On maximums of Gaussian fields. Application to processes of Bessel type.

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This talk is based on joint works with Enkelejd Hashorva, Sergey Kobelkov and Igor Rodionov.

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- General setting. Result.
- Bessel process and Bessel Bridge
- Fractional Bessel processes.

Let $S \subset \mathbb{R}^d$ be the closure of a bounded open set containing zero, and let $X(\mathbf{t})$, $\mathbf{t} \in S$, be a zero mean a.s. continuous Gaussian random field with covariance function $R(\mathbf{s}, \mathbf{t}) = \mathbf{E}X(\mathbf{s})X(\mathbf{t})$; denote by $\sigma^2(\mathbf{t}) = R(\mathbf{t}, \mathbf{t})$, its variance function. We study the asymptotic behavior of the probability

$$P(S; u) = \mathbf{P}(\max_{\mathbf{t} \in S} X(\mathbf{t}) > u) \tag{1}$$

as $u \to \infty$.

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Non-stationary Gaussian processes and fields are more subtle to deal with since both the local properties of the variance function at its point of global maximum and those of the covariance function have to be carefully formulated. Our aim is to show maximum capabilities of the Pickands' Double Sum Method applying to Gaussian fields with unique point of global maximum of theirs variance.

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First results here belongs to V. Piterbarg and V. Prisyazhnuk, 1978, for processes, and V. Piterbarg, 1988, for fields. The Pickands' method have been generalized to Gaussian random fields, with similar, i. e. power like, behavior of the variance and covariance functions at the unique maximum point of the variance. However, while the power behavior of the covariance function, with possible slight generalization to regular variation of it, is natural and quite essential for the Pickand's method, the required in that works power like behavior of the variance looks somewhat artificial.

Conditions

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We need actually a slightly stronger condition than a.s. continuity of sample paths. Denote $\mathbb{B}_\varepsilon := \{\mathbf{t} : |\mathbf{t}| \leq \varepsilon\}$.

Condition A1 $X(\mathbf{t})$ *is a.s. continuous. Moreover, there exists* $\varepsilon > 0$ *such that Dudley's integral for the standardized field* $\bar{X}(\mathbf{t}) = X(\mathbf{t})/\sigma(\mathbf{t})$, $\mathbf{t} \in \mathbb{B}_\varepsilon$ *is finite.*

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Condition A2 $\sigma(\mathbf{t})$ *reaches its absolute maximum on* S *at only* $\mathbf{0}$.

Assume that $\sigma(\mathbf{0}) = 1$. Maximum point can be also made arbitrary. It follows now that $R(\mathbf{s}, \mathbf{t}) \le 1$ and the equality holds only for $\mathbf{s} = \mathbf{t} = \mathbf{0}$.

Conditions

Condition A3. Local stationarity at $\mathbf{0}$. *There exists a covariance function $r(\mathbf{t})$ of a homogeneous random field with $r(\mathbf{t}) < 1$ for all $\mathbf{t} \neq \mathbf{0}$ such that*

$$\lim_{\mathbf{s},\mathbf{t}\to\mathbf{0},\mathbf{s}\neq\mathbf{t}} \frac{1 - R(\mathbf{s},\mathbf{t})}{1 - r(\mathbf{t}-\mathbf{s})} = 1.$$

For vectors $\mathbf{a} = (a_1, ..., a_d)$, $\mathbf{b} = (b_1, ..., b_d)$ define $\mathbf{ab} = (a_1 b_1, ..., a_d b_d)$. For a set T we write $\mathbf{a}T = \{\mathbf{at}, \mathbf{t} \in T\}$.

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Condition A4 *There exists a basis in $\mathbb{R}^d$, a vector function $\mathbf{q}(u) = (q_1(u), ..., q_d(u))$, $u > 0$, $q_i(u) > 0$, $i = 1, ..., d$, $\forall u$, and a positive for all $\mathbf{t} \neq \mathbf{0}$ function $h(\mathbf{t})$ such that for any $\mathbf{t}$ written in these coordinates,*

$$\lim_{u\to\infty} u^2(1 - r(\mathbf{q}(u)\mathbf{t})) = h(\mathbf{t}), \qquad (2)$$

uniformly in $\mathbf{t}$ from any closed set.

Pickands' condition

Condition **A4** is the crucial point in the use of Picands' method. By using standard technique for regularly varying functions and properties of covariation functions it follows

Proposition. *For any vector* $\mathbf{f}$ *the function* $1 - r(t\mathbf{f})$ *regularly varies at zero with degree* $\alpha(\mathbf{f}) \in (0, 2]$. *Moreover, if* $\alpha(\mathbf{f}) = 2$, *then*

$$\lim_{t \to 0} t^{-2}(1 - r(t\mathbf{f})) \in (0, \infty).$$

Behavior of variance

Now assume a behavior of $\sigma(\mathbf{t})$ near its point of global maximum. As in previous works on non-stationary processes and fields, the crucial point is the behavior of the ratio

$$\frac{1-\sigma^2(\mathbf{q}(u)\mathbf{t})}{1-r(\mathbf{q}(u)\mathbf{t})}$$

as $u \to \infty$. In view of Condition **A4** we assume the following.
Condition A5 *For any* $\mathbf{t}$ *there exists the limit*

$$h_1(\mathbf{t}) := \lim_{u\to\infty} u^2(1-\sigma^2(\mathbf{q}(u)\mathbf{t})) \in [\mathbf{0}, \infty],$$

where the basis in $\mathbb{R}^d$ *from Condition* **A4** *is used.*

$\lim_{u\to\infty} u^2(1-\sigma^2(\mathbf{q}(u)\mathbf{t})) = h_1(\mathbf{t})$

In case when $h_1(\mathbf{t}) = 0$, for all $\mathbf{t}$ we speak on the *stationary-like case*. The asymptotic behavior of the probability is similar to the behavior for stationary fields.

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If $h_1(\mathbf{t}) = \infty$, for all $\mathbf{t}$, we refer to the *Talagrand case*, since M. Talagrand has shown that in most general conditions, for any closed set S and a Gaussian a. s. continuous function $X(t)$, $t \in S$, having unique point of maximum of variance, say, at $t_0 \in S$,

$$P(S;u) = \mathbf{P}(X(t_0) > u)(1 + o(1)), \quad u \to \infty.$$

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Typically one has a combination of the three cases. Denote correspondingly,

$$\mathcal{K}_0 := \{\mathbf{t} \in S \setminus \{\mathbf{0}\} : h_1(\mathbf{t}) = 0\},$$
$$\mathcal{K}_c := \{\mathbf{t} \in S \setminus \{\mathbf{0}\} : h_1(\mathbf{t}) \in (0,\infty)\},$$
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Properties of these sets together with all above Conditions follow asymptotic behavior of the probability $P(S;u)$.
Additionally we have to assume a simple structure of $\mathcal{K}_0$:
Condition *The set $\mathcal{K}_0$ consists of finite number of smooth (two times continuously differentiable) disjoint manifolds.*

Main result

Denote $\Psi(x) = \mathbf{P}(X(\mathbf{0}) > x)$.

Theorem

Let S be a bounded open set in $\mathbb{R}^d$ containing zero, and $X(\mathbf{t})$, $\mathbf{t} \in S$, be an a.s. continuous zeromean Gaussian field satisfying Conditions **A1** - **A6**. *Then for the probability $P(S, u)$ given by (1) the following asymptotic relations take places as $u \to \infty$.*

- *If* $\dim \mathcal{K}_0 > 0$, *then* $P(S; u) = Q(u)\Psi(u)(1 + o(1))$, *where* $Q(u)$ *is a regularly varying function which depends of the structure of* $\mathcal{K}_0$, *as well as of* $\mathbf{q}(u)$ *and* $\mathbf{h}(u)$.

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- *If* $\dim \mathcal{K}_0 = \dim K_c = 0$, *then* $P(S; u) = \Psi(u)(1 + o(1))$.

APPLICATIONS: High excursions of Bessel and related random processes

Let $\mathbf{X}(t) := (X_1(t), X_2(t), ..., X_d(t)) \in \mathbb{R}^d$, $d > 1$, $t \in [0, T]$, be a Gaussian vector process with independent identically distributed zero-mean components $X_i(\cdot)$, $i = 1, \ldots, d$. We assume here that the variance of a component reaches its maximum at only one point of the time interval. We call the norm in $\mathbb{R}^d$ of this process, a β-*process*, and apply the above general result to evaluation of the exact asymptotic behavior of the probability

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This setting includes several important special cases.
One is when the coordinates X_i are standard Brownian motions, so that the β-process is the classical Bessel process, another one is the Bessel bridge, which components are standard Brownian bridges.

Bessel and related processes

The Bessel process and the Bessel bridge are diffusion processes; they are usually studied with stochastic calculus, see monographs by D. Revuz D.& M. Yor, by A. Shiryaev, and many related monographs and reviews. The Bessel process is related also to many other models in financial mathematics such as the log-normal model, the Cox-Ingersoll-Ross (square-root) model, the Heston model, and many others described in the mentioned fundamental monographs.

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Maximum trajectories distributions for the Bessel process and the Bessel bridge are represented as series, see e. g. A. Estrella, 2003, for the Bessel process and celebrated works by I. Gikhman, 1957, and J. Kiefer, 1959, for the Bessel bridge (Gikhman-Kiefer formula). One can use this series representation in various evaluations but the fact that all the terms of these series give contributions to the asymptotic in question makes the task much harder.

Bessel and related processes

I. I. Gikhman used also rather subtle PDE techniques to derive not only Gikhman-Kiefer representation but, for odd dimensions, a decomposition in cylinder functions series. With this series decomposition, similarly to the Kolmogorov-Smirnov series, the first member gives the desired asymptotic behavior so that one can obtain the tail asymptotic behavior of the maximum distribution for the Bessel bridge.

I. I. Gikhman gives, as an example, an explicit expression of the series in case $d = 4$.

The latest advances in this direction are presented in the above mentioned monographs.

Applications of Main result

In order to apply the above general result, introduce the Gaussian field

$$Y(t, \mathbf{v}) = X_1(t)v_1 + X_2(t)v_2 + ... + X_d(t)v_d, \tag{4}$$

given on the cylinder

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By duality, for any closed $S \subset [0, T]$,

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Now, instead of formulation a general results about the β-process, which (**almost**) directly follows from the above Theorem 1, we give three examples.

Applications. Examples

1. Bessel process. Assume, that components of the process $\mathbf{X}$ are (independent) standard Brownian motions on $[0,1]$. That is $\beta(t)$ is the standard Bessel process, $BESQ^d(t)$. We have,

Corollary. *For the standard Bessel process* $\beta(t)$*,*

$$P(\max_{[0,1]} BESQ^d(t) > u) = \frac{\pi^{(d-1)/2}}{2^{d/2-1}\Gamma(d/2)} u^{d-2} e^{-u^2/2}(1+o(1))$$

as $u \to \infty$.

Fractional Bessel process

2. Fractional Bessel process.

Let $X_1(t) \overset{d}{=} B_H(t),\ t \in (0,1]$ be the fractional Brownian motion with the Hurst parameter H, $H \in (0,1)$. Similarly to the previous example, we call the process $\beta(t)$, the *fractional Bessel process*, $BESQ_H^d(t)$. Remark that $\beta_{1/2}(t) = BESQ^d(t)$, the standard Bessel process.

Corollary. *For the fractional Bessel process with Hurst parameter* $H > 1/2$,

$$P(\max_{[0,1]} BESQ_H^d(t) > u) = \frac{\pi^{(d-1)/2}}{2^{d/2-1}\Gamma(d/2)} u^{d-2} e^{-u^2/2}(1+o(1));$$

For the fractional Bessel process with Hurst parameter $H < 1/2$

$$= \frac{H_{2H}}{H 2^{(d+1/H)/2}\Gamma(d/2)} u^{1/H+d-4} e^{-u^2/2}(1+o(1))$$

as $u \to \infty$.

Bessel bridge

3.Bessel bridge. The Bessel process $BESQ^d(t)$, $t \in (0,1]$, conditioned on $BESQ^d(1) = x$, is called the Bessel bridge, $BESQ^d_1(0,x)$. If $x = 0$, then the components of the process $\mathbf{X}(t)$ are also equal to 0 at $t = 1$, becoming the standard Brownian bridges, hence in this case the Bessel bridge is the β-process with $X_1(t) \overset{d}{=} B(t) - tB(1)$, $t \in [0,1]$. Denote it by $BESQ^d_{br}(t)$.

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Corollary. *For the Bessel bridge on* $[0,1]$, *starting and ending at zero,*

$$P(\max_{[0,1]} BESQ_{br}^d(t) > u) = \frac{2^{d/2}\sqrt{\pi}}{\Gamma(d/2)} u^{d-1} e^{-2u^2}(1 + o(1))$$

as $u \to \infty$.

Issues

1. E. Hashorva, S.G. Kobelkov, V.I. Piterbarg, On maximum of Gaussian process with unique maximum point of its variance, Fundam. Appl. Math. (2020), in press. See also *arXiv:1901.09753v1[math.PR]*
2. S.G. Kobelkov, V.I. Piterbarg, On maximum of Gaussian random fields having unique maximum point of its variance, Extremes (2019) Vol. 22, no. 4. P. 413–432.
3. Piterbarg V. I., Rodionov I. V. High excursions of Bessel process and other processes of Bessel type // Doklady Mathematics. (2019) Vol. 100, no. 1. P. 346–348.

Many thanks for your attention!