Mean value formulas for degenerate Kolmogorov equations

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Agenda

- Introduction;
- Mean value formulas for harmonic functions;
- Mean value formulas for the heat equation;
- Mean value formulas for Kolmogorov equations (work in progress with E. Malagoli).

Proptotype equation

Langevin equation. Let $(W_t)_{t\geq 0}$ denote a Wiener process in $\mathbb{R}^n$.

$$\begin{cases} X_t = x_0 + W_t \\ Y_t = y_0 + \int_0^t X_s ds, \end{cases} \qquad \begin{cases} dX_t = dW_t \\ dY_t = X_t dt, \end{cases}$$

The density Γ of (X_t, Y_t), with $(x_0, y_0) = (0,0)$, is

$$\Gamma(x,y,t) = \frac{c_n}{t^{2n}} \exp\left(-\frac{|x|^2}{t} - 3\frac{\langle x,y\rangle}{t^2} - 3\frac{|y|^2}{t^3} \right).$$

It satisfes the Kolmogorov equation

$$\partial_t \Gamma(x,y,t) = \frac{1}{2}\Delta_x \Gamma(x,y,t) + \langle x, \nabla_y \Gamma(x,y,t)\rangle.$$

The very beginning

Theorem (Cauchy)

Let Ω *be an open set of* $\mathbb{C}$ *and let* $f : \Omega \to \mathbb{C}$ *be holomorphic. Let* $z \in \Omega$ *and let* $\gamma : I \to \Omega$ *be a smooth, closed path. Then*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w - z}.$$

Mean value formulas for harmonic functions

Mean value formulas

Let Ω be an open set of $\mathbb{R}^n$ and let $u \in C^2(\Omega)$.

If $\Delta u = 0$ in Ω, and $\overline{B(x,r)} \subset \Omega$, then

- $$u(x) = \frac{1}{\mathscr{H}^{n-1}(\partial B(x,r))} \int_{\partial B(x,r)} u(y) d\mathscr{H}^{n-1}(y),$$
- $$u(x) = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(y) dy.$$

Mean value formulas

Let Ω be an open set of $\mathbb{R}^n$ and let $u \in C^2(\Omega)$.

If $\Delta u \geq 0$ in Ω, and $\overline{B(x,r)} \subset \Omega$, then

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Applications

Let Ω be an open set of $\mathbb{R}^n$ and let $u \in C(\Omega)$.

- Regularity. If u satisfies the mean value formula, then $u \in C^\infty(\Omega)$ and it is harmonic.
- Strong maximum principle. If $\Delta u \geq 0$ in Ω, and $u(x_0) = \sup_\Omega u$, then u is constant.
- Harnack inequality. There exists a positive constant C_H such that

$$\sup_{B(x,r)} u \leq C_H \inf_{B(x,r)} u$$

whenever $B(x, 4r) \subset \Omega$ and $\Delta u = 0$ in Ω.

Proof of the mean value formula

Let Ω be a smooth open set of $\mathbb{R}^n$ and let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

$$\int_\Omega (u(x)\Delta v(x) - v(x)\Delta u(x))\, dx =$$
$$\int_\Omega \operatorname{div}\left(u(x)\nabla v(x) - v(x)\nabla u(x)\right) dx =$$
$$\int_{\partial\Omega} \langle u(x)\nabla v(x) - v(x)\nabla u(x), \nu(x)\rangle d\mathscr{H}^{n-1}(x).$$

Proof of the mean value formula (cont.)

Choose $\Omega = B(x_0, r) \backslash B(x_0, \varepsilon)$, $v(x) = c_r - \Gamma(x - x_0)$, and u harmonic.

$$0 = \int_\Omega \left(u(x) \Delta v(x) - v(x) \Delta u(x) \right) dx =$$

$$\int_{\partial B(x_0,r)} \langle u(x) \nabla v(x) - v(x) \nabla u(x), \nu(x) \rangle d\mathscr{H}^{n-1}(x) -$$

$$\int_{\partial B(x_0,\varepsilon)} \langle u(x) \nabla v(x) - v(x) \nabla u(x), \nu(x) \rangle d\mathscr{H}^{n-1}(x) \to$$

$$\int_{\partial B(x_0,r)} \underbrace{-\langle \nabla \Gamma(x - x_0), \nu(x) \rangle}_{(\omega_n r^{n-1})^{-1}} u(x) d\mathscr{H}^{n-1}(x) - u(x_0).$$

as we let $\varepsilon \to 0$.

Proof of the Harnack inequality

Let $u \geq 0$ be harmonic in Ω. Let $x, y, z \in \Omega$ and $r > 0$ be such that $B(x, 4r) \subset \Omega$, and $y, z \in B(x, r)$. Then

$$B(y, r) \subset B(x, 2r) \subset B(z, 3r) \subset B(x, 4r) \subset \Omega.$$

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Hence

$$u(y) = \frac{1}{\mu(B(y, r))} \int_{B(y,r)} u(w) dw \leq \frac{\mu(B(z, 3r))}{\mu(B(y, r))} \frac{1}{\mu(B(z, 3r))} \int_{B(z,3r)} u(w) dw = 3^n u(z).$$

Mean value formulas for the heat equation

Mean value formulas

Theorem ([Pini] - 1954)

Let Ω be an open set of $\mathbb{R}^2$ and let $u \in C^{2,1}(\Omega)$ satisfying the heat equation $\partial_t u(x,t) = \partial_x^2 u(x,t)$. Then

$$u(x,t) = \frac{1}{\mu(\Omega_r(x,t))} \int_{\Omega_r(x,t)} M_r(x,t,y,s) u(y,s) dy\, ds,$$

whenever $\Omega_r(x,t) \subset \Omega$.

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Here

$$\Omega_r(x,t) := \left\{ (y,s) \in \mathbb{R}^2 \mid \Gamma(x,t,y,s) > \tfrac{1}{r} \right\} = \left\{ (y,s) \in \mathbb{R}^2 \mid (x-y)^2 < 4(t-s) \cdot \left(\log(r) - \tfrac{1}{2} \log(4\pi(t-s)) \right) \right\}.$$

Strong maximun principle

Note that $M_r(x,t,y,s) = \frac{(x-y)^2}{4(t-s)^2} \geq 0$.

Corollary. Let Ω be an open set of $\mathbb{R}^2$ and let $u \in C^{2,1}(\Omega)$ be such that

- $\partial_t u(x,t) \leq \partial_x^2 u(x,t)$;
- $u(x_0,t_0) = \sup_\Omega u$.

Then u is constant in the *propagation set* $\mathscr{A}_{(x_0,t_0)}(\Omega)$.

Bounded kernels

Remark ([Kuptsov] - 1980) Note that $M_r(x,t,y,s) = \frac{|x-y|^2}{4(t-s)^2} \leq \frac{1}{t-s}\left(\log(r) - \frac{1}{2}\log(4\pi(t-s))\right)$ in the set $\Omega_r(x,t)$.
Proceed as follows:

- add some further variable $\widetilde{x} \in \mathbb{R}^m$ an set $\widetilde{u}(x,\widetilde{x},t) := u(x,t)$,
- note that $\partial_t \widetilde{u} = \Delta_x \widetilde{u} + \Delta_{\widetilde{x}} \widetilde{u}$,
- use the representation formula for $\widetilde{u}$ and integrate on $\mathbb{R}^m$.

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We then find a new representation formula with a kernel $\widetilde{M}_r$ which is bounded, provided that $m \ge 2$.

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We obtain an elementary proof of the parabolic Harnack inequality.

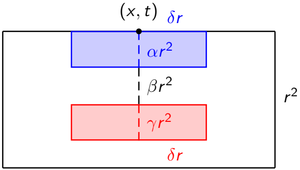
Parabolic equations

Theorem ([Pini] - 1954, [Hadamard] - 1954)

Let $Q_r(x,t) = B(x,r)\times]t-r^2,t[\subset \mathbb{R}^{n+1}$*,*
and let $\alpha,\beta,\gamma,\delta \in]0,1[$ *with* $\alpha+\beta+\gamma<1$*.*
Then there exists $C = C(\alpha,\beta,\gamma,\delta,n)$ *such that*

$$\sup_{Q_r^-(x,t)} u \le C \inf_{Q_r^+(x,t)} u$$

for every $u : Q_r(x,t) \to \mathbb{R}, u \ge 0,$ *satisfying* $u_t = \Delta u$*.*



Mean value formulas for degenerate Kolmogorov equations

Mean value formulas

Theorem ([Kuptsov] - 1983, [Garofalo, Lanconelli] - 1990, [P. Lanconelli] - 1994)

Let Ω be an open set of $\mathbb{R}^{2n+1}$ and let $u \in C^{2,1}(\Omega)$ satisfying

$$\partial_t u(x,y,t) = \Delta_x u(x,y,t) + \langle x, \nabla_y u(x,y,t)\rangle.$$

Then

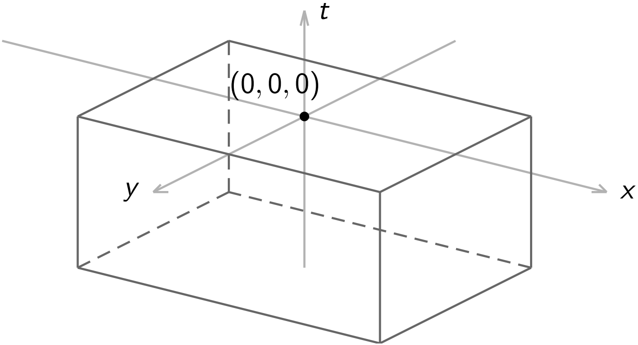
$$u(x,y,t) = \frac{1}{\mu(\Omega_r(x,y,t))} \int_{\Omega_r(x,y,t)} M_r(x,y,t,\xi,\eta,\tau) u(\xi,\eta,\tau) d\xi\, d\eta\, d\tau.$$

$$\Omega_r(x,y,t) := \left\{ (\xi,\eta,\tau) \in \mathbb{R}^{2n+1} \mid \Gamma(x,y,t,\xi,\eta,\tau) > \tfrac{1}{r} \right\} = \left\{ (\xi,\eta,\tau) \in \mathbb{R}^{2n+1} \mid \tfrac{|x-\xi|^2}{t-\tau} + 3\tfrac{|y-\eta+(t-\tau)(x+\xi)|^2}{(t-\tau)^3} < (\log(r) - 2n\log(c_n(t-\tau))) \right\}.$$

Strong minimum principle

Theorem ([Bony] (1969))

Let $u : Q \to \mathbb{R}$ be a non-negative solution to $u_{xx} + xu_y = u_t$. If $u(0,0,0) = 0$, then...



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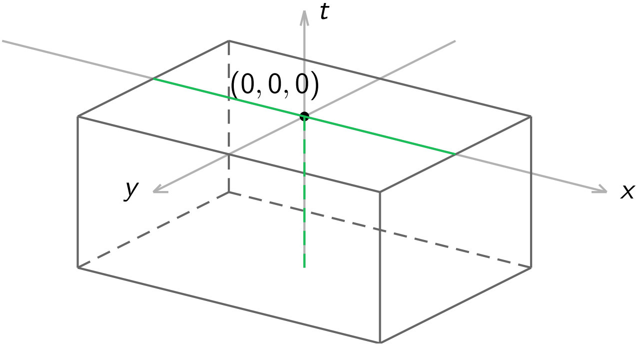

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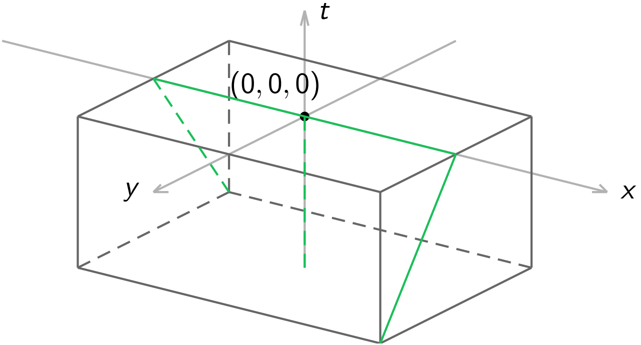

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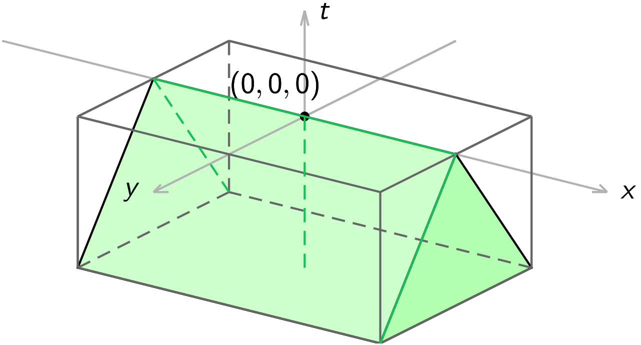

...$u \equiv 0$ in the *Propagation set* $\mathcal{A}_{(0,0,0)}$.

Hölder continuous coefficients

$$\mathscr{L}u = \mathrm{Tr}\left(A(x,y,t)D_x^2 u\right) + \langle x, D_y u\rangle - \partial_t u, \quad (x,y,t) \in \mathbb{R}^{2n+1}.$$

$$\partial_t u(x,y,t) = \sum_{j,k=1}^{n} a_{jk}(x,y,t)\partial^2_{x_j x_k} u(x,y,t) + \sum_{j=1}^{n} x_j \partial_{x_j} u(x,y,t),$$

for every $(x,y,t) \in \Omega$ (open set of $\mathbb{R}^{2n+1}$).

$A(x,y,t) := \left(a_{jk}(x,y,t)\right)_{j,k=1,\dots,n}$

symmetric, uniformly positive, with bounded, Hölder continuous coefficients.

Fundamental Solution

Exitence of a Fundamental Solution via the Parametrix method

- [P.] (1994)
- [Di Francesco, Pascucci] (2005),
- [Di Francesco, P.] (2006),
- [Konakov, Menozzi, Molchanov] (2010),
- [Menozzi] (2011),
- [de Raynal, Menozzi, Honoré] (2018).

Properties of the Fundamental Solution

$\Gamma = \Gamma(x, y, t, \xi, \eta, \tau)$ belongs to $C^{2,\alpha}$

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- for every $T > 0$ we have

  $$c^- \Gamma^-(x, y, t, \xi, \eta, \tau) \leq \Gamma(x, y, t, \xi, \eta, \tau) \leq c^+ \Gamma^+(x, y, t, \xi, \eta, \tau)$$

  for $0 < t - \tau < T$
- for every $\varepsilon > 0$ there exists $K > 0$ such that

  $$(1-\varepsilon) Z(x, y, t, \xi, \eta, \tau) \leq \Gamma(x, y, t, \xi, \eta, \tau) \leq (1+\varepsilon) Z(x, y, t, \xi, \eta, \tau)$$

  whenever $Z(x, y, t, \xi, \eta, \tau) > K$.

Mean value formulas

Theorem ([P] - 1994)

Let Ω *be an open set of* $\mathbb{R}^{2n+1}$ *and let* $u \in C^{2,1}(\Omega)$ *satisfying*

$$\partial_t u(x,y,t) = Tr\left(A(x,y,t)D_x^2 u\right) + \langle x, \nabla_y u(x,y,t)\rangle.$$

with *smooth* *coefficients. If Then*

$$u(x,y,t) = \frac{1}{\mu(\Omega_r(x,y,t))}\int_{\Omega_r(x,y,t)} M_r(x,y,t,\xi,\eta,\tau)u(\xi,\eta,\tau)d\xi\, d\eta\, d\tau.$$

Many thanks for your attention!

Invariance properties

Consider a solution u to the equation

$$u_{xx} + xu_y = u_t$$

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- Dilation: $v(x,y,t) = u(rx, r^3 y, r^2 t)$
  satisfies $v_{xx} + x v_y = v_t$
- Translation: $w(x,y,t) = u(x+\xi, y+\eta-\xi t, t+\tau)$
  satisfies $w_{xx} + x w_y = w_t$

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Invariance group:

$$(x,y,t) \circ (\xi,\eta,\tau) := (x+\xi, y+\eta-\xi t, t+\tau)$$

$$\delta_r(x,y,t) := (rx, r^3 y, r^2 t)$$

Lie group

$$\mathbb{G} = \left(\mathbb{R}^{2n+1}, \circ, (\delta_r)_{r>0}\right)$$

- Translation: $(x, y, t) \circ (\xi, \eta, \tau) = (\xi + x, \eta + y - t\xi, t + \tau)$;
- Dilation; $\delta_r(x, y, t) = \left(rx, r^3 y, r^2 t\right)$;

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- Translation: $(x, y, t) \circ (\xi, \eta, \tau) = (\xi + x, \eta + y - t\xi, t + \tau)$;
- Dilation; $\delta_r(x, y, t) = \left(rx, r^3 y, r^2 t\right)$;
- Homogeneous norm: $\|(x, y, t)\| := |x| + |y|^{1/3} + |t|^{1/2}$;
- Distance: $d((x, y, t), (\xi, \eta, \tau)) := \|(\xi, \eta, \tau)^{-1} \circ (x, y, t)\|$;
- Ball: $B_r(x_0, y_0, t_0) := \left\{(x, y, t) \mid d((x, y, t), (x_0, y_0, t_0)) < r\right\}$.

Hölder spaces

$$\mathscr{L} u = \mathrm{Tr}\left(A(x,y,t)D_x^2 u\right) + \langle x, D_y u\rangle - \partial_t u$$

We say that $u \in C^{\alpha}(\Omega)$ if

$$|u(x,y,t) - u(\xi,\eta,\tau)| \le M\, d((x,y,t),(\xi,\eta,\tau))^{\alpha}$$

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We say that $u \in C^{2,\alpha}(\Omega)$ if

$$u, \partial_{x_j} u, \partial^2_{x_j x_k} u \in C^{\alpha}(\Omega), \quad j,k = 1,\dots,n,$$

and the *directional* ("Lie") derivative $Yu = \langle x, D_y u\rangle - \partial_t u$ belongs to $C^{\alpha}(\Omega)$.

$$Yu(x,y,t) = \frac{d}{ds} u(x, y + xs, t - s)_{|s=0}$$

Hypoellipticity

Theorem ([Hörmander] - 1967)

Let u be a (distributional) solution to $X_1^2 u + \dots, X_m^2 u + Yu = f$ in $\Omega \subset \mathbb{R}^N \times \mathbb{R}$. If

$$span\Big\{ Y, X_1, \dots, X_m, [X_i, X_j], [X_i, Y], \dots, [X_i, \dots, [X_j, X_l]] \Big\} = \mathbb{R}^{N+1}$$

Then

$$f \in C^\infty(\Omega) \quad \Rightarrow \quad u \in C^\infty(\Omega).$$

Commutators: $[X_i, X_j]f := X_i X_j f - X_j X_i f$

Kolmogorov operator

$$\mathscr{L} := \partial_x^2 + x\partial_y - \partial_t = X^2 + Y$$

- $X = \partial_x, \quad Y = x\partial_y - \partial_t,$

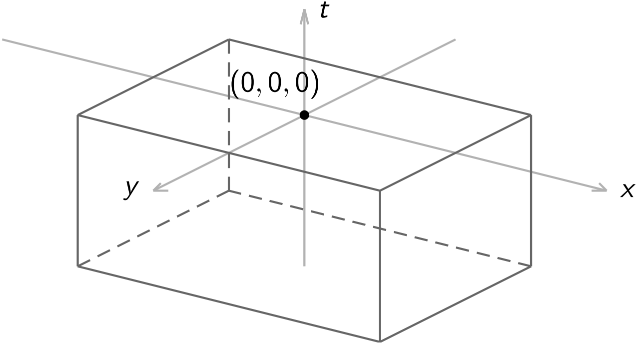
$$Y \sim \begin{pmatrix} 0 \\ x \\ -1 \end{pmatrix} \qquad X \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad [X, Y] = XY - YX \sim \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[X, Y]f := \partial_x(x\partial_y - \partial_t)f - (x\partial_y - \partial_t)\partial_x f = \partial_y f$$

Cauchy-Dirichlet problem

$$u_{xx} + x u_y = u_t, \qquad Q :=]-1,1[^2 \times]-1,0[.$$

Fichera's boundary regularity



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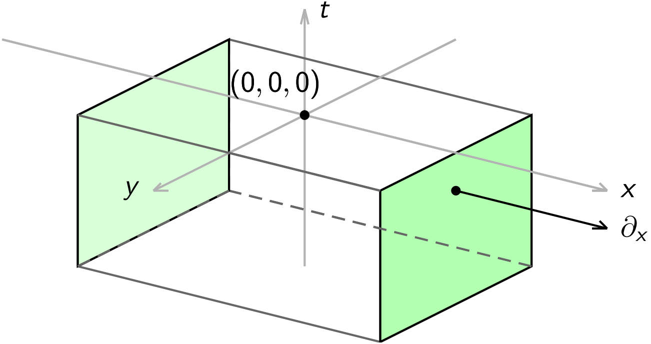

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