Optimal Bail-Out Dividend Problem with Transaction Cost and Capital Injection Constraint

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Problem formulation

De Finetti's Problem with Fixed Transaction Cost and Capital Injection

The insurance company's surplus under a strategy $\pi := (L^{\pi}, R^{\pi})$ is given by the process

$$X^{\pi}_t = X_t - L^{\pi}_t + R^{\pi}_t, \qquad t \geq 0,$$

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[a] $X = \{X_t : t \geq 0\}$ is a Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{F} := \{\mathcal{F}_t : t \geq 0\}$ be the completed and right-continuous filtration generated by X.

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- We assume that L^{π} and R^{π} are non-negative, and non-decreasing $\mathbb{P}_x$-a.s., start at zero and are adapted to the filtration $\mathbb{F}$.

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Admissible strategies

- $\pi \in \Theta$ iff X^π is non-negative and

$$\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}R_t^\pi\right] < \infty, \quad \text{for } x \geq 0.$$

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- When there is a fixed transaction cost $\delta > 0$, we only consider the class of admissible strategies $\pi = \{L^\pi, R^\pi\} \in \Theta$ such that

$$L_t^\pi = \sum_{0 \leq s \leq t} \Delta L_s^\pi, \quad t \geq 0,$$

where $\Delta L_t^\pi := L_{t+}^\pi - L_t^\pi$.

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where $\Delta L_t^\pi := L_{t+}^\pi - L_t^\pi$. We denote this class by Θ_δ and in the case $\delta = 0$, we take $\Theta_0 \equiv \Theta$.

Expected net present value (NPV)

Given an initial capital $x \geq 0$ and a policy $\pi = \{L^\pi, R^\pi\} \in \Theta_\delta$, with $\delta \geq 0$,

$$v^\pi_{\delta,\Lambda}(x) := \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}\left(L^\pi_t - \delta \sum_{0\leq s\leq t} \mathbf{1}_{\{\Delta L^\pi_s > 0\}}\right)\right] - \Lambda \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}R^\pi_t\right], \tag{1}$$

where $q > 0$, $\delta \geq 0$, and $\Lambda \geq 0$ is the unit cost per capital injected.

Dividend problem

Find an optimal admissible strategy $\pi^* = (L^{\pi^*}, R^{\pi^*}) \in \Theta_\delta$ such that

$$v_{\delta,\Lambda}^{\pi^*}(x) = V_{\delta,\Lambda}(x) := \sup_{\pi \in \Theta_\delta} v_{\delta,\Lambda}^{\pi}(x).$$

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- Junca M., Moreno-Franco H. A., Pérez J.L.. 2019. Optimal bail-out dividend problem with transaction cost and capital injection constraint. *Risks.*
- Wang W., Wang Y., Li X., Wu X.. 2019. Dividend and Capital Injection Optimization with Transaction Cost for Spectrally Negative Lévy Risk Processes. *Arxiv.*

Optimal dividends with capital injection constraint

We aim to solve

$$V_\delta(x,K) := \sup_{\pi\in\Theta_\delta} \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}\left(L_t^\pi - \delta \sum_{0\le s\le t} \mathbf{1}_{\{\Delta L_s^\pi>0\}}\right)\right]$$
$$\text{s.t. } \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}R_t^\pi\right] \le K, \quad \text{for any } x\ge 0 \text{ and } K \ge 0. \tag{2}$$

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In another context:

- Hernández C., Junca M., and Moreno-Franco H.. 2018. A time of ruin constrained optimal dividend problem for spectrally one-sided Lévy processes. *Insurance: Mathematics and Economics.*
- Junca M. Moreno-Franco H.A., Pérez J.L., Yamazaki K.. 2019. Optimality of refraction strategies for a constrained dividend problem. *Adv. App. Prob.*.

Lagrange multipliers

In order to solve the problem (2), we use Lagrange multipliers to reformulate our problem. For $\Lambda \geq 0$ we define the function

$$v^{\pi}_{\delta,\Lambda}(x, K) := v^{\pi}_{\delta,\Lambda}(x) + \Lambda K,$$

where $v^{\pi}_{\delta,\Lambda}$ is the expected NPV defined in (1).

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$$v^{\pi}_{\delta,\Lambda}(x,K) := v^{\pi}_{\delta,\Lambda}(x) + \Lambda K,$$

where $v^{\pi}_{\delta,\Lambda}$ is the expected NPV defined in (1).
Note that $V_{\delta}(x,K) = \sup_{\pi\in\Theta_{\delta}} \inf_{\Lambda\geq 0} v^{\pi}_{\delta,\Lambda}(x,K)$ since

$$\inf_{\Lambda\geq 0} v^{\pi}_{\delta,\Lambda}(x,K) = \begin{cases} \mathbb{E}_x\Big[\int_0^{\infty} \mathrm{e}^{-qt}\,\mathrm{d}\Big(L^{\pi}_t \\ \quad -\delta\sum_{0\leq s\leq t}\mathbf{1}_{\{\Delta L^{\pi}_s>0\}}\Big)\Big], & \text{if } \mathbb{E}_x\left[\int_0^{\infty}\mathrm{e}^{-qt}\,\mathrm{d}R^{\pi}_t\right]\leq K, \\ -\infty, & \text{otherwise.} \end{cases}$$

Main goal

The dual problem of (2) is defined as

$$V_\delta^D(x,K) := \inf_{\Lambda\geq 0} \sup_{\pi\in\Theta_\delta} v_{\delta,\Lambda}^{\pi}(x,K) = \inf_{\Lambda\geq 0} \left\{ \Lambda K + \sup_{\pi\in\Theta_\delta} v_{\delta,\Lambda}^{\pi}(x) \right\}.$$

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The main goal is to prove that

$$V_\delta^D(x,K) \leq V_\delta(x,K).$$

Some assumptions and properties of the process X

- We omit the case when X has monotone trajectories.
- The Laplace exponent of X is given by

$$\psi(\theta) := \log \mathbb{E}[\theta X_1] = \gamma\theta + \frac{\sigma^2}{2}\theta^2 - \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta z} - \theta z \mathbf{1}_{\{0<z\leq 1\}}\right)\Pi(\mathrm{d}z),$$

  where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and the Lévy measure of X, Π, is a measure defined on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge z^2)\Pi(\mathrm{d}z) < \infty.$$

- we assume that $\psi'(0+) = \mathbb{E}[X_1] > -\infty$.
- We assume that either X has unbounded variation or Π is absolutely continuous with respect to the Lebesgue measure.

Some assumptions and properties of the process X

- The process X has bounded variation paths if and only if

$$\sigma = 0 \quad \text{and} \quad \int_{(0,1]} z\Pi(\mathrm{d}z) < \infty.$$

In this case, X can be written as

$$X_t = ct - \widetilde{S}_t, \qquad t \geq 0, \tag{3}$$

where $c := \gamma + \int_{(0,1]} z\Pi(\mathrm{d}z)$ and $\widetilde{S} = \{\widetilde{S}_t : t \geq 0\}$ is a drift-less subordinator.

- It is necessary that the constant c is greater than zero. Note that the Laplace exponent of X, with X as in Eq. (3), is given as follows,

$$\psi(\theta) = c\theta - \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\theta z}\right)\Pi(\mathrm{d}z), \quad \theta \geq 0.$$

Review of scale functions

- Let $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ be scale function of X, which takes value zero on $(-\infty, 0)$, is strictly increasing functions on $[0, \infty)$ and is defined by its Laplace transform,

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

  where $\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$.
- Define, for $x \in \mathbb{R}$,

$$\overline{W}^{(q)}(x) := \int_0^x W^{(q)}(y)\mathrm{d}y,$$

$$Z^{(q)}(x) := 1 + q\overline{W}^{(q)}(x),$$

$$\overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z)\mathrm{d}z.$$

- For $x \leq 0$, $\overline{W}^{(q)}(x) = 0$, $Z^{(q)}(x) = 1$ and $\overline{Z}^{(q)}(x) = x$.
- We have $W^{(q)} \in C^1(\mathbb{R}\backslash\{0\})$.
- We know

$$W^{(q)}(0) = \begin{cases} 0, & \text{if } X \text{ is of unbounded variation,} \\ \dfrac{1}{c}, & \text{if } X \text{ is of bounded variation,} \end{cases}$$

$$W^{(q)\prime}(0+) := \lim_{x \downarrow 0} W^{(q)\prime}(x)$$

$$= \begin{cases} \dfrac{2}{\sigma^2}, & \text{if } \sigma > 0, \\ \infty, & \text{if } \sigma = 0 \text{ and } \Pi(0,\infty) = \infty, \\ \dfrac{q + \Pi(0,\infty)}{c^2}, & \text{if } \sigma = 0 \text{ and } \Pi(0,\infty) < \infty. \end{cases}$$

- We have that $Z^{(q)}$ is a strictly log-convex function on $(0,\infty)$, for $q > 0$ and $\overline{W}^{(q)}$ is a log-concave function on $(0,\infty)$.

One- and two-sided exit formulae

We define the stopping times τ_{a^-} and τ_{a^+},

$$\tau_a^- := \inf\{t > 0 : X_t < a\} \quad \text{and} \quad \tau_a^+ := \inf\{t > 0 : X_t > a\}, \quad a \in \mathbb{R};$$

here and further on, let $\inf \varnothing = \infty$. Then, for $a > b$ and $x \leq a$,

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_b^-\}}\right] = \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)},$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_b^-}\mathbf{1}_{\{\tau_a^+ > \tau_b^-\}}\right] = Z^{(q)}(x-b) - Z^{(q)}(a-b)\frac{W^{(q)}(x-b)}{W^{(q)}(a-b)}.$$

Reflected Lévy processes

Let Y be the Lévy process reflected from below 0, i.e.,

$$Y_t := X_t + R_t^0 := X_t + \sup_{0 \le s \le t} (-X_s \vee 0), \quad \text{for } t \ge 0. \tag{4}$$

Taking $\kappa_b := \inf\{t > 0 : Y_t \in (b, \infty)\}$, with $b > 0$, we know that

$$\mathbb{E}_x\left[\mathrm{e}^{-q\kappa_b} \right] = \frac{Z^{(q)}(x)}{Z^{(q)}(b)}, \quad x \le b.$$

and

$$\mathbb{E}_x\left[\int_{[0,\kappa_b]} \mathrm{e}^{-qt}\, \mathrm{d}R_t^0 \right] = -k^{(q)}(x) + \frac{Z^{(q)}(x)}{Z^{(q)}(b)} k^{(q)}(b), \quad x \le b,$$

where

$$k^{(q)}(x) := \overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}.$$

Capital injection and fixed transaction cost

Dividend problem

Find an optimal admissible strategy $\pi^* = (L^{\pi^*}, R^{\pi^*}) \in \Theta_\delta$ such that

$$v^{\pi^*}_{\delta,\Lambda}(x) = V_{\delta,\Lambda}(x) := \sup_{\pi\in\Theta_\delta} v^{\pi}_{\delta,\Lambda}(x),$$

where

$$v^{\pi}_{\delta,\Lambda}(x) := \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}\left(L^\pi_t - \delta \sum_{0\leq s\leq t} \mathbf{1}_{\{\Delta L^\pi_s>0\}}\right)\right] - \Lambda\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}R^\pi_t\right].$$

Remark

Assume that $\Lambda \geq 1$, otherwise the value function will go to infinity.

Reflected (c_1, c_2)-policies

- Let (c_1, c_2) be a pair such that $0 \leq c_1 < c_2$. Let Y be the Lévy process reflected as before (Eq. (4)).
- So we set $X_t^{c_1,c_2} = Y_t$, for $t \leq T_1^{c_1,c_2}$, where $T_1^{c_1,c_2} = \inf\{t > 0 : Y_t > c_2\}$.
- The process then jumps downward by $c_1 - Y_{T_1^{c_1,c_2}}$ so that $X_{T_1^{c_1,c_2}}^{c_1,c_2} = c_1$.
- Now, for $T_1^{c_1,c_2} \leq t < T_2^{c_1,c_2} = \inf\{t > T_1^{c_1,c_2} : X_t^{c_1,c_2} > c_2\}$, X^{c_1,c_2} is the reflected process from below at 0 of $X_t - (c_1 - X_{T_1^{c_1,c_2}})$, and $X_{T_2^{c_1,c_2}} = c_1$.
- By repeating this procedure, we can construct the process inductively. The process X^{c_1,c_2} clearly admits the decomposition

$$X_t^{c_1,c_2} = X_t - L_t^{(c_1,c_2),0} + R_t^{(c_1,c_2),0}, \qquad t \geq 0.$$

Value function of reflected (c_1, c_2)-policies

The expected NPV associated with a reflected (c_1, c_2)-policy is given by

$$v_{\delta,\Lambda}^{c_1,c_2}(x) = \begin{cases} Z^{(q)}(x) G_\Lambda(c_1, c_2) + \Lambda k^{(q)}(x), & \text{if } x \le c_2, \\ x - c_1 - \delta + v_{\delta,\Lambda}^{c_1,c_2}(c_1), & \text{if } x > c_2, \end{cases}$$

where

$$k^{(q)}(x) = \overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q},$$

$$G_\Lambda(c_1, c_2) := \frac{c_2 - c_1 - \delta - \Lambda\left(\overline{Z}^{(q)}(c_2) - \overline{Z}^{(q)}(c_1)\right)}{Z^{(q)}(c_2) - Z^{(q)}(c_1)}, \qquad (5)$$

for all $c_2 > c_1 \ge 0$.

Choice of Optimal Thresholds

- Note that G_Λ is C^2 on $\mathcal{A} := \{(c_1, c_2) \in \mathbb{R}^2_+ : c_1 < c_2\}$, and

$$\lim_{c_2 \downarrow c_1} G_\Lambda(c_1, c_2) = -\infty, \text{ for } c_1 \geq 0 \text{ fixed},$$

$$\lim_{|c_1|+|c_2| \to \infty} G_\Lambda(c_1, c_2) = \lim_{c_2 \to \infty} G_\Lambda(c_1, c_2) = -\frac{\Lambda}{\Phi(q)}.$$

- The function G_Λ attains its maximum on $\mathcal{A}$.

Proposition

There exists a unique pair $(c_1^\Lambda, c_2^\Lambda)$ in

$$\mathcal{B} := \{(c_1^*, c_2^*) \in \mathcal{A} : G_\Lambda(c_1^*, c_2^*) \geq G_\Lambda(c_1, c_2) \text{ for all } (c_1, c_2) \in \mathcal{A}\}.$$

Furthermore, $0 \leq c_1^\Lambda \leq a_\Lambda < c_2^\Lambda < \infty$ and the value function associated with the $(c_1^\Lambda, c_2^\Lambda)$-policy is

$$v_{\delta,\Lambda}^{c_1^\Lambda, c_2^\Lambda}(x) = \begin{cases} Z^{(q)}(x)\zeta_\Lambda(c_2^\Lambda) + \Lambda k^{(q)}(x), & \text{if } x \leq c_2^\Lambda, \\ x - c_2^\Lambda + v_{\delta,\Lambda}^{c_1^\Lambda, c_2^\Lambda}(c_2^\Lambda), & \text{if } x > c_2^\Lambda, \end{cases}$$

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where

$$\zeta_\Lambda(a) := \frac{1 - \Lambda Z^{(q)}(a)}{qW^{(q)}(a)}, \ a > 0,$$

and

$$a_\Lambda = \sup\{a \geq 0 : \zeta_\Lambda(a) \geq \zeta_\Lambda(x), \text{ for all } x \geq 0\}.$$

Previous results to the verification theorem

- The function $v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}$ is $C^2((0,\infty)\backslash\{c_2^\Lambda\})$ and $C^1(0,\infty)$.
- Let $\mathcal{L}$ be the operator defined as follows,

$$\mathcal{L}F(x) := \gamma F'(x) + \frac{\sigma^2}{2}F''(x) + \int_{(0,\infty)} \left(F(x-z) - F(x) + F'(x)z\mathbf{1}_{\{0<z\leq 1\}}\right)\Pi(\mathrm{d}z), \quad x>0.$$

  Then,
  - $(\mathcal{L}-q)v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(x) = 0$ for $x < c_2^\Lambda$.
  - $(\mathcal{L}-q)v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(x) \leq 0$ for $x > c_2^\Lambda$.
- For $x>0$, we have that $\frac{\mathrm{d}}{\mathrm{dx}}v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(x) \leq \Lambda$.
- For $x \geq y \geq 0$, we have that $v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(x) - v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(y) \geq x - y - \delta$.

Verification theorem

Recalling that the functions $V_{\delta,\Lambda}(x) := \sup_{\pi\in\Theta_\delta} v^{\pi}_{\delta,\Lambda}(x)$ and

$$v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(x) = \begin{cases} Z^{(q)}(x)\zeta_\Lambda(c_2^\Lambda) + \Lambda k^{(q)}(x), & \text{if } x \le c_2^\Lambda, \\ x - c_2^\Lambda + v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(c_2^\Lambda), & \text{if } x > c_2^\Lambda, \end{cases}$$

we have $v_{\delta,\Lambda}^{c_1^\Lambda,c_2^\Lambda}(x) = V_{\delta,\Lambda}(x)$ for all $x \ge 0$. Hence, the $(c_1^\Lambda, c_2^\Lambda)$-policy is optimal.

Optimal Dividends with Capital Injection Constraint

We aim to solve

$$V_\delta(x,K) := \sup_{\pi\in\Theta_\delta} \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}\left(L_t^\pi - \delta \sum_{0\le s\le t} \mathbf{1}_{\{\Delta L_s^\pi > 0\}}\right)\right]$$
$$\text{s.t. } \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}R_t^\pi\right] \le K, \quad \text{for any } x\ge 0 \text{ and } K\ge 0. \tag{6}$$

Main goal

The dual problem of (6) is defined as

$$V_\delta^D(x,K) := \inf_{\Lambda\ge 0}\sup_{\pi\in\Theta_\delta} v_{\delta,\Lambda}^\pi(x,K)$$
$$= \inf_{\Lambda\ge 0}\left\{\Lambda K + \sup_{\pi\in\Theta_\delta} v_{\delta,\Lambda}^\pi(x)\right\} = \inf_{\Lambda\ge 1}\left\{\Lambda K + \sup_{\pi\in\Theta_\delta} v_{\delta,\Lambda}^\pi(x)\right\}. \tag{7}$$

The main goal is to prove that $V_\delta^D(x,K) \le V_\delta(x,K)$.

Previous results

- The curve $\Lambda \mapsto (c_1^\Lambda, c_2^\Lambda)$ is continuous and unbounded, for $\Lambda \in [1, \infty)$.

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- The curve $\Lambda \mapsto (c_1^\Lambda, c_2^\Lambda)$ is continuous and unbounded, for $\Lambda \in [1,\infty)$.
- Let $\overline{\Psi}_x(c_1,c_2)$ be the expected present value of the injected capital under a (c_1,c_2)-reflected policy, i.e.,

$$\overline{\Psi}_x(c_1,c_2) := \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}R_t^{(c_1,c_2),0}\right]$$
$$= \begin{cases} Z^{(q)}(x)\dfrac{\overline{Z}^{(q)}(c_2)-\overline{Z}^{(q)}(c_1)}{Z^{(q)}(c_2)-Z^{(q)}(c_1)} - k^{(q)}(x), & \text{if } 0 \le x \le c_2, \\ \dfrac{\overline{Z}^{(q)}(c_2)Z^{(q)}(c_1)-\overline{Z}^{(q)}(c_1)Z^{(q)}(c_2)}{Z^{(q)}(c_2)-Z^{(q)}(c_1)} - \dfrac{\psi'(0+)}{q}, & \text{if } x > c_2. \end{cases}$$

- Let $x \geq 0$ be fixed. Then
  - If $c_1 \geq 0$ is fixed, then the function $\overline{\Psi}_x(c_1, c_2)$ is strictly decreasing for all $c_2 > c_1$, and

$$\lim_{c_2 \to \infty} \overline{\Psi}_x(c_1, c_2) = -k^{(q)}(x) + \frac{Z^{(q)}(x)}{\Phi(q)} =: \underline{K}_x,$$

  - If $c_2 > 0$ is fixed, $\overline{\Psi}_x(c_1, c_2)$ is strictly decreasing for all $c_1 \in [0, c_2)$.
  - For each $K \in (\underline{K}_x, \overline{K}_x)$, with $\overline{K}_x := \overline{\Psi}_x(0, c_2^1)$, there exist $\underline{c} \leq \overline{c}$ such that the level curve

$$L_K(\overline{\Psi}_x) := \{(c_1, c_2) : \overline{\Psi}_x(c_1, c_2) = K\}$$

    is continuous, contained in the set $[0, \underline{c}] \times [\underline{c}, \overline{c}]$ and contains the points $(0, \overline{c})$ and $(\underline{c}, \overline{c})$.

Theorem

Assume $\delta > 0$ and let V_δ and V_δ^D as in Equations (6) and (7), respectively, then $V_\delta = V_\delta^D$. Furthermore, if x, K are such that

- $K < \underline{K}_x$, then $V_\delta(x, K) = -\infty$;
- $K = \underline{K}_x$, then $V_\delta(x, K) = 0$;
- $K \geq \overline{K}_x$, then $V_\delta(x, K) = V_{\delta,1}(x) + K$; and
- $K \in (\underline{K}_x, \overline{K}_x)$, then there exists $\Lambda^* \geq 1$ such that

$$V_\delta(x, K) = \Lambda^* K + V_{\delta,\Lambda^*}(x)$$
$$= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt}\,\mathrm{d}\left(L_t^{(c_1^{\Lambda^*}, c_2^{\Lambda^*}),0} - \delta \sum_{0\leq s<t} \mathbf{1}_{\left\{\Delta L_s^{(c_1^{\Lambda^*}, c_2^{\Lambda^*}),0} > 0\right\}}\right)\right].$$

Thank you