U.$$