Superposition principle for non-local Fokker-Planck operators

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Introduction

- Consider the following SDE in $\mathbb{R}^d$:

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t, \tag{1.1}$$

where $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are measurable functions, W_t is a standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- Assume that X_t solves the above SDE in the sense that

$$\mathbb{E}\left(\int_0^T (|b_t(X_t)| + |a_t(X_t)|)\mathrm{d}t\right) < \infty,$$

where $a_t(x) = \frac{1}{2}(\sigma_t\sigma_t^T)(x)$, and

$$X_t = X_0 + \int_0^t b_s(X_s)\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s.$$

- Let

$$\mathscr{A}_t f(x) := \mathrm{tr}(a_t(x) \cdot \nabla^2 f(x)), \ \mathscr{B}_t f(x) := b_t(x) \cdot \nabla f(x). \qquad (1.2)$$

- Let μ_t be the marginal law of X_t. By Itô's formula, μ_t solves the following Fokker-Planck equation in the distributional sense

$$\partial_t \mu_t = \left(\mathscr{A}_t + \mathscr{B}_t\right)^* \mu_t, \qquad (1.3)$$

where $\mathscr{A}_t^*$ and $\mathscr{B}_t^*$ stand for the adjoint operators of $\mathscr{A}_t$ and $\mathscr{B}_t$, respectively, that is, for any $f \in C_c^2(\mathbb{R}^d)$,

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(\mathscr{A}_s(f) + \mathscr{B}_s(f))\mathrm{d}s.$$

In order to make the right hand have sense, it suffices to require

$$\int_0^t \int_{B_N} (|a_s(x)| + |b_s(x)|)\mu_s(\mathrm{d}x)\mathrm{d}s < \infty, \ \forall N > 0.$$

- Question: (Superposition principle) For any probability measure solution μ_t of FPE (**??**), is there a solution X_t to SDE (**??**) so that

$$\mu_t = \text{Law of } X_t, \quad \forall t \geqslant 0?$$

- In the deterministic case, i.e., $\sigma \equiv 0$, Ambrosio [2004] first studied the above problem, and use it to establish the well-posedness of ODE with BV velocity field.
- In stochastic case, when b and σ are bounded measurable, Figalli [2008] showed that for any probability measure solution μ_t of FPE (**??**), there is a martingale solution for SDE (**??**) so that

$$\mu_t = \mathbb{P} \circ X_t^{-1}.$$

- Trevisan [2016] showed the same result under the following natural assumption:

$$\int_0^T \int_{\mathbb{R}^d} (|b_t(x)| + |a_t(x)|)\mu_t(\mathrm{d}x)\mathrm{d}t < \infty.$$

However, the above assumption is not satisfied if

$$|b_t(x)| + |\sigma_t(x)| \leqslant C(1 + |x|) \text{ and } \int_{\mathbb{R}^d} |x|\mu_t(\mathrm{d}x) = \infty.$$

- Recently, Bogachev, Röckner and Shaposhnikov [2019] improved Trevisan's result to the following more general assumption:

$$\int_0^T \int_{\mathbb{R}^d} \frac{|\langle x, b_t(x)\rangle| + |a_t(x)|}{1 + |x|^2} \mu_t(\mathrm{d}x)\mathrm{d}t < \infty.$$

- Let $(X_t)_{t\geqslant 0}$ be a Feller process in $\mathbb{R}^d$ with infinitesimal generator $(\mathscr{L}, \mathrm{Dom}(\mathscr{L}))$.
- One says $\mathscr{L}$ satisfies a maximum principle if for all $f \in \mathrm{Dom}(\mathscr{L})$ reaching a maximum at point $x_0 \in \mathbb{R}^d$, then $\mathscr{L} f(x_0) = 0$.
- Suppose that $C_c^\infty(\mathbb{R}^d) \subset \mathrm{Dom}(\mathscr{L})$. By Courrège's theorem, $\mathscr{L}$ satisfies the maximum principle if and only if

$$\begin{aligned}\mathscr{L} f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial^2_{ij} f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x) \\ + \int_{\mathbb{R}^d} \left(f(x+z) - f(x) - 1_{|z|\leqslant 1} z \cdot \nabla f(x)\right) \nu_x(\mathrm{d}z),\end{aligned} \tag{1.4}$$

where a, b are measurable functions and $\nu_x(\mathrm{d}z)$ is a family of Lévy measures.

- Let μ_t be the marginal law of X_t. By Dykin's formula, we have

$$\partial_t \mu_t = \mathscr{L}^* \mu_t.$$

- We naturally ask that for any probability measure solution μ_t to the above Fokker-Planck equation, is it possible to find some process X so that μ_t is just the law of X_t for each $t \geqslant 0$?

Main result

- Let $\{\nu_{t,x}\}_{t\geqslant 0, x\in\mathbb{R}^d}$ be a family of Lévy measures over $\mathbb{R}^d$, that is, for each $t \geqslant 0$ and $x \in \mathbb{R}^d$,

$$g_t^\nu(x) := \int_{B_\ell} |z|^2 \nu_{t,x}(\mathrm{d}z) < \infty, \quad \nu_{t,x}(B_\ell^c) < \infty, \tag{2.1}$$

  where $\ell > 0$ is a fixed number, and $B_\ell := \{z \in \mathbb{R}^d : |z| < \ell\}$. Without loss of generality we may assume

$$\ell \leqslant 1/\sqrt{2}.$$

- We introduce the following Lévy type operator: for any $f \in C_b^2(\mathbb{R}^d)$,

$$\mathscr{N}_t f(x) := \mathscr{N}_t^\nu f(x) := \mathscr{N}^{\nu_{t,x}} f(x) := \int_{\mathbb{R}^d} \Theta_f(x; z) \nu_{t,x}(\mathrm{d}z), \tag{2.2}$$

  where

$$\Theta_f(x; z) := f(x+z) - f(x) - \mathbf{1}_{|z|\leqslant\ell} \cdot \nabla f(x). \tag{2.3}$$

- Let us consider the following non-local Fokker-Planck equation:

$$\partial_t \mu_t = \mathscr{L}_t^* \mu_t = (\mathscr{A}_t + \mathscr{B}_t + \mathscr{N}_t)^* \mu_t. \tag{2.4}$$

Definition 1 (Weak solution)

Let $\mu : \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}^d)$ be a continuous curve. We call μ_t a weak solution of the non-local Fokker-Planck equation (**??**) if for any $R > 0$ and $t > 0$,

$$\left\{ \begin{aligned} &\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{B_R}(x)\Big(|a_s(x)| + |b_s(x)| + g_s^\nu(x)\Big)\mu_s(\mathrm{d}x)\mathrm{d}s < \infty, \\ &\int_0^t \int_{\mathbb{R}^d} \Big(\nu_{s,x}(B^c_{\ell\vee(|x|-R)}) + \mathbf{1}_{B_R}(x)\nu_{s,x}(B^c_\ell)\Big)\mu_s(\mathrm{d}x)\mathrm{d}s < \infty, \end{aligned} \right\} \tag{2.5}$$

and for all $f \in C_c^2(\mathbb{R}^d)$ and $t \geqslant 0$,

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(\mathscr{L}_s f)\mathrm{d}s. \tag{2.6}$$

- Let $\mathbb{D}$ be the space of all $\mathbb{R}^d$-valued cádlág functions on $\mathbb{R}_+$.
- Let $X_t(\omega) = \omega_t$ be the canonical process.
- For $t \geqslant 0$, let $\mathcal{B}_t^0(\mathbb{D}) = \sigma\{X_s, s \in [0, t]\}$ and

$$\mathcal{B}_t := \mathcal{B}_t(\mathbb{D}) := \cap_{s \geqslant t} \mathcal{B}_t^0(\mathbb{D}), \ \ \mathcal{B} := \mathcal{B}(\mathbb{D}) := \mathcal{B}_\infty(\mathbb{D}).$$

Definition 2 (Martingale Problem)

Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and $s \geqslant 0$. We call $\mathbb{P} \in \mathcal{P}(\mathbb{D})$ a martingale solution of $\mathscr{L}_t$ with initial distribution μ_0 at time s if

(i) $\mathbb{P}(X_t = X_s, t \in [0, s]) = 1$ and $\mathbb{P} \circ X_s^{-1} = \mu_0$.

(ii) For any $f \in C_c^2(\mathbb{R}^d)$, M_t^f is a $\mathcal{B}_t$-martingale under $\mathbb{P}$, where

$$M_t^f := f(X_t) - f(X_s) - \int_s^t \mathscr{L}_r f(X_r) \mathrm{d}r, \ t \geqslant s.$$

All the martingale solutions associated with $\mathscr{L}_t$ with initial law μ_0 at time s will be denoted by $\mathcal{M}_s^{\mu_0}(\mathscr{L}_t)$. In particular, if $\mu_0 = \delta_x$ (the Dirac measure concentrated on x), we shall write $\mathcal{M}_s^x(\mathscr{L}_t)$ for simplify.

(H) We make the following assumption:

$$\Gamma^{\nu}_{a,b} := \sup_{t,x}\left[\frac{|a_t(x)| + g^{\nu}_t(x)}{1+|x|^2} + \frac{|b_t(x)|}{1+|x|} + \hbar^{\nu}_t(x)\right] < \infty, \tag{2.7}$$

where $g^{\nu}_t(x)$ is defined by (**??**) and

$$\hbar^{\nu}_t(x) := \int_{B^c_\ell} \log\left(1 + \tfrac{|z|}{1+|x|}\right)\nu_{t,x}(\mathrm{d}z), \tag{2.8}$$

and if $\nu_{t,x}$ is symmetric, then we may let

$$\hbar^{\nu}_t(x) := \int_{|z|>1+|x|} \log\left(1 + \tfrac{|z|}{1+|x|}\right)\nu_{t,x}(\mathrm{d}z). \tag{2.9}$$

Theorem 3 (Superposition principle)

Under **(H)**, *for any weak solution* $(\mu_t)_{t\geqslant 0}$ *of FPE* (**??**) *in the sense of Definition* **??**, *there is a martingale solution* $\mathbb{P} \in \mathcal{M}^{\mu_0}_0(\mathscr{L}_t)$ *such that*

$$\mu_t = \mathbb{P} \circ X_t^{-1}, \quad \forall t \geqslant 0.$$

Example 1

Let $\nu_{t,x}(\mathrm{d}z) = \kappa_t(x,z)\mathrm{d}z/|z|^{d+\alpha}$ with $\alpha \in (0,2)$ and

$$|\kappa_t(x,z)| \leqslant c(1+|x|)^{\alpha\wedge 1}(\mathbf{1}_{\alpha\neq 1} + \mathbf{1}_{\alpha=1}/\log(1+|x|)),$$

that is, $\mathscr{N}_t$ is an α-stable like operator. Then $\sup_{t,x} \hbar_t^\nu(x) < \infty$.

Example 2

If $\kappa_t(x,z)$ is symmetric, that is, for every $z \in \mathbb{R}^d$, $\kappa_t(x,z) = \kappa_t(x,-z)$, and $|\kappa_t(x,z)| \leqslant c(1+|x|)^\alpha$, $\alpha \in (0,2)$. Then $\sup_{t,x} \hbar_t^\nu(x) < \infty$.

Example 3

Consider the following SDE driven by symmetric α-stable process:

$$\mathrm{d}X_t = \sigma_t(X_{t-})\mathrm{d}L_t^{(\alpha)} + b_t(X_t)\mathrm{d}t,$$

where $\sigma_t(x)$ and $b_t(x)$ are linear growth.

Corollary 4

Under **(H)***, the well-posedness of Fokker-Planck equation (***??***) is equivalent to the well-posedness of martingale problem associated with $\mathscr{L}_t$. More precisely, we have the following equivalences:*

- **(Existence)** *For any $\nu \in \mathcal{P}(\mathbb{R}^d)$, the non-local FPE (**??**) admits a solution $(\mu_t)_{t\geqslant 0}$ with initial value $\mu_0 = \nu$ if and only if $\mathcal{M}_0^\nu(\mathscr{L})$ has one element.*
- **(Uniqueness)** *The following two statements are equivalent.*
  - (i) *For each $(s, \nu) \in \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d)$, the non-local FPE (**??**) has at most one solution $(\mu_t)_{t\geqslant s}$ starting $\mu_s = \nu$ at time s.*
  - (ii) *For each $(s, \nu) \in \mathbb{R}_+ \times \mathcal{P}(\mathbb{R}^d)$, $\mathcal{M}_s^\nu(\mathscr{L})$ has at most one element.*

- In the constant nonlocal noise, Z. [2013] used the superposition principle to show the uniqueness of non-local FPEs.
- Fournier and Xu [2018] proved a non-local version to the superposition principle in a special case, that is,

$$\mathscr{N}_t^{\nu} f(x) = \int_{\mathbb{R}^d} [f(x+z) - f(x)]\nu_{t,x}(\mathrm{d}z),$$

and $(\mu_t)_{t\geqslant 0}$ have finite first order moments, i.e.,

$$\int_{\mathbb{R}^d} |x|\mu_t(\mathrm{d}x) < \infty, \quad \forall t \geqslant 0.$$

Proof of Theorem ??: continuous elliptic coefficients

Theorem 5 (Stroock [1975])

Suppose that the following conditions are satisfied:

(A) $a_t(x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{S}^d_+$ *is continuous and* $a_t(x)$ *is invertible;*

(B) $b_t(x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ *is locally bounded and measurable;*

(C) *for any* $A \in \mathcal{B}(\mathbb{R}^d)$, $(t, x) \mapsto \int_A (1 \wedge |z|^2)\nu_{t,x}(\mathrm{d}z)$ *is continuous;*

(D) $\bar{\Gamma}^\nu_{a,b} := \sup_{t,x} \left(\frac{|a_t(x)| + \langle x, b_t(x)\rangle^+ + g^\nu_t(x)}{1+|x|^2} + 2\hbar^\nu_t(x) \right) < \infty.$

Then for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *there is a unique martingale solution* $\mathbb{P}_{s,x} \in \mathcal{M}^x_s(\mathscr{L}_t)$. *Moreover, we have the following conclusions:*

- *For each* $A \in \mathcal{B}(\mathbb{D})$, $(s, x) \mapsto \mathbb{P}_{s,x}(A)$ *is Borel measurable.*
- *The strong Markov property holds: for every* $f \in C_b(\mathbb{R}_+ \times \mathbb{R}^d)$ *and any finite stopping time* τ,

$$\mathbb{E}^{\mathbb{P}_{0,x}}(f(\tau + t, X_{\tau+t})|\mathcal{B}_\tau) = \mathbb{E}^{\mathbb{P}_{s,y}}(f(s + t, X_{s+t}))|_{(s,y)=(\tau,X_\tau)}.$$

Lemma 6 (Lyapunov's type estiamte)

Let $\psi \in C^2(\mathbb{R};\mathbb{R}_+)$ *with* $\lim_{r\to\infty}\psi(r)=\infty$ *and*

$$0<\psi'\leqslant 1,\quad \psi''\leqslant 0. \tag{3.1}$$

Fix $y\in\mathbb{R}^d$ *and define a Lyapunov function* $V_y(x):=\psi(\log(1+|x-y|^2))$. *Then for all* $t\geqslant 0$ *and* $x\in\mathbb{R}^d$, *we have*

$$\mathscr{L}_t V_y(x)\leqslant 2\left(\frac{|a_t(x)|+\langle x-y,b_t(x)\rangle^+ + g^\nu_t(x)}{1+|x-y|^2}+2H^\nu_t(x,y)\right), \tag{3.2}$$

where $g^\nu_t(x)$ *is defined by* (**??**), *and*

$$H^\nu_t(x,y):=\int_{B^c_\ell}\log\left(1+\tfrac{|z|}{1+|x-y|}\right)\nu_{t,x}(\mathrm{d}z). \tag{3.3}$$

Theorem 7 (Continuous elliptic coefficients)

Assume that **(A)**-**(D)** *hold. Then for any* $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, *there are unique solution* $(\mu_t)_{t\geqslant 0}$ *to FPE* (**??**) *and unique martingale solution* $\mathbb{P}_{0,\mu_0} \in \mathcal{M}_0^{\mu_0}(\mathscr{L})$ *so that*

$$\mu_t = \mathbb{P}_{0,\mu_0} \circ X_t^{-1}.$$

Proof of Theorem ??: General case

- Let μ_t be a solution of the following FPEs:

$$\partial_t \mu = \mathscr{L}_t^* \mu = (\mathscr{A}_t + \mathscr{B}_t + \mathscr{N}_t)^* \mu_t.$$

- We want to show that there is a martingale solution $\mathbb{P} \in \mathcal{M}_0^{\mu_0}(\mathscr{L}_t)$ so that

$$\mu_t = \mathbb{P} \circ X_t^{-1},$$

- We use the following convention: for $t \leqslant 0$,

$$\mu_t(\mathrm{d}x) := \mu_0(\mathrm{d}x), \ a_t(x) = 0, \ b_t(x) = 0, \ \nu_{t,x}(\mathrm{d}z) = 0.$$

Regularization

- Let ρ_ε be a family of mollifies in $\mathbb{R}^{d+1}$ with supports in B_ε.
- For a locally finite signed measure $\zeta_t(\mathrm{d}x)\mathrm{d}t$ over $\mathbb{R}^{d+1}$, define

$$\rho_\varepsilon * \zeta(t,x) := \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(t-s, x-y)\zeta_s(\mathrm{d}y)\mathrm{d}s.$$

- Let $\phi(x) := (2\pi)^{-d/2}\mathrm{e}^{-|x|^2/2}$ be the normal distribution density and

$$\Delta\phi + \mathrm{div}(x\cdot\phi) = 0.$$

- For $\varepsilon \in (0,\ell)$, define approximation sequence $\mu_t^\varepsilon \in \mathcal{P}(\mathbb{R}^d)$ by

$$\mu_t^\varepsilon(x) := (1-\varepsilon)(\rho_\varepsilon * \mu)(t,x) + \varepsilon\phi(x). \tag{4.1}$$

We have the following easy consequence.

Lemma 8

- *For each* $t \geqslant 0$ *and* $\varepsilon \in (0, \ell)$, *we have*

$$0 < \mu_t^\varepsilon(x) \in C^\infty(\mathbb{R}_+; C_b^\infty(\mathbb{R}^d)), \quad \int_{\mathbb{R}^d} \mu_t^\varepsilon(x)\mathrm{d}x = 1.$$

- *For each* $t \geqslant 0$, μ_t^ε *weakly converges to* μ_t, *that is, for any* $f \in C_b(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to \infty} \int_{\mathbb{R}^d} f(x)\mu_t^\varepsilon(x)\mathrm{d}x = \int_{\mathbb{R}^d} f(x)\mu_t(\mathrm{d}x).$$

Ctd.

- μ_t^ε solves the following Fokker-Planck equation:

$$\partial_t \mu_t^\varepsilon = (\mathscr{A}_t^\varepsilon + \mathscr{B}_t^\varepsilon + \mathscr{N}_t^\varepsilon)^* \mu_t^\varepsilon =: (\mathscr{L}_t^\varepsilon)^* \mu_t^\varepsilon,$$

where $\mathscr{A}_t^\varepsilon$, $\mathscr{B}_t^\varepsilon$ and $\mathscr{N}_t^\varepsilon$ are defined as in the introduction in terms of

$$a_t^\varepsilon(x) := \frac{(1-\varepsilon)[\rho_\varepsilon * (a\mu)](t,x) + \varepsilon\phi(x)\mathbb{I}}{\mu_t^\varepsilon(x)},$$

$$b_t^\varepsilon(x) := \frac{(1-\varepsilon)[\rho_\varepsilon * (b\mu)](t,x) + \varepsilon\phi(x)x}{\mu_t^\varepsilon(x)},$$

and

$$\nu_{t,x}^\varepsilon(\mathrm{d}z) := \frac{1-\varepsilon}{\mu_t^\varepsilon(x)} \int_{\mathbb{R}^{d+1}} \rho_\varepsilon(t-s, x-y)\nu_{s,y}(\mathrm{d}z)\mu_s(\mathrm{d}y)\mathrm{d}s.$$

Ctd.

- The following uniform estimates hold: for any $\varepsilon \in (0, \ell)$,

$$\sup_{t,x}\left[\frac{|a^\varepsilon_t(x)| + g^{\nu^\varepsilon}_t(x)}{1+|x|^2} + \frac{|b^\varepsilon_t(x)|}{1+|x|}\right] \leqslant 1 + 2\sup_{t,x}\left[\frac{|a_t(x)| + g^{\nu}_t(x)}{1+|x|^2} + \frac{|b_t(x)|}{1+|x|}\right] \tag{4.2}$$

  and

$$\sup_{t,x} H^{\nu^\varepsilon}_t(x,y) \leqslant \sup_{t,x} H^{\nu}_t(x,y), \quad y \in \mathbb{R}^d. \tag{4.3}$$

Lemma 9 (Probabilistic representation for μ^ε)

For any $\varepsilon \in (0, \ell)$ *and* $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, *there is a unique martingale solution* $\mathbb{P}^\varepsilon_{s,x} \in \mathcal{M}^x_s(\mathscr{L}^\varepsilon_t)$. *In particular, there is also a martingale solution* $\mathbb{Q}^\varepsilon \in \mathcal{M}^{\mu^\varepsilon_0}_0(\mathscr{L}^\varepsilon_t)$ *so that for each* $t \geqslant 0$,

$$\mu^\varepsilon_t(x)\mathrm{d}x = \mathbb{Q}^\varepsilon \circ X_t^{-1}(\mathrm{d}x).$$

Tightness

Lemma 10 (Moments of initial law)

*For $\mu_0^\varepsilon \in \mathcal{P}(\mathbb{R}^d)$ being defined by (**??**), there exits a function $\psi \in C^2(\mathbb{R}_+)$ with the properties*

$$\psi \geqslant 0, \quad \psi(0) = 0, \quad 0 < \psi' \leqslant 1, \quad -2 \leqslant \psi'' \leqslant 0, \quad \lim_{r\to\infty} \psi(r) = +\infty,$$

and such that

$$\sup_{\varepsilon\in[0,\ell)} \int_{\mathbb{R}^d} \psi\big(\log(1+|x|^2)\big)\mu_0^\varepsilon(\mathrm{d}x) < \infty. \tag{4.4}$$

Lemma 11

*Let $H_t^\nu(x,y)$ be defined by (**??**). We have*

$$H_t^\nu(x,y) \leqslant 2(1+|y|)\hbar_t^\nu(x), \ \ \forall t \geqslant 0, x, y \in \mathbb{R}^d. \tag{4.5}$$

Lemma 12

The family of probability measures $(\mathbb{Q}^\varepsilon)_{\varepsilon\in(0,\ell)}$ *is tight in* $\mathcal{P}(\mathbb{D})$.

Proof.

By Aldous' criterion, it suffices to check the following two conditions:

(i) For any $N > 0$ and $T > 0$, it holds that

$$\lim_{N\to\infty} \sup_{\varepsilon} \mathbb{Q}^\varepsilon \left(\sup_{t\in[0,T]} |X_t| > N \right) = 0.$$

(ii) For any $T, \delta_0 > 0$ and stopping time $\tau < T - \delta_0$, it holds that

$$\lim_{\delta\to 0} \sup_{\varepsilon} \sup_{\tau} \mathbb{Q}^\varepsilon (|X_{\tau+\delta} - X_\tau| > \lambda) = 0, \quad \forall \lambda > 0.$$

□

Lemma 13 (Stochastic Gronwall's inequality)

Let $\xi(t)$ and $\eta(t)$ be two non-negative càdlàg adapted processes, A_t a continuous non-decreasing adapted process with $A_0 = 0$, M_t a local martingale with $M_0 = 0$. Suppose that

$$\xi(t) \leqslant \eta(t) + \int_0^t \xi(s)\mathrm{d}A_s + M_t, \ \forall t \geqslant 0.$$

Then for any $0 < q < p < 1$ and stopping time $\tau > 0$, we have

$$\left[\mathbb{E}(\xi(\tau)^*)^q\right]^{1/q} \leqslant \left(\tfrac{p}{p-q}\right)^{1/q} \left(\mathbb{E}\mathrm{e}^{pA_\tau/(1-p)}\right)^{(1-p)/p} \mathbb{E}\left(\eta(\tau)^*\right),$$

where $\xi(t)^* := \sup_{s\in[0,t]} \xi(s)$.

Remark: Continuous martingale due to Scheutzow (2013). Discontinuous martingale Xie and Z. (2016).

Limits

- We rewrite

$$\mathscr{B}_t f(x) + \mathscr{N}_t f(x) = \tilde{b}_t(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \Theta_f^\pi(x; z) \nu_{t,x}(\mathrm{d}z)$$
$$=: \widetilde{\mathscr{B}_t} f(x) + \widetilde{\mathscr{N}_t} f(x),$$

where

$$\tilde{b}_t(x) := b_t(x) + \int_{\mathbb{R}^d} \left[\pi(z) - z\mathbf{1}_{|z|\leqslant\ell}\right] \nu_{t,x}(\mathrm{d}z),$$

and

$$\Theta_f^\pi(x; z) := f(x+z) - f(x) - \pi(z) \cdot \nabla f(x).$$

- Here, $\pi : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth symmetric function satisfying

$$\pi(z) = z, \ \ |z| \leqslant \ell, \ \ \pi(z) = 0, \ \ |z| > 2\ell.$$

Lemma 14

For any $f \in C_c^2(\mathbb{R}^d)$ *with support in* B_R*, there is a constant* $C = C(f) > 0$ *such that for all* $x \in \mathbb{R}^d$ *and* $z, z' \in \mathbb{R}^d$ *with* $|z'| \leqslant |z|$*,*

$$|\Theta_f^\pi(x; z) - \Theta_f^\pi(x; z')| \leqslant C(|z - z'| \wedge 1)(\mathbf{1}_{B_{R+\ell}}(x)\mathbf{1}_{|z| \leqslant \ell}|z| + \mathbf{1}_{|z| > \ell \vee (|x| - R)}),$$

where $\Theta_f^\pi(x; z) := f(x + z) - f(x) - \pi(z) \cdot \nabla f(x)$.

The following approximation result will be crucial for taking weak limits.

Lemma 15

For any $\delta \in (0,1)$ *and* $T > 0$*, there is a family of Lévy measures* $\eta_{t,x}(\mathrm{d}z)$ *such that for any* $f \in C_c^2(\mathbb{R}^d)$*,*

$$\int_0^T \int_{\mathbb{R}^d} \sup_{x \in B_1(y)} |\widetilde{\mathscr{N}}^{\nu_{s,y}} f(x) - \widetilde{\mathscr{N}}^{\eta_{s,y}} f(x)| \mu_s(\mathrm{d}y)\mathrm{d}s \leqslant \delta,$$

and

$$\sup_{s,y} \|\widetilde{\mathscr{N}}^{\eta_{s,y}} f\|_\infty < \infty, \quad (s,y,x) \mapsto \widetilde{\mathscr{N}}^{\eta_{s,y}} f(x) \text{ is continuous.}$$

Key point.

By the randomization of kernel functions, there is a measurable function

$$h_{t,x}(\theta) : [0, T] \times \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d \cup \{\infty\}$$

such that

$$\nu_{t,x}(A) = \int_0^\infty \mathbf{1}_A(h_{t,x}(\theta))\mathrm{d}\theta, \quad \forall A \in \mathscr{B}(\mathbb{R}^d).$$

In particular, we have

$$\widetilde{\mathscr{N}}^{\nu_{s,y}} f(x) = \int_0^\infty \Theta_f^\pi(x; h_{s,y}(\theta))\mathrm{d}\theta =: \widetilde{\mathscr{N}}^{h_{s,y}} f(x).$$

□

Application to fractional porous mediam equation

- Consider the following fractional porous media equation:

$$\partial_t u = \Delta^{\alpha/2}(|u|^{m-1}u), \quad u(0,x) = \varphi(x), \tag{5.1}$$

where $m > 1$, $\alpha \in (0,2)$ and $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is the usual fractional Laplacian.

Definition 16

A function u is called a weak or L^1-energy solution of FPME (**??**) if

- $u \in C([0,\infty); L^1(\mathbb{R}^d))$ and $|u|^{m-1}u \in L^2_{loc}((0,\infty); \dot{H}^{\alpha/2}(\mathbb{R}^d))$;
- for every $f \in C^1_0(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^d} u \cdot \partial_t f \mathrm{d}x\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} (|u|^{m-1}u) \cdot \Delta^{\alpha/2} f \mathrm{d}x\mathrm{d}t;$$

- $u(0,x) = \varphi(x)$ almost everywhere.

Theorem 17 (Pablo, Quirós, Rodrïguez and Vázquez (2012))

*Let $\alpha \in (0,2)$ and $m > 1$. For every $\varphi \in L^1(\mathbb{R}^d)$, there exists a unique weak solution u for equation (**??**). Moreover, u enjoys properties:*

- (i) *if $\varphi \geqslant 0$, then $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}^d$;*
- (ii) *$\partial_t u \in L^\infty((s,\infty); L^1(\mathbb{R}^d))$ for every $s > 0$;*
- (iii) *for all $t \geqslant 0$, $\int_{\mathbb{R}^d} u(t,x)\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x$;*
- (iv) *if $\varphi \in L^\infty(\mathbb{R}^d)$, then for every $t > 0$,*

$$\|u(t,\cdot)\|_\infty \leqslant \|\varphi\|_\infty;$$

- (v) *for some $\beta \in (0,1)$, $u \in C^\beta((0,\infty) \times \mathbb{R}^d)$.*

- Let L_t be an α-stable process with Lévy measure $\mathrm{d}z/|z|^{d+\alpha}$.
- Consider the following distribution dependent SDE

$$\mathrm{d}Y_t = \rho_{Y_t}(Y_{t-})^{\frac{m-1}{\alpha}}\mathrm{d}L_t, \quad \rho_{Y_0}(x) = \varphi(x), \tag{5.2}$$

where $\rho_{Y_t}(x)$ denotes the distributional density of Y_t.

Definition 18

Let $(\Omega, \mathcal{F}, \mathbf{P}; (\mathcal{F}_t)_{t\geqslant 0})$ be a stochastic basis and (Y, L) two $\mathcal{F}_t$-adapted càdlàg processes. For $\mu \in \mathcal{P}(\mathbb{R}^d)$, we call $(\Omega, \mathcal{F}, \mathbf{P}; (\mathcal{F}_t)_{t\geqslant 0}; Y, L)$ a solution of (**??**) with initial law μ if

(i) L is an α-stable process with Lévy measure $\mathrm{d}z/|z|^{d+\alpha}$;

(ii) for each $t \geqslant 0$, $\mathbf{P} \circ Y_t^{-1}(\mathrm{d}x) = \rho_{Y_t}(x)\mathrm{d}x$;

(iii) Y_t solves the following SDE:

$$Y_t = Y_0 + \int_0^t \rho_{Y_s}(Y_{s-})^{\frac{m-1}{\alpha}}\mathrm{d}L_s.$$

Theorem 19

*Let $\varphi \geqslant 0$ be bounded and satisfy $\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x = 1$. Let u be the unique weak solution to FPME (**??**) given by Theorem **??** with initial value φ. Then there exists a weak solution Y to DDSDE (**??**) such that*

$$\rho_{Y_t}(x) = u(t,x), \quad \forall t \geqslant 0.$$

Thank you for your attention!

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