Weak well-posedness

for a class of degenerate SDEs driven by stable processes

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Our Setting

We consider the following SDE on $\mathbb{R}^{nd}$ starting at x:

$$dX_t = \mathbf{F}(t, X_t)dt + BdZ_t \quad \text{where} \quad B := \begin{bmatrix} I_{d\times d} \\ 0_{(n-1)d\times d} \end{bmatrix},$$

- Z_t is a Lévy process on $\mathbb{R}^d$ such that its Lévy measure ν is
  - symmetric, α-stable for some α in $(0, 2)$, so that

$$\nu(dy) = \frac{\mu(ds)d\rho}{\rho^{1+\alpha}} \quad \text{where} \quad y = \rho s;$$

  - non-degenerate, in the sense that $\Psi_Z(\lambda) \asymp |\lambda|^\alpha$;
- drift $\mathbf{F} = (F_1, \dots, F_n)$ such that $F_i(t, x) = F_i(t, x_{i-1}, \dots, x_n)$ and
  - $x_j \mapsto F_i(t, x_{i-1}, \dots, x_n)$ is β_i^j-Hölder, uniformly in time;
  - the matrixes $D_{x_{i-1}}\mathbf{F}_i(t, x)$ have full rank for any (t, x).

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Idea of Proof

Weak well-posedness $\rightsquigarrow$ martingale formulation.

Martingale Problem

A probability measure $\mathbb{P}^\alpha$ on $\mathcal{D}([0,\infty),\mathbb{R}^{nd})$ is a solution of martingale problem starting at x if:

- $\mathbb{P}^\alpha(X_0 = x) = 1$;
- Let $\mathcal{L}$ generator of $\{X_t\}_{t\geq 0}$. For any u in $\mathrm{dom}(\partial_t + \mathcal{L})$,

$$\left(u(t,X_t) - u(0,X_0) - \int_0^t (\partial_s + \mathcal{L})u(s,X_s)\,ds\right)_{t\geq 0} \text{ is a } \mathbb{P}^\alpha\text{-martingale.}$$

- Existence through Euler "discretization" scheme. For $s \in [t_i, t_{i+1}]$,

$$X_s^m = X_{t_i}^m + \int_{t_i}^s \Big(\mathbf{F}(t_i, X_{t_i}^m) + D_{sd}\mathbf{F}(t_i, X_{t_i}^m)(X_u^m - X_{t_i}^m)\Big)\,du + B(Z_s - Z_{t_i}).$$

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Uniqueness

Let $\mathbb{P}^\alpha$, $\tilde{\mathbb{P}}^\alpha$ be two solutions of the martingale problem starting at x.

- fix $T > 0$ and for any f in a "rich enough" class of functions $\mathcal{F}$, consider

$$\begin{cases} (\partial_t + \mathcal{L})u = -f & \text{on } [0, T] \times \mathbb{R}^{nd}; \\ u(T, \cdot) = 0 & \text{on } \mathbb{R}^{nd}; \end{cases}$$

- If u is smooth enough,

$$\big(u(t, X_t) - u(0, x) + \int_0^t f(s, X_s)\, ds\big)_{0 \le t \le T} \text{ is a } \mathbb{P}^\alpha\text{-martingale;}$$

- Then,

$$\mathbb{E}[\int_0^T f(s, X_s)\, ds] = u(0, x) = \tilde{\mathbb{E}}[\int_0^T f(s, X_s)\, ds].$$

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Linear Example

For the moment, we focus on the linear example:

$$\mathbf{F}(t,x) = Ax \quad \text{where} \quad A_{i,j} = \begin{cases} \text{non-singular}, & \text{if } j = i-1 \\ 0_{d\times d}, & \text{otherwise.} \end{cases}$$

Given $f\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ and $g\colon \mathbb{R}^{nd}\to\mathbb{R}$, we are interested in the associated parabolic IPDE:

$$\begin{cases} (\partial_t + \mathcal{L}^{ou})u = -f & \text{on } [0,T]\times\mathbb{R}^{nd}; \\ u(T,\cdot) = g(\cdot) & \text{on } \mathbb{R}^{nd} \end{cases}$$

where $\mathcal{L}^{ou} = \langle Ax, D_x\rangle + \mathcal{L}_\alpha$ and $\mathcal{L}_\alpha$ is the generator of Z_t, i.e.

$$\mathcal{L}_\alpha\phi(t,x) := \text{p.v.}\int_{\mathbb{R}^d}\left[\phi(t,x+By)-\phi(t,x)\right]\nu(dy).$$

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Which is the more suitable functional space for Schauder Estimates?

Roughly speaking, the degenerate structure of A gives the dynamic different speed pace depending on the direction x_i we consider. Indeed,

Analytical point of view

We search a dilation operator δ_λ such that

$$\mathcal{L}^{ou}u = 0 \implies \mathcal{L}^{ou}(u \circ \delta_\lambda) = 0.$$

Due to the structure of A and the α-stability of ν, this is true for

$$\delta_\lambda(t, x) = (\lambda^\alpha t, x_1, \lambda^{1+\alpha}x_2, \dots, \lambda^{1+\alpha(n-1)}x_n)$$

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Probabilistic point of view

Fixed x in $\mathbb{R}^{nd}$, we introduce the associated SDE:

$$\begin{cases} dX_t = AX_t dt + BdZ_t, \quad t \geq 0 \\ X_0 = x. \end{cases}$$

Focusing only on the random part, a solution of the above SDE has essentially the following structure

$$\overline{X}_t = \Big(Z_t, \int_0^t Z_u \, du, \dots, \underbrace{\int_0^{u_n=t} du_{n-1} \dots \int_0^{u_2} du_1}_{n-1 \text{ times}} Z_{u_1}\Big)$$

The intrinsic time scale of this multivariate process is then

$$\big(t^{\frac{1}{\alpha}}, t^{\frac{1+\alpha}{\alpha}}, \dots, t^{\frac{1+\alpha(n-1)}{\alpha}}\big).$$

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Anisotropic Hölder Spaces

The previous reasoning suggest us to introduce the homogeneous quasi-distance corresponding to the dilation operator δ_λ

$$d_P((t,x);(s,x')) := (s-t)^{\frac{1}{\alpha}} + d(x,x')$$

where

$$d(x,x') := \sum_{i=1}^{n} |(x-x')_i|^{\frac{1}{1+\alpha(i-1)}}$$

$\rightsquigarrow$ More suitable space for our estimates is the anisotropic Hölder space associated to d:

$$\|\phi(t,x)\|_{L^\infty(C^\beta_{b,d})} := \sup_{t\in[0,T]} \sum_{i=1}^{n} \sup_{z\in\mathbb{R}^{d(n-1)}} \|\phi(t,z,\cdot_i)\|_{C_b^{\frac{\beta}{1+\alpha(i-1)}}}.$$

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Representation of Solution

- Through Fourier techniques,

$$\overline{X}_t \overset{(law)}{=} \mathbb{M}_t S_t$$

where S_t is a non-degenerate, α-stable process in $\mathbb{R}^{nd}$ and

$$(\mathbb{M}_t)_{i,j} = \begin{cases} t^{i-1} I_{d\times d}, & \text{if } j = i \\ 0, & \text{otherwise.} \end{cases}$$

- the solution X_t of the associated SDE has a smooth density

$$p_\alpha(t,x,y) := \frac{1}{\det(\mathbb{M}_t)} p_S(t, \mathbb{M}_t^{-1}(e^{At}x - y));$$

- We can represent a solution u of the considered IPDE through

$$u(t,x) = P_{T-t}g(x) + \int_t^T P_{s-t}f(s,x)\,ds$$

where P_t is the transition semigroup associated to p_α.

Representation of Solution

- Through Fourier techniques,

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$$p_\alpha(t,x,y) := \frac{1}{\det(\mathbb{M}_t)} p_S\big(t, \mathbb{M}_t^{-1}(e^{At}x - y)\big);$$

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Smoothing effects in time

To control u in the Holder norm, we firstly need to bound the derivatives of the associated Markov semigroup:

- The derivatives of the solution density induce additional singularities in time

$$|D_{x_k} p_\alpha(t,x,y)| \le Cq(t,x,y)t^{-\frac{1+\alpha(k-1)}{\alpha}}$$

where $q(t,\cdot)$ is a density with suitable properties;

- The associated semigroup gives a "non-standard" regularization effect in time to the system

$$|D_{x_k}[P_t\phi](t,x)| \le C\|\phi\|_{C^\beta_{b,d}} t^{\frac{\beta-(1+\alpha(k-1))}{\alpha}}$$

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Given f in $L^\infty([0,T];C^{\beta}_{b,d}(\mathbb{R}^{nd}))$ and g in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$,

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Any mild solution u of the considered IPDE satisfies

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Perturbative Approach I

- Depending on some frozen parameters (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, choose a flow

$$\theta_{t,\tau}(\xi) := \xi + \int_{\tau}^{s} A\theta_{u,\tau}(\xi) + \tilde{\mathbf{F}}(u, \theta_{u,\tau}(\xi))\, du;$$

- Introduce the time inhomogeneous drift $\tilde{\mathbf{F}}(t, \theta_{t,\tau}(\xi))$ frozen along the considered flow;
- Rewrite the IPDE of interest in the following way:

$$\begin{aligned}\partial_t u(t,x) + \langle Ax + \tilde{\mathbf{F}}(t, \theta_{t,\tau}(\xi)), D_x u(t,x)\rangle + L_\alpha u(t,x) \\ = -f(t,x) + \langle \tilde{\mathbf{F}}(t, \theta_{t,\tau}(\xi)) - \tilde{\mathbf{F}}(t,x), D_x u(t,x)\rangle.\end{aligned}$$

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Control of the Error

- Decompose error: $\sum_{i=1}^{d}\left[\tilde{F}_i(t,\theta_{t,\tau}(\xi)) - \tilde{F}_i(t,x)\right] D_{x_i} u(t,x)$;
- on the first component: direct computation;
- on the degenerate variables: Besov duality + Besov control of the density;

$\Rightarrow$ Schauder estimates holds for the perturbed system under the additional constraints

$$\frac{\alpha-\beta}{1+\alpha(n-2)} > 1-\alpha \quad \text{and} \quad \tilde{F}_i \in L^\infty([0,T]; C_d^{1+\alpha(i-2)+\beta}(\mathbb{R}^{nd})).$$

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Through Linearization of **F** and the perturbative approach,

Mild Well-Posedness of PDE

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"Almost" Sharp Threshold

Fixed i in $[\![1, n]\!]$, j in $[\![i, n]\!]$, uniqueness in law may fail for the original SDE if:

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$\rightarrow$ Construction of "ad hoc" Peano example starting at 0 (for $d = 1$)

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Focusing only on the i-th component of the chain:

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where $\mathcal{I}_t^k(Z)$ is the k-th iterated time integral on $[0,t]$ of $\{Z_s\}_{s\geq 0}$.

The deterministic Peano example has infinite solutions of the form:

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The additional random component recovers the well-posedness if the noise is strong enough.

The critical exponents can be explained through comparison in small time between:

$$\text{(fluctuations of noise) } t^{i-1+\frac{1}{\alpha}} > t^{\frac{(j-i)\beta_i^j+1}{1-\beta_i^j}} \text{ (extremal solutions)}.$$

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