Household epidemic models and McKean–Vlasov SDEs

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Introduction

- We consider an SIS household epidemic model. In our model, the population consists of N households, with sizes $\nu_1, \nu_2, \ldots, \nu_N$, where the ν_i's are i.i.d. $\mathbb{N}$–valued random variables. Let $X_i^N(t)$ denote the number of infectious individuals in the i–th household at time t.
- We suppose that each infected individual can infect another individual within the same household at rate λ_L, for some $\lambda_L > 0$ (the individual to be infected is chosen uniformly from those in the household, and if he (she) is already infected, nothing happens).
- Moreover, each infected individual infects another individual *uniformly chosen* in the total population at rate λ_G (equivalently, first a household is chosen according to the size–biased distribution, then an individual is chosen uniformly in that household).
- Finally, each infected individual becomes susceptible at rate γ, for some $\gamma > 0$.
- Main reference on household epidemic models : Ball, Mollison, Scalia–Tomba (1997). Our point of view is new.

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- Let $\mathcal{X} = \{(n,k) \in \mathbb{N} \times \mathbb{Z}_+ : n \geq 1, 0 \leq k \leq n\}$.
- Let us formalize :

Definition (SIS household epidemic model)

Fix $\lambda_L > 0$, $\lambda_G > 0$ and $\gamma > 0$. Let $\{(\nu_i, X_i(0)), i \geq 1\}$ be i.i.d. $\mathcal{X}$-valued random variables such that $\mathbb{E}[\nu_1^2] < +\infty$ and let $(P_{inf,i}(t), t \geq 0, i \geq 1)$ and $(P_{r,i}(t), t \geq 0, i \geq 0)$ be mutually independent standard Poisson processes, which are also independent of $\{(\nu_i, X_i(0)), i \geq 1\}$. For $N \geq 1$, let $(X_1^N(t), \ldots, X_N^N(t), t \geq 0)$ be the solution of the system of SDEs :

$$X_i^N(t) = X_i(0) + P_{inf,i}\left(\int_0^t \left(1 - \frac{X_i^N(s)}{\nu_i}\right)\left[\lambda_L X_i^N(s) + \lambda_G \frac{\nu_i}{\overline{\nu}^N} \frac{1}{N}\sum_{j=1}^N X_j^N(s)\right] ds\right) - P_{r,i}\left(\gamma \int_0^t X_i^N(s) ds\right), \quad \text{where } \overline{\nu}^N = N^{-1}(\nu_1 + \cdots + \nu_N).$$

We call this process the SIS household model with N households.

- Note : interactions between households is of mean field type.

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The law of large numbers limit : "propagation of chaos"

Theorem (Propagation of chaos in the SIS household model)

Assume that $\{(\nu_i, X_i(0)), i \geq 1\}$ *are independent and identically distributed* $\mathcal{X}$*-valued random variables such that* $\mathbb{E}[\nu_1^2] < +\infty$*. For all* $N \geq 1$*, let* $(X_i^N(t), t \geq 0, 1 \leq i \leq N)$ *be the solution of the above system of equations. Then the random measure* μ^N *defined as*

$$\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{(\nu_i, X_i^N(\cdot))}$$

converges weakly to μ *as* $N \to \infty$ *in probability. Moreover, for any* $k \geq 1$*, in* $\mathcal{P}\left(D([0,T], \mathcal{X})^k\right)$*,*

$$\mathcal{L}\left((\nu_1, X_1^N(\cdot)), \ldots, (\nu_k, X_k^N(\cdot))\right) \Rightarrow \mu^{\otimes k} \text{ as } N \to \infty\,.$$

This is reminiscent of Sznitman (1991).

The limiting equation

- The limit $\mu_{n,k}(t)$ solves (with $\overline{\mu}(t) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} k\mu_{n,k}(t)$, $\overline{\pi} := \mathbb{E}[\nu]$) $\mu_{n,k}(0) = \mathbb{P}(\nu_1 = n, X_1(0) = k)$,

$$\frac{d\mu_{n,k}(t)}{dt} = \mu_{n,k-1}(t)\left(1 - \frac{k-1}{n}\right)\left[\lambda_L(k-1) + \lambda_G \frac{n}{\overline{\pi}}\overline{\mu}(t)\right]$$
$$- \mu_{n,k}(t)\left\{\left(1 - \frac{k}{n}\right)\left[\lambda_L k + \lambda_G \frac{n}{\overline{\pi}}\overline{\mu}(t)\right] + \gamma k\right\} + \mu_{n,k+1}(t)\gamma(k+1),$$

- and it is the law of the solution of the "McKean–Vlasov" SDE

$$X(t) = X(0) + P_{inf}\left(\int_0^t \left(1 - \frac{X(s)}{\nu}\right)\left[\lambda_L X(s) + \lambda_G \frac{\nu}{\overline{\pi}}\mathbb{E}[X(s)]\right] ds\right)$$
$$- P_{rec}\left(\gamma \int_0^t X(s)ds\right).$$

- We have $\mu_{n,k}(t) = \mathbb{P}(\nu = n, X(t) = k)$. Solving the equation for $\mu(t)$ would be easy if the support of π, the law of the ν_i's, would be finite.

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The forced process

- If we replace $\mathbb{E}[X(t)]$ by a given function $m(t)$, we can solve the two above equations, and obtain a process $X_t(m)$. The unknown function $\mathbb{E}[X(t)]$ is a fixed point of the mapping $m \mapsto \mathbb{E}[X_t(m)]$, or equivalently of the mapping $m \mapsto \bar{\mu}(\cdot, m)$.
- To prove existence and uniqueness of that fixed point, we prove the monotonicity of the mapping $m \mapsto X_t(m)$, bound the difference $\mathbb{E}[X_t(m_2)] - \mathbb{E}[X_t(m_1)]$ when $m_1 \leq m_2$, and use an increasing and a decreasing sequence of m's.

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Graphical construction of the forced process

- Let $c(dk)$ denote the counting measure on $\mathbb{N}$. We define the three mutually independent Poisson Random Measures, condit upon (ν, X_0)
  - Π_{rec} is a PRM on $\mathbb{R}_+ \times [1,\nu]$ with mean measure $\gamma\, dt \times c(dk)$;
  - Π_L is a PRM on $\mathbb{R}_+ \times [1,\nu]^2$ with mean measure $\frac{\lambda_L}{\nu}\, dt \times c(dk) \times c(dk)$;
  - Π_G is a PRM on $\mathbb{R}_+ \times [1,\nu] \times [0,\bar{\pi}]$ with mean measure $\frac{\lambda_G}{\bar{\pi}} a\, dt \times c(dk) \times du$, $\Pi_G = \{(t_k, i_k, u_k), k \geq 1, 0 < t_1 < t_2 < \cdots\}$.
- Define the random set $I^k(t) \in [1,\nu]$ as follows : $I^k(t) = \emptyset$ for $t < t_k$; $I^k(t_k) = \{i_k\}$; $\forall (t,i,j) \in \Pi_L$, if $i \in I^k(t^-)$, $I^k(t) = I^k(t^-) \cup \{j\}$; $\forall (t,i) \in \Pi_{rec}$, $I^k(t) = I^k(t^-) \cap \{i\}^c$.
- With that construction,

$$X_t(m) := \left| I^0(t) \cup \bigcup_{k \geq 1} \{ I^k(t) : u_k \leq m(t_k) \} \right|$$

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Large time behavior of the McKean–Vlasov SDE

- $I(t)$ being the number of individuals infected at time t in a local epidemic, with $\pi^+(n) = n\mathbb{P}(\nu = n)/\overline{\pi}$,

$$R_0 = \lambda_G \sum_n \pi^+(n)\mathbb{E}\left[\int_0^\infty I(t)dt \middle| I(0) = 1, \nu = n\right]$$

$$= \frac{\lambda_G}{\gamma} \sum_{n=1}^{\infty} \pi^+(n) \left(1 + \sum_{\ell=1}^{n-1} \left(\frac{\lambda_L}{\gamma}\right)^{\ell} \prod_{j=1}^{\ell} \left(1 - \frac{j}{n}\right)\right).$$

- We have

Theorem

Assume that $\mathbb{E}[\nu^2] < +\infty$.

i) *If $R_0 > 1$, then there exists a unique probability distribution μ_∞ on $\mathbb{N} \times \mathbb{Z}_+$ such that, if $\mathbb{P}(X(0) \geq 1) > 0$, $(\nu, X(t))$ converges in distribution to μ_∞ as $t \to \infty$. Moreover μ_∞ is non-trivial in the sense that $\mu_\infty \neq \pi \otimes \delta_0$.*

ii) *If $R_0 \leq 1$, then $X(t) \to 0$ in probability as $t \to \infty$.*

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Hint for the large time behaviour

- The infinite time limit of μ_t must be of the form $\mu_\infty(m)$, where $m = \overline{\mu}_\infty(m)$.
- We show that $m \mapsto \overline{\mu}_\infty(m)$ is increasing and concave, and that $\frac{d\overline{\mu}_\infty}{dm}(0) = R_0$.
- Finally we use a comparison with a supercritical continuous time non Markov branching process to make sure that in case $R_0 > 1$, $\overline{\mu}_\infty > 0$. For that purpose, we establish the

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If $R_0 > 1$ *and* $\mathbb{E}[X(0)] > 0$, *then* $\liminf_{t\to\infty} \mathbb{E}[X(t)] > 0$.

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Particular examples

- If $\pi = \delta_1$, every household is of size 1, the model reduces to the standard SIS model, with parameters λ_G and γ. In that case, $R_0 = \lambda_G/\gamma$.
- The same is true if $\lambda_L = 0$, while π is general.
- Suppose that the sizes of all households are very large. Then the process $I(t)$ which describes a local epidemic within a typical household could be well approximated by a branching process with birth rate λ_L and death rate γ. R_0 is then close to $+\infty$ if $\lambda_L \geq \gamma$, and by $\lambda_G/(\gamma - \lambda_L)$ otherwise, so that $R_0 > 1$ is equivalent to $\lambda_G + \lambda_L > \gamma$.

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Central Limit Theorem 1

- We now define, for each $N \geq 1$, $U_t^N = \sqrt{N}(\mu^N(t) - \mu(t))$, and $\overline{U}_t^N = \sum_{n\geq 1}\sum_{k=1}^n kU_{n,k}^N(t)$, $\overline{V}^N = \sum_{n\geq 1} n\sum_{k=0}^n U_{n,k}^N(0)$. Note that we also have $\overline{U}_t^N = \sqrt{N}(\overline{\mu}^N(t) - \overline{\mu}(t))$.
- U^N solves

$$
\begin{aligned}
(U_t^N, \phi) &= (U_0^N, \phi) + \sqrt{N}\mathcal{Y}^{N,\phi}(t) \\
&+ \int_0^t \left\{ (U_s^N, \mathcal{L}\phi(\cdot,\cdot,\overline{\nu}^N, \overline{\mu}_s^N)) + \mu_s^N(\mathcal{G}_{SPR}\phi(\cdot,\cdot)) \left(\frac{\overline{U}_s^N}{\overline{\pi}} - \frac{\overline{\mu}_s^N \overline{V}^N}{\overline{\nu}^N \overline{\pi}} \right) \right\} ds,
\end{aligned}
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\begin{aligned}
\mathcal{L}\phi(n,x,y,m) &= [\phi(n,x+1) - \phi(n,x)] \left(1 - \frac{x}{n}\right) [\lambda_L x + \lambda_G \frac{n}{y} m] \\
&+ [\phi(n,x-1) - \phi(n,x)]\gamma x
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Central Limit Theorem 1

- We now define, for each $N \geq 1$, $U_t^N = \sqrt{N}(\mu^N(t) - \mu(t))$, and $\overline{U}_t^N = \sum_{n\geq 1}\sum_{k=1}^n kU_{n,k}^N(t)$, $\overline{V}^N = \sum_{n\geq 1} n\sum_{k=0}^n U_{n,k}^N(0)$. Note that we also have $\overline{U}_t^N = \sqrt{N}(\overline{\mu}^N(t) - \overline{\mu}(t))$.
- U^N solves

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Central Limit Theorem 2

- As $N \to \infty$, $U^N \Rightarrow U$, where U is the Ornstein–Uhlenbeck process, solution of the SDE

$$(U_t, \phi) = (U_0, \phi) + \mathcal{Y}^{\phi}(t) + \int_0^t \left\{ (U_s, \mathcal{L}\phi(\cdot,\cdot,\overline{\pi},\overline{\mu}_s)) + (\mu_s, \mathcal{G}_{SPR}\phi(\cdot,\cdot)) \left(\frac{\overline{U}_s}{\overline{\pi}} - \frac{\overline{\mu}_s \overline{V}}{\overline{\pi}^2} \right) \right\} ds,$$

  for all test function ϕ, $t > 0$, and where $\overline{U}_t := \sum_{n \geq 1} \sum_{k=1}^{n} k U_{n,k}(t)$, $\overline{V} := \sum_{n \geq 1} \sum_{k=0}^{n} U_{n,k}(0)$.
- The collection of Gaussian martingales $\mathcal{Y}_{n,k}(t)$ can be represented with the help of a collection of mutually independent Brownian motions (see next page).
- So far proved only if $\text{supp}(\pi)$ is finite. *Work in progress : we expect to be able to treat the case where π has a finite 7–th moment.*

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Central Limit Theorem 3

$$\mathcal{Y}_{n,k}(t) = \int_0^t \sqrt{\mu_{n,k}(s)\left(1-\frac{k}{n}\right)(\lambda_L k + \lambda_G \frac{n}{\pi}\overline{\mu}_s)}dW_s^{n,k,i}$$
$$- \int_0^t \sqrt{\mu_{n,k-1}(s)\left(1-\frac{k-1}{n}\right)(\lambda_L (k-1) + \lambda_G \frac{n}{\pi}\overline{\mu}_s)}dW_s^{n,k-1,i}$$
$$+ \int_0^t \sqrt{\mu_{n,k}(s)\gamma k}dW_s^{n,k,r} - \int_0^t \sqrt{\mu_{n,k+1}(s)\gamma(k+1)}dW_s^{n,k+1,r},$$

with the convention that $\mu_s^{n,-1} = \mu_s^{n,n+1} = 0$ and the various Brownian motions are mutually independent.

Two remarks

- We note that $\sum_{n\ge1}\sum_{k=0}^{n} U^N_{n,k}(t) = 0$ a.s., since it is the difference between the total masses of two probability measures. Hence the same is true in the limit : $\sum_{n\ge1}\sum_{k=0}^{n} U_{n,k}(t) = 0$ a.s., i.e. the process $U(t)$ takes its values in the set of vectors which are orthogonal to the vector $\mathbf{1}$.
- We see that

$$\frac{d}{dt}\sum_{k=0}^{n}\mu_{n,k}(t) = \frac{d}{dt}\sum_{k=0}^{n}\mu^N_{n,k}(t) = \frac{d}{dt}\sum_{k=0}^{n}U_{n,k}(t) = 0\,.$$

This follows readily from the fact that the sizes of the households are fixed over time. Let now $V_n := \sum_{k=0}^{n} U_{n,k}(0)$. We clearly have that $V_n = \lim_{N\to\infty}\sqrt{N}\sum_{i=1}^{N}(\mathbf{1}_{\nu_i=n} - \pi_n)$. It is easy to check that the vector V whose n–th component is V_n follows the Gaussian law $N(0,\Gamma)$, where $\Gamma = \text{diag}(\pi) - \pi\otimes\pi$. Since $\Gamma\mathbf{1} = 0$, $\sum_{n\ge1} V_n = 0$ a.s., as could be predicted from the previous remark.

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More general results on Fluctuations

- At least in the case where $\text{supp}(\pi)$ is finite, one can also study the Moderate and Large deviations from the Law of Large Numbers.
- However, for complex models as the household model, it is hard to get explicit expressions for the associated rate functions.
- On the next slide, we go back to the simple SIS homogeneous model, call Z_t^N the proportion of infectious individuals in the population, and $z*$ that proportion in the endemic equilibrium (assuming that $\lambda > \gamma$) of the deterministic LLN ODE.

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Comparison between CLT, MD and LD (for the homogeneous model)

- In the central limit theorem, the asymptotic variance is γ/λ. From standard estimates on Gaussian r.v.'s, for any $\eta > 0$, there exist t and N large enough such that

$$\mathbb{P}(\sqrt{N}(Z_t^N - z^*) \le -a) \le \exp\left\{-\left(\frac{\lambda a^2}{2\gamma} - \eta\right)\right\}.$$

- The moderate deviations result says that similarly, for $0 < \alpha < 1/2$,

$$\mathbb{P}(N^\alpha(Z_t^N - z^*) \le -a) \le \exp\left\{-N^{1-2\alpha}\left(\frac{\lambda a^2}{2\gamma} - \eta\right)\right\},$$

- The large deviations result says that, if $a < \gamma/\lambda$,

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and for small $a > 0$, $a + \left(\frac{\gamma}{\lambda} - a\right)\log\left(1 - a\frac{\lambda}{\gamma}\right) \sim \frac{\lambda a^2}{2\gamma}$.

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THANK YOU FOR YOUR ATTENTION !